Engin 322
Exam 3 Solution

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Solution

(1) $S_X(\omega) = S_{X_1}(\omega) + S_{X_2}(\omega) \quad \begin{cases} S_{X_1}(\omega) = \dfrac{\omega^2+4}{\omega^4+5\omega^2+4} \\ S_{X_2}(\omega) = 8\pi\,\delta(\omega) + 2\pi\,\delta(\omega-4) + 2\pi\,\delta(\omega+4) \end{cases}$

$$\overline{X^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, S_{X_1}(\omega) + \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \left[8\pi\,\delta(\omega) + 2\pi\,\delta(\omega-4) + 2\pi\,\delta(\omega+4)\right]}_{4+1+1\;=\;6}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \frac{\omega^2+4}{\omega^4+5\omega^2+4} \underset{s=j\omega}{=} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{-s^2+4}{s^4-5s^2+4} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{1}{(s+1)(-s+1)}$$

$c(s) = 1 \rightarrow c_0 = 1$
$d(s) = s+1 \rightarrow d_0 = 1;\ d_1 = 1 \quad ; \quad n = 1 \xrightarrow[\text{Table 7-1}]{} = \dfrac{c_0^2}{2d_0d_1} = \dfrac{1}{2}$

$\rightarrow \overline{X^2} = \frac{1}{2} + 6 = 6.5$

(2) (a) $\overline{Y} = \overline{X}\int_0^{\infty} d\lambda\,(2-\lambda)\left[u(\lambda) - u(\lambda-2)\right] = \overline{X}\underbrace{\int_0^2 d\lambda\,(2-\lambda)}_{4-\frac{4}{2}} = 2\overline{X} = 2\sqrt{7}$

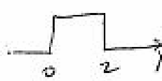
$R_X(\tau\to\infty) = 7 = \overline{X}^2 \rightarrow \overline{X} = \sqrt{7}$

(b) $$\overline{Y^2} = \int_0^{\infty}\int_0^{\infty} d\lambda_1 d\lambda_2\; 2\,\delta(\lambda_2-\lambda_1)\,(2-\lambda_1)\left[u(\lambda_1)-u(\lambda_1-2)\right](2-\lambda_2)\left[u(\lambda_2)-u(\lambda_2-2)\right]$$
$$+ 7\int_0^2 d\lambda_1 (2-\lambda_1)\int_0^2 d\lambda_2 (2-\lambda_2)$$
$$= \int_0^{\infty} d\lambda_1\; 2\,(2-\lambda_1)^2\left[u(\lambda_1) - u(\lambda_1-2)\right]^2 + 7\Big[\underbrace{\int_0^2 d\lambda_1\,(2-\lambda_1)}_{2}\Big]^2$$
$$= 2\int_0^2 d\lambda_1\,(\underbrace{2-\lambda_1}_{\gamma})^2 + 7\Big[\int_0^2 d\lambda_1\,(\underbrace{2-\lambda_1}_{\gamma})\Big]^2$$
$$= -2\int_2^0 d\gamma\,\gamma^2 + 7\left[-\int_2^0 d\gamma\,\gamma\right]^2 = \frac{16}{3} + 7\cdot 2^2$$
$$= \frac{16}{3} + 28 = \frac{100}{3}$$

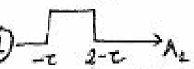
(c) $R_Y(\tau) = \int_0^\infty \int_0^\infty d\lambda_1 d\lambda_2 \, 2\,\delta(\lambda_2-\lambda_1-\tau)(2-\lambda_1)[u(\lambda_1)-u(\lambda_1-2)](2-\lambda_2)[u(\lambda_2)-u(\lambda_2-2)]$

$$+ \underbrace{7\int_0^2 d\lambda_1 (2-\lambda_1)\int_0^2 d\lambda_2 (2-\lambda_2)}_{28}$$

$$= 28 + 2\int_0^\infty d\lambda_1 \, (2-\lambda_1)\underbrace{[u(\lambda_1)-u(\lambda_1-2)]}\,(2-\lambda_1-\tau)\underbrace{[u(\lambda_1+\tau)-u(\lambda_1+\tau-2)]}$$



and

① 

$\tau > 0$

overlap b/w 0 and $2-\tau$ if $\boxed{2>\tau>0}$

② 

$\tau < 0$

overlap b/w $|\tau|$ and 2 if $\boxed{-2<\tau<0}$

① $= 28 + 2\int_0^{2-\tau} d\lambda_1 \, (2-\lambda_1)(2-\lambda_1-\tau) = 28 + 2\int_0^{2-\tau} d\lambda_1 \underbrace{(2-\lambda_1)^2}_{\gamma} - 2\tau\int_0^{2-\tau} d\lambda_1 \underbrace{(2-\lambda_1)}_{\gamma}$

$= 28 - 2\int_2^\tau d\gamma\, \gamma^2 + 2\tau\int_2^\tau d\gamma\, \gamma = 28 - 2\left(\frac{\tau^3}{3}-\frac{8}{3}\right) + 2\tau\left(\frac{\tau^2}{2}-\frac{4}{2}\right)$

$= 28 + \frac{\tau^3}{3} - 4\tau + \frac{16}{3} = \boxed{\frac{100}{3} + \frac{\tau^3}{3} - 4\tau} \quad (2>\tau>0)$

② $= 28 + 2\int_{|\tau|}^2 d\lambda_1 \underbrace{(2-\lambda_1)^2}_{\gamma^2} - 2\tau\int_{|\tau|}^2 d\lambda_1 \underbrace{(2-\lambda_1)}_{\gamma} = 28 - 2\int_{2-|\tau|}^0 d\gamma\,\gamma^2 + 2\tau\int_{2-|\tau|}^0 d\gamma\,\gamma$

$= 28 + 2\frac{(2-|\tau|)^3}{3} - 2\tau\frac{(2-|\tau|)^2}{2} = 28 + \frac{2}{3}\left(8 - 12|\tau| + 6|\tau|^2 - |\tau|^3\right)$

$-\tau\left(4 - 4|\tau| + |\tau|^2\right) = 28 + \frac{16}{3} - 8|\tau| + 4|\tau|^2 - \frac{2}{3}|\tau|^3 - 4\tau + 4\tau|\tau| - \tau|\tau|^2$

$= \frac{16}{3} + 28 + 8\tau + 4\tau^2 + \frac{2}{3}\tau^3 - 4\tau - 4\tau^2 - \tau^3 = \boxed{\frac{100}{3} + 4\tau - \frac{\tau^3}{3}}$

$|\tau| = -\tau$ in $-2<\tau<0$

$(-2<\tau<0)$

Combining ① & ② into one expression:

$$\boxed{R_Y(\tau) = \frac{100}{3} + 4|\tau| - \frac{|\tau|^3}{3}, \quad |\tau|<2}$$