Properties of the Spectral density:

$$\boxed{\overline{X^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \, S_X(\omega)}$$

We can obtain a statistical information such as the moment of second order by integrating the spectral density $S_X(\omega)$ and divide by 2π.

Proof: using Parseval theorem: $\int_{-T}^{T} dt \; x_T^2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \, |\hat{F}_X(\omega)|^2$

$\begin{cases} x_T(t) : \text{signal} \\ \hat{F}_X(\omega) : \text{its Fourier transform.} \end{cases}$

Let's apply $\lim_{T\to\infty} E\left\{\frac{1}{2T} \times \text{Parseval Theorem}\right\}$:

$$\underbrace{E\left\{ \lim_{T\to\infty} \underbrace{\frac{1}{2T}\int_{-T}^{T} dt \; x_T^2(t)}_{\langle x_T^2 \rangle} \right\}}_{\underbrace{\langle x^2 \rangle}_{\overline{X^2}}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \underbrace{\lim_{T\to\infty} E\left\{\frac{|\hat{F}_X(\omega)|^2}{2T}\right\}}_{S_X(\omega)}$$

$$\overline{X^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \, S_X(\omega) \quad \checkmark$$

HW7: Ch6 3-2:

$X(t) = Y \cos(\omega_0 t + \theta)$

$\left\{ \begin{matrix} Y \\ \omega_0 \\ \theta \end{matrix} \right\}$ stat. indep. random variables

$\downarrow$

$\bar{Y} = 3$; $\sigma_Y^2 = 9$

θ uniform from $-\pi$ to π

w_0 : uniform b/w -6 & 6

$\bar{\theta} = 0$

$\sigma_\theta^2 = \frac{(\theta_2 - \theta_1)^2}{12} = \frac{\pi^2}{3}$

$\bar{w}_0 = 0$

$\sigma_{w_0}^2 = 12$

$\sigma_Y^2 = \overline{Y^2} - \bar{Y}^2$

$9 + \bar{Y}^2 = \overline{Y^2} \Rightarrow \boxed{\overline{Y^2} = 18}$

a) Is this process stationary? → if $t \to t + T$

$X(t) \to X(t+T) = X(t)$

$w_0 T = 2\pi$ is period of a cosine.

Yes.

Is this process ergodic?

No.

b) $\bar{X}$? $\overline{X^2}$?

Since process described by X is not ergodic → have to use ensemble average:

$\bar{X} = E[X] = E[Y \cos(w_0 t + \theta)]$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$= E[Y \cos w_0 t \cos\theta - Y \sin w_0 t \sin\theta]$

$= E[Y] \cdot \{ E[\cos w_0 t] E[\cos\theta] - E[\sin w_0 t] E[\sin\theta] \} =$

$\downarrow$

Y & w_0 & θ are stat. indep

$= 3 \cdot \{ 0 \cdot 0 - 0 \cdot 0 \} = 0$

V & W stat indep

$E[VW] = E[V] \cdot E[W]$

$$\overline{X^2} = E[X^2] = E[Y^2 \cos^2(w_0 t + \theta)]$$

a) App A: $2\cos^2 A$
$= 1 + \cos 2A$

$\rightarrow \cos^2 A = \frac{1}{2}(1 + \cos 2A)$

b) $\cos(A+B) = \cos A \cos B$
$- \sin A \sin B.$

$$\overset{a)}{=} E\left[Y^2 \frac{1}{2}(1 + \cos(2w_0 t + 2\theta))\right] \overset{b)}{=} E\left[\frac{Y^2}{2}(1 + \cos 2w_0 t \cos 2\theta - \sin 2w_0 t \sin 2\theta)\right]$$

$$= \frac{1}{2}E[Y^2] = \frac{18}{2} = 9.$$

$$\boxed{\overline{X^2} = 9}$$

c) $E[X(t_1)X(t_2)] = R_X(\tau)$ $(\tau = t_2 - t_1)$

$\hookrightarrow E[X(t)X(t+\tau)]$?

$$E[\underbrace{Y\cos(w_0 t + \theta)}_{X(t)} \cdot \underbrace{Y\cos(w_0 t + w_0\tau + \theta)}_{X(t+\tau)}] = E[Y^2(\cos w_0 t \cos\theta - \sin w_0 t \sin\theta).$$

$$(\cos(w_0 t + \theta)\cos w_0\tau - \boxed{\sin(w_0 t + \theta)} \sin w_0\tau)] =$$

$$= E\left[Y^2\{(\cos w_0 t \cos\theta - \sin w_0 t \sin\theta)^2 \cos w_0\tau - (\cos w_0 t \cos\theta - \sin w_0 t \sin\theta) \cdot (\cos w_0 t \sin\theta + \sin w_0 t \cos\theta) \sin w_0\tau\}\right]$$

$\sin(A+B) = \cos A \sin B + \sin A \cos B$

$\rightarrow \sin(w_0 t + \theta) = \cos w_0 t \sin\theta + \sin w_0 t \cos\theta$

$$= E\left[Y^2\{\cos^2 w_0 t \cos^2\theta \cos w_0\tau + \sin^2 w_0 t \sin^2\theta \cos w_0\tau - 2\cos w_0 t \sin w_0 t \cos\theta \sin\theta \cos w_0\tau\right.$$
$$- \cos^2 w_0 t \cos\theta \sin\theta \sin w_0\tau - \cos w_0 t \sin w_0 t \cos^2\theta \sin w_0\tau$$
$$\left. + \sin w_0 t \cos w_0 t \sin^2\theta \sin w_0\tau + \sin^2 w_0 t \sin\theta \cos\theta \sin w_0\tau\}\right]$$

stat indep.

$$= E[Y^2]\{\underbrace{E[\cos^2 w_0 t]}_{\frac{1}{2}} \underbrace{E[\cos^2\theta]}_{\frac{1}{2}} E[\cos w_0\tau] + \underbrace{E[\sin^2 w_0 t]}_{\frac{1}{2}} \underbrace{E[\sin^2\theta]}_{\frac{1}{2}} E[\cos w_0\tau]\}$$

$$R_X(\tau) = \overline{Y^2}\left\{\frac{1}{2}E[\cos w_0\tau]\right\} = \frac{18}{2}E[\cos w_0\tau]$$

$$\underset{\uparrow}{E[\cos w_0\tau]} = \int_{-6}^{6} dw_0 \underbrace{\frac{1}{12}}_{f(w_0)} \cdot \cos w_0\tau = \frac{1}{12}\int_{-6}^{6} dw_0 \cos w_0\tau$$

$$= \frac{1}{12}\left[\frac{\sin w_0\tau}{\tau}\right]_{-6}^{6}$$

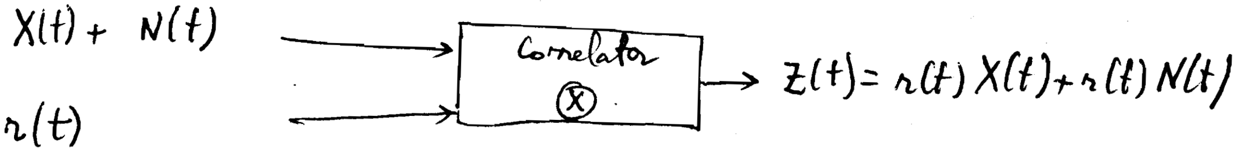
$$= \frac{1}{12\tau}\left[\sin 6\tau - \sin(-6\tau)\right]$$

$$= \frac{2\sin 6\tau}{12\tau} = \frac{\sin 6\tau}{6\tau}$$

$$\rightarrow \boxed{R_X(\tau) = \frac{18}{2}\frac{\sin 6\tau}{6\tau} = \frac{3\sin 6\tau}{2\tau}}$$

HW8 Ch6 : 8.2

$$X(t) = 0.01 \sin(100t + \theta) \quad ; \quad r(t) = 10 \cos(100t + \phi)$$



a)
$$E[Z(t)] = E[\underbrace{r(t)X(t)}_{\substack{\text{constant}\\ \text{for this ensemble}\\ \text{average wrt noise}}}] + E[\underbrace{r(t)N(t)}_{\substack{r \,\&\, N \text{ are}\\ \text{stat.}\\ \text{indep.}}}] = r(t)X(t) + \underbrace{E[r(t)]\underbrace{E[N(t)]}_{\bar{N}=0}}_{0}$$

↓ wrt to Noise

ϕ is fixed and unknown.

$$= r(t)X(t) = 10 \times 0.01 \sin(100t + \theta) . \cos(100t + \phi)$$

$$= \frac{0.1}{2}\left[\sin(200t + \theta + \phi) + \underbrace{\sin(\theta - \phi)}\right]$$

Trig: $\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

b) For what value of ϕ is $E[Z(t)]$ largest ?

When $\sin(\underbrace{\theta - \phi}) = 1 \rightarrow \theta - \phi = \frac{\pi}{2} \rightarrow \boxed{\phi = \theta - \frac{\pi}{2}}$

HW9 (Ch7):

② .2 Stationary random process has a spectral density

$$\hat{S}_X(\omega) = \begin{cases} 1 - \dfrac{|\omega|}{8\pi} & |\omega| \leq 8\pi \\ 0 & \text{elsewhere} \end{cases}$$

Find the mean square value of this process: $\overline{X^2}$

$$\overline{X^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \hat{S}_X(\omega) = \frac{1}{2\pi}\int_{-8\pi}^{8\pi} d\omega \left(1 - \frac{|\omega|}{8\pi}\right)$$

$$\Rightarrow |\omega| = \begin{cases} \omega & \omega \geq 0 \\ -\omega & \omega < 0 \end{cases}$$

$$= \frac{1}{2\pi}\int_{-8\pi}^{0} d\omega \left(1 + \frac{\omega}{8\pi}\right) + \frac{1}{2\pi}\int_{0}^{8\pi} d\omega \left(1 - \frac{\omega}{8\pi}\right)$$

$$= \frac{1}{2\pi}\left[8\pi + \frac{0 - (8\pi)^2}{16\pi}\right] + \frac{1}{2\pi}\left[8\pi - \frac{(8\pi)^2 - 0}{16\pi}\right]$$

$$= 8 - 2\frac{(8\pi)^2}{32\pi^2} = 8 - 4 = 4$$

---

③ .1 Valid spectral density?

$$\hat{S}_X(\omega) = \lim_{T\to\infty} \frac{E\{|\hat{F}_X(\omega)|^2\}}{2T}$$

$$\overline{X^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \hat{S}_X(\omega)$$

1) even in ω: $\hat{S}_X(-\omega) = \hat{S}_X(\omega)$

2) $\hat{S}(\omega) \geq 0$

a) $\dfrac{1}{\omega^2 + 3\omega + 1}$

↳ Not a valid spectral density, b/c it's not even in ω (when I switch the sign of ω, I get a different function)

b) $\dfrac{\omega^2 + 16}{\omega^4 + 9\omega^2 + 18}$ → Yes.

c) $10e^{-\omega^2}$ → Yes.

d) $\frac{w^2+4}{w^4-4w^2+1}$: No $\begin{cases} \text{even } ✓ \\ \text{positive : not always.} \end{cases}$

e) $\left(\frac{1-\cos w}{w}\right)^2$: Yes $\begin{cases} \text{even : } ✓ \\ \text{positive : always } ✓ \end{cases}$

f) $\underbrace{\delta(w)}_{\substack{\text{even} \\ \text{\& positive}}} + \frac{w^3}{w^4+1}$: No : $\left((-w)^3 = -w^3\right)$

(4.1) $\hat{S}_x(w) = \frac{16(w^2+36)}{w^4+13w^2+36}$

a) Write this spectral density as a function of complex frequency s

$S \equiv jw \quad \begin{cases} w^2 = -s^2 \\ w^4 = s^4 \end{cases}$

$$\hat{S}_x(s) = \frac{16(-s^2+36)}{s^4-13s^2+36}$$

b) Poles & zeros: ↳ roots of numerator : $-s^2+36 = 0 \to s = \pm 6$

↳ $(-s+6)(s+6)$

↓ roots of denominator

↓

$s^4-13s^2+36 = 0$

$(\overrightarrow{-s^2+36}) = -(s-6)(s+6)$

$= (-s+6)(s+6)$

$$s^2 = \frac{13 \pm \sqrt{13^2-144}}{2} = \frac{13 \pm 5}{2} \begin{cases} 9 \to s = \pm 3 \\ 4 \to s = \pm 2 \end{cases}$$

$$\hat{S}_x(s) = \frac{16(-s+6)(s+6)}{(s-3)(s+3)(s-2)(s+2)} \quad \begin{cases} \text{zeros : } \pm 6 \\ \text{poles : } \pm 3 ;\ \pm 2 \end{cases}$$

c) $f = 1\,\text{Hz} \rightarrow \omega = 2\pi \frac{\text{rad}}{s} \rightarrow \hat{S}_x(\omega = 2\pi) = \dfrac{16(4\pi^2+36)}{16\pi^4 + 13\times4\times\pi^2+36}$

$= 0.573$

d) Scale $\hat{S}_x(s)$ such that $\hat{S}_x(0)$ is same but

$\hat{S}_x'(j200\pi) = \hat{S}_x(j2\pi)$

$f = 100\,\text{Hz} \rightarrow \omega = 2\pi\times100\ s^{-1}$

$$\hat{S}_x'(s) = \frac{16\left(-\frac{s}{100}+6\right)\left(\frac{s}{100}+6\right)}{\left(\frac{s}{100}-3\right)\left(\frac{s}{100}+3\right)\left(\frac{s}{100}-2\right)\left(\frac{s}{100}+2\right)}$$

(5.4) $$\hat{S}_x(\omega) = \frac{\omega^2+10}{\omega^4+5\omega^2+4} + 8\pi\,\delta(\omega) + 2\pi\,\delta(\omega-3) + 2\pi\,\delta(\omega+3)$$

↳ even
always positive.

Find $\overline{X^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \hat{S}_x(\omega)$

$$= \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{\omega^2+10}{\omega^4+5\omega^2+4}} + \underbrace{4\underbrace{\int_{-\infty}^{\infty} d\omega\,\delta(\omega)}_{1} + 1\underbrace{\int_{-\infty}^{\infty} d\omega\,\delta(\omega-3)}_{1} + 1\underbrace{\int_{-\infty}^{\infty} d\omega\,\delta(\omega+3)}_{1}}_{6}$$

write in term of complex freq $\boxed{s = j\omega} \rightarrow \omega = -js = \frac{s}{j}$

$\rightarrow d\omega = \frac{ds}{j}$

$$\frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} ds \frac{-s^2+10}{s^4-5s^2+4}$$

$$\underbrace{\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} ds\ \hat{S}_x(s)} \quad \text{where } \hat{S}_x(s) = \frac{c(s)\,c(-s)}{d(s)\,d(-s)}$$

$c(s) = c_{n-1}s^{n-1} + c_{n-2}s^{n-2} + \dots c_0$

$d(s) = d_n s^n + d_{n-1}s^{n-1} + \dots d_0$

↓

$n = 1 \quad = \dfrac{c_0^2}{2d_0d_1}$

$n = 2 \rightarrow \dfrac{c_1^2 d_0 + c_0^2 d_2}{2d_0d_1d_2}$

order of d

$n = 3 \rightarrow \dfrac{c_2^2 d_0 d_1 + (c_1^2 - 2c_0c_2)d_0d_3 + c_0^2 d_2 d_3}{2d_0d_3(d_1d_2 - d_0d_3)}$

$$\boxed{\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} ds \, \frac{(-s+\sqrt{10})(s+\sqrt{10})}{(s-2)(s+2)(s-1)(s+1)}} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} ds \, \frac{\overbrace{(s+\sqrt{10})}^{c(s)}\;\overbrace{(-s+\sqrt{10})}^{c(-s)}}{\underbrace{(s+2)(s+1)}_{d(s)}\;\underbrace{(-s+2)(-s+1)}_{d(-s)}}$$

$$s^4 - 5s^2 + 4 = 0$$

$$s^2 = \frac{5 \pm \sqrt{9}}{2} = \begin{cases} 4 \rightarrow s = \pm 2 \\ 1 \rightarrow s = \pm 1 \end{cases}$$

$$\rightarrow \quad c(s) = s + \sqrt{10} \quad \begin{cases} c_0 = \sqrt{10} \\ c_1 : 1 \end{cases}$$

$$d(s) = (s+2)(s+1) = s^2 + 3s + 2 \begin{cases} d_0 : 2 \\ d_1 : 3 \\ d_2 = 1 \end{cases}$$

$$\hookrightarrow n = 2 \rightarrow \frac{c_1^2 d_0 + c_0^2 d_2}{2 d_0 d_1 d_2}$$

$$= \frac{1 \times 2 + 10 \times 1}{2 \times 1 \times 2 \times 3} = \frac{12}{12} = 1$$

$$\rightarrow \overline{X^2} = 1 + 6 = 7.$$

(6.2) $$\underbrace{R_X(\tau)}_{\substack{\perp \\ \text{even in } \tau \ \checkmark}} = 16\, e^{-5|\tau|} \cos 20\pi\tau + \underbrace{8 \cos 10\pi\tau}$$

a) Find variance of X :

$$\sigma_X^2 = \underset{\substack{\uparrow}}{\overline{X^2}} - \underset{\substack{\uparrow}}{\bar{X}}^2 = R_X(0) - R_X(\infty)$$

From $R_X(\tau)$.

$$= 24 - 0 = 24$$

$$\rightarrow R_X(\tau) = \bar{X}^2 + \underbrace{R_N(\tau)}_{8\cos(10\pi\tau)}$$

$$R_X(\tau \rightarrow \infty) = \bar{X}^2 + R_N(\tau \rightarrow \infty)$$

b) c) on-line.

(7.1) on-line.

Chapter 8: Response of Linear Systems to Random Inputs

Mean value of system output: (from Linear S.T.I: $Y(t) = X(t) * h(t)$)

$$\overline{Y} = \overline{\int_0^\infty X(t-\lambda)h(\lambda)\,d\lambda} = \overline{X}\int_0^\infty h(\lambda)\,d\lambda .$$

Mean-square value of system input:

$$\overline{Y^2} = \overline{\int_0^\infty X(t-\lambda_1)h(\lambda_1)\,d\lambda_1 \int_0^\infty X(t-\lambda_2)\,h(\lambda_2)\,d\lambda_2}$$

$$= \int_0^\infty\int_0^\infty d\lambda_1\,d\lambda_2\;\overline{X(t-\lambda_1)X(t-\lambda_2)}\;h(\lambda_1)h(\lambda_2)$$

$$= \int_0^\infty\int_0^\infty d\lambda_1\,d\lambda_2\;R_X(\lambda_2-\lambda_1)\,h(\lambda_1)h(\lambda_2)$$

Autocorrelation function of system output:

$$R_Y(\tau) = \overline{Y(t)\,Y(t+\tau)} = \int_0^\infty\int_0^\infty d\lambda_1\,d\lambda_2\;\overline{X(t-\lambda_1)X(t+\tau-\lambda_2)}\;h(\lambda_1)h(\lambda_2)$$

$$= \int_0^\infty\int_0^\infty d\lambda_1\,d\lambda_2\;R_X(\lambda_2-\lambda_1-\underset{\substack{\uparrow\\ \text{difference w.r.t. } \overline{Y^2}}}{\tau})\,h(\lambda_1)h(\lambda_2)$$

Cross-correlation function between input & output:

$$R_{XY}(\tau) = \overline{X(t)\,Y(t+\tau)} = \overline{X(t)\int_0^\infty d\lambda\;X(t+\tau-\lambda)h(\lambda)}$$

$$= \int_0^\infty d\lambda\;\overline{X(t)X(t+\tau-\lambda)}\;h(\lambda) = \int_0^\infty d\lambda\;R_X(\tau-\lambda)\,h(\lambda)$$

$$R_{YX}(\tau) = \overline{Y(t)\,X(t+\tau)} = \int_0^\infty d\lambda\;\overline{X(t-\lambda)X(t+\tau)}\;h(\lambda) = \int_0^\infty d\lambda\;R_X(\tau\underset{\substack{\uparrow\\ \text{difference w.r.t.}\\ R_{XY}}}{+}\lambda)h(\lambda)$$

Frequency-domain analysis:

$$\text{Fourier}\left\{\, Y(t) = X(t) * \underset{\substack{\uparrow\\ \text{"impulse response"}}}{h(t)} \,\right\} \rightarrow Y(\omega) = X(\omega)\underset{\substack{\uparrow\\ \text{"transfer function"}}}{H(\omega)} \text{ or } Y(f) = X(f)H(f)$$

$$\text{Laplace}\left\{\, Y(t) = X(t) * h(t) \,\right\} \longrightarrow Y(s) = X(s)\,H(s)$$

Ch7 important formulas:

$$\overline{X^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\ \hat{S}_X(\omega)$$

$$\hat{S}_X(\omega) = \int_{-\infty}^{\infty} d\tau\ R_X(\tau)\, e^{-j\omega\tau}$$

$$\int_{-\infty}^{\infty} d\tau\, e^{-j\omega\tau} = 2\pi\delta(\omega)$$

(6.2) (Cont.)
Ch7

$$\hat{S}_X(\omega) =$$

$$= \int_{-\infty}^{\infty} d\tau\ 16\, e^{-5|\tau|} \cos 20\pi\tau\ e^{-j\omega\tau} + 8\int_{-\infty}^{\infty} d\tau\ \cos 10\pi\tau\, e^{-j\omega\tau}$$

write $\cos 20\pi\tau = \dfrac{e^{j20\pi\tau} + e^{-j20\pi\tau}}{2}$, then combine to do exponential integrals:

$$\hat{S}_X(\omega) = \boxed{8\int_{-\infty}^{\infty} d\tau\ e^{-5|\tau|}\left[e^{-j(\omega-20\pi)\tau} + e^{-j(\omega+20\pi)\tau}\right]}$$

$$\boxed{+4\int_{-\infty}^{\infty} d\tau\ \left[e^{-j(\omega-10\pi)\tau} + e^{-j(\omega+10\pi)\tau}\right]} \rightarrow 8\pi\left[\delta(\omega-10\pi) + \delta(\omega+10\pi)\right]$$

$$8\int_{-\infty}^{0} d\tau \left\{ e^{[5-j(\omega-20\pi)]\tau} + e^{[5-j(\omega+20\pi)]\tau} \right\}$$

$$+ 8\int_{0}^{\infty} d\tau \left\{ e^{[-5-j(\omega-20\pi)]\tau} + e^{[-5-j(\omega+20\pi)]\tau} \right\}$$

$$= 8\left[\frac{e^{[5-j(\omega-20\pi)]\tau}}{5-j(\omega-20\pi)}\right]_{-\infty}^{0} + 8\left[\frac{e^{[5-j(\omega+20\pi)]\tau}}{5-j(\omega+20\pi)}\right]_{-\infty}^{0}$$

$$+ 8\left[\frac{e^{[-5-j(\omega-20\pi)]\tau}}{-5-j(\omega-20\pi)}\right]_{0}^{\infty} + 8\left[\frac{e^{[-5-j(\omega+20\pi)]\tau}}{-5-j(\omega+20\pi)}\right]_{0}^{\infty}$$

$$= 8\left[\frac{1}{5-j(\omega-20\pi)}\right] + 8\,\frac{1}{5-j(\omega+20\pi)}$$

$$+ 8\,\frac{+1}{+5+j(\omega-20\pi)} + 8\,\frac{1}{5+j(\omega+20\pi)}$$

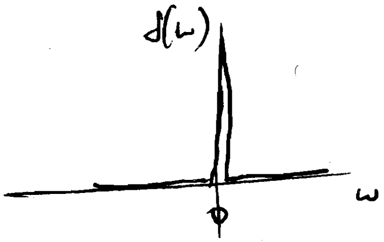
$$(a-b)(a+b) = a^2 - b^2$$

$$= 8 \frac{10}{25 + (\omega - 20\pi)^2} + 8 \frac{10}{25 + (\omega + 20\pi)^2}$$

→ Combining results:

$$\boxed{\hat{S}_X(\omega) = \frac{80}{25 + (\omega - 20\pi)^2} + \frac{80}{25 + (\omega + 20\pi)^2} + 8\pi\,\delta(\omega - 10\pi) + 8\pi\,\delta(\omega + 10\pi)}$$

c) $$\boxed{\hat{S}_X(0) = \frac{80}{25 + 400\pi^2} \times 2} + 8\pi \underbrace{\delta(-10\pi)}_{0} + 8\pi \underbrace{\delta(10\pi)}_{0}$$



(Ch 7) (7.1) $$\hat{S}_X(\omega) = \frac{9}{\omega^2 + 64}$$ uniform spectral density.

a) Write another $\hat{S}_N'$ for a bandlimited white noise with

$$\hat{S}_N'(0) = \frac{9}{64} \quad \text{and} \quad \overline{N^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \, \frac{9}{\omega^2 + 64} = \frac{1}{2\pi}\left[\tan^{-1}\frac{\omega}{8}\right]_{-\infty}^{\infty}$$

$$= \frac{9}{16}$$

$$\hat{S}_N'(\omega) = \begin{cases} \frac{9}{64} & |\omega| \le 4\pi \\ 0 & |\omega| \ge 4\pi \end{cases}$$

b) $$R_X(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \; \hat{S}_X(\omega)\, e^{+j\omega\tau}$$ (inverse Fourier Transf. of the spectral density)

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \, \frac{9}{\omega^2 + 64}\, e^{j\omega\tau} = \frac{9}{16} e^{-8|\tau|}$$

⑫

c) $$R_N(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\ \hat{S}'_N(\omega)\, e^{+j\omega\tau} = \frac{9}{32\pi 4}\int_{-4\pi}^{4\pi} d\omega\ e^{j\omega\tau}$$

$$= \frac{9}{32\pi 4}\left[\frac{e^{j\omega\tau}}{j\tau}\right]_{-4\pi}^{4\pi} = \frac{9}{32\pi j\tau 4}\left(e^{j4\pi\tau} - e^{-j4\pi\tau}\right)$$

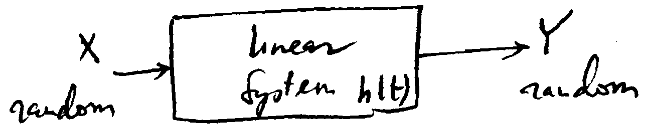
$$= \frac{9}{4\times 16\pi\tau}\sin(4\pi\tau)$$

d) $$R_X(0) = \frac{9}{16}$$

$$R_N(0) = \frac{9}{4\times 16\cancel{\pi}}\, 4\cancel{\pi} = \frac{9}{16}$$

HW10. Ch8:

③.3 $$R_X(\tau) = 16\, e^{-2|\tau|} + 16 \ ; \quad h(t) = \delta(t) - 2e^{-2t}u(t)$$ impulse response



a) $$\overline{Y} = \overline{X}\int_0^{\infty} h(\lambda)\, d\lambda = \overline{X}\Big[\underbrace{\int_0^{\infty} dt\, \delta(t)}_{1} - 2\int_0^{\infty} dt\, e^{-2t}\underbrace{1}_{u(t)}\Big]$$

$$= \overline{X}\left[\ 1 - 2\left[\frac{e^{-2t}}{-2}\right]_0^{\infty}\ \right] = \overline{X}.0 = 0$$

$$\left(\overline{X} = \sqrt{R_X(\tau\to\infty)} = \sqrt{16} = 4\ \right)$$

b) $$\overline{Y^2} \underset{\text{(Notes)}}{=} \int_0^{\infty}\!\!\int_0^{\infty} d\lambda_1\, d\lambda_2\ \underbrace{R_X(\lambda_2-\lambda_1)}_{2\text{ terms}}\ \underbrace{h(\lambda_1)}_{2\text{ terms}}\ \underbrace{h(\lambda_2)}_{2\text{ terms}}$$

$$= \int_0^{\infty}\!\!\int_0^{\infty} d\lambda_1\, d\lambda_2 \left(16e^{-2|\lambda_2-\lambda_1|} + 16\right)\left(\delta(\lambda_1) - 2e^{-2\lambda_1}u(\lambda_1)\right)\left(\delta(\lambda_2) - 2e^{-2\lambda_2}u(\lambda\right.$$

$$\downarrow$$

$$= 16\int_0^{\infty}\!\!\int_0^{\infty} d\lambda_1 d\lambda_2 \Big\{ e^{-2|\lambda_2-\lambda_1|}\delta(\lambda_1)\delta(\lambda_2) - 2e^{-2|\lambda_2-\lambda_1|}e^{-2\lambda_2}u(\lambda_2)\delta(\lambda_1) - 2e^{-2|\lambda_2-\lambda_1|}e^{-2\lambda_1}u(\lambda_1)$$

$$+ 4e^{-2|\lambda_2-\lambda_1|}e^{-2\lambda_1}e^{-2\lambda_2}u(\lambda_1)u(\lambda_2) + \delta(\lambda_1)\delta(\lambda_2) - 2\delta(\lambda_1)e^{-2\lambda_2}u(\lambda$$

$$-2e^{-2\lambda_1}u(\lambda_1)\delta(\lambda_2) + 4e^{-2\lambda_1}e^{-2\lambda_2}u(\lambda_1)u(\lambda_2)\ \Big\}$$

$$= 16 \left\{ \underbrace{1 - \frac{1}{2} - \frac{1}{2} + 1 - 1 - 1 + 1}_{0} + 4\int_0^\infty d\lambda_2 \int_0^{\lambda_2} d\lambda_1 e^{-4\lambda_2} + 4\int_0^\infty d\lambda_2 \int_{\lambda_2}^\infty d\lambda_1 e^{-4\lambda_1} \right\}$$

$$= 64 \left\{ \underbrace{\int_0^\infty d\lambda_2 \, \lambda_2 e^{-4\lambda_2}}_{\frac{1}{16}} + \underbrace{\int_0^\infty d\lambda_2 \, \frac{e^{-4\lambda_2}}{4}}_{\frac{1}{16}} \right\}$$

$$\boxed{\overline{Y^2} = 8}$$

c) $\sigma_Y^2 = \overline{Y^2} - \bar{Y}^2 = 8 - 0^2 = 8$

Will post: 4.1 ; 4.4 & 7.1

8.4.1 $\quad H(s) = \dfrac{1000}{0.1s+1000} = \dfrac{10^4}{s+10^4} \Rightarrow h(t) = 10^4 e^{-10^4 t} u(t)$

a) $\quad R_Y(\tau) = S_0 \int_0^\infty h(\lambda)\, h(\lambda+\tau)\, d\lambda = 10^{-4} \times 10^8 \int_0^\infty e^{-10^4 \lambda} e^{-10^4(\lambda+\tau)}\, d\lambda$

$= 0.5 e^{-10^4 \tau} \quad ; \quad \tau \geqslant 0$

By symmetry $\quad R_Y(\tau) = 0.5 e^{-10^4|\tau|}$

b) $\quad \overline{Y^2} = R_Y(0) = \frac{1}{2}$

8.4.4

$$R_Y(\tau) = \int_0^\infty \int_0^\infty \left[2\,\delta(\lambda_2 - \lambda_1 - \tau) + 9\right] \cdot h(\lambda_1)\, h(\lambda_2)\, d\lambda_2\, d\lambda_1$$

$$= 2 \cdot \int_0^1 \int_0^1 \delta(\lambda_2 - \lambda_1 - \tau)(1-\lambda_1)(1-\lambda_2)\, d\lambda_1\, d\lambda_2 + 9\left[\int_0^1 (1-\lambda_1)\, d\lambda_1\right]^2$$

$$= 2 \cdot \int_\tau^1 (1-\lambda_1)(1-\lambda_1-\tau)\, d\lambda_1 + 9 \cdot \left(\tfrac{1}{2}\right)^2$$

$$= \frac{2}{3} - 3\tau + 4\tau^2 - \frac{5}{3}\tau^3 + \frac{9}{4} = \frac{35}{12} - 3\tau + 4\tau^2 - \frac{5}{3}\tau^3 \qquad 0 \le \tau \le 1$$

a) $\overline{Y} = \frac{9}{4}$

b) $\overline{Y^2} = \frac{35}{12}$

c) $R_Y(\tau) = \frac{35}{12} - 3|\tau| + 4\tau^2 - \frac{5}{3}|\tau|^3 \; ; \quad -1 \le \tau \le 1$

$\quad = \frac{9}{4} \qquad ;$ otherwise

8.7.1

a) $$\frac{V_o}{V_{in}} = \frac{1\times10^3 + \frac{1}{sc}}{3\times10^3 + \frac{1}{sc}} = \frac{s+1}{3s+1} = H(s)$$

b) $$Y(s) = X(s)\,H(s) = \frac{s\,(s+1)}{(3s+1)(s+4)}$$

$$|Y(s)|^2 = Y(s).\,Y(-s) = \frac{s\,(s+1)\,(-s)(-s+1)}{(3s+1)\,(s+4)\,(-3s+1)(-s+4)}$$

$$= \frac{s^2\,(s^2-1)}{(9s^2-1)(s^2-16)}$$