Ch6 (Cont.)

2) X is also ergodic: $\underbrace{\text{statistical average}}_{\substack{\text{requires information}\\ \text{on prob. distribution}\\ \text{via the density function}}}$ is given by the $\underbrace{\text{time average}}_{\text{simple integral}}$

$$R_X(\tau) = \underbrace{E[X(t_1)X(t_2)]}_{\substack{\text{(statistical or}\\ \text{ensemble average)}}} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} dt \; x(t)\, x(t+\tau)$$

$$\equiv \langle x(t)\, x(t+\tau) \rangle \quad \text{(time average)}$$

$$\equiv \mathcal{R}_X(\tau)$$

"straight R" to go with ensemble average

"Curly R" to go with time average

Properties of the Auto-correlation function:

1) $R_X(0) = R_X(t_1, t_1) = E[X_1 X_1] = \overline{X^2}$

Autocorrelation function at $\tau=0$ is the moment of second order

2) $\boxed{R_X(\tau) = \underset{t_2 - t_1 \equiv \tau}{R_X(t_1, t_2)} = R_X(t_1+T, t_2+T)}$ — stationary

$\boxed{T = -t_2}$ $\quad = R_X(t_1 - t_2, 0) = \boxed{R_X(-\tau)}$

Auto correlation function is symmetric or even in τ

3) $|R_X(\tau)| \leq \underset{\uparrow \text{max}}{R_X(0)}$

Proof: $E\left[\underbrace{(X(t_1) + X(t_2))^2}_{\text{non-negative}}\right] \geq 0$ (ensemble average of a non-negative variable is non-negative)

$$\hookrightarrow \underbrace{E[X^2(t_1)]}_{\overline{X^2}} + \underbrace{E[X^2(t_2)]}_{\overline{X^2}} + \underbrace{2E[X(t_1)X(t_2)]}_{2R_X(\tau)} \geq 0$$

$$\overline{X^2} + \overline{X^2} + 2R_X(\tau) \geq 0$$

$\hookrightarrow$ Prop #1

$R_X(0)$

$$\hookrightarrow \quad 2R_X(0) + 2R_X(\tau) \geq 0 \rightarrow \boxed{R_X(\tau) \geq -R_X(0)}$$

Similarly: $E\left[\underbrace{(X(t_1) - X(t_2))^2}_{\text{non-negative}}\right] \geq 0$

$$\hookrightarrow 2R_X(0) - 2R_X(\tau) \geq 0 \rightarrow \boxed{R_X(\tau) \leq R_X(0)}$$

$$\rightarrow -R_X(0) \leq R_X(\tau) \leq R_X(0) \rightarrow \boxed{|R_X(\tau)| \leq R_X(0)} \checkmark$$

---

4) If $X(t) = \overline{X} + N(t) \rightarrow \boxed{R_X(\tau) = \overline{X}^2 + R_N(\tau)}$

$\hookrightarrow$ Noise $\left\{ \overline{N} = 0 \quad \text{or} \quad E[N(t)] = 0 \right.$

Proof: $R_X(\tau) = E\left[(\overline{X} + N(t_1))(\overline{X} + N(t_2))\right]$

$$= E\left[\overline{X}^2 + \overline{X}N(t_1) + \overline{X}N(t_2) + N(t_1)N(t_2)\right]$$

$$= \underset{\downarrow \text{ number}}{E[\overline{X}^2]} + \overline{X}\underbrace{E[N(t_1)]}_{0} + \overline{X}\underbrace{E[N(t_2)]}_{0} + \underbrace{E[N(t_1)N(t_2)]}_{R_N(\tau)}$$

$$= \overline{X}^2 + R_N(\tau) \quad \checkmark$$

Consequence: $R_X(\tau \to \infty) = \overline{X}^2 + \underbrace{R_N(\tau \to \infty)}_{0} = \overline{X}^2$

also $\rightarrow R_X(\tau = 0) \overset{1)}{=} \overline{X^2}$

Summary: $X(t) = \bar{X} + N(t) \rightarrow$

Statistical	Autocorrelation
$\overline{X^2}$	$R_X(0)$
$\bar{X}^2$	$R_X(\infty)$

there is the importance or usefulness of the auto-correlation functions

We can obtain statistical information such as : moment of 2nd order and the mean (ensemble average) w/o knowing the probability distribution (or density function) if we know autocorrelation function at $\tau=0$ & $\tau=\infty$. Furthermore, for ergodic & stationary processes these autocorrelations are equal to time autocorrelations (using time average)

Cross-correlation functions:

$$X_1 = X(t_1) \quad ; \quad Y_2 = Y(t_2) \quad ; \quad t_2 = t_1 + \tau$$

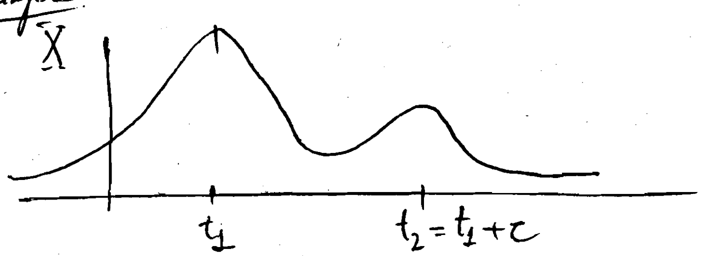
X, Y are stationary & ergodic random variables

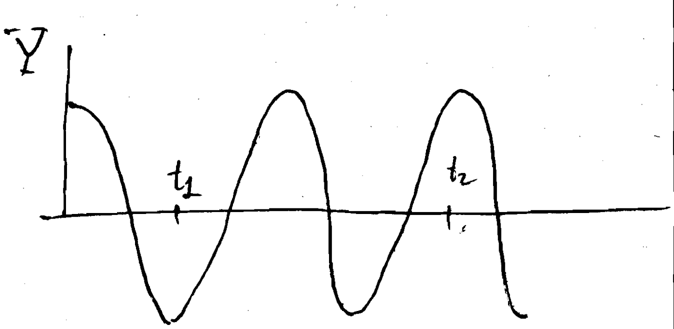
$$R_{XY}(\tau) = E[X_1 Y_2] = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dy_2 \; x_1 y_2 \, f(x_1) f(y_2)$$

$$R_{YX}(\tau) = E[Y_1 X_2] = \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dx_2 \; x_2 y_1 \, f(x_2) f(y_1)$$

$$Y_1 = Y(t_1); \; X_2 = X(t_2)$$

Example:





$$\boxed{R_{XY}(\tau) \neq R_{YX}(\tau)}$$

Cross correlation functions using time average:

$$\mathcal{R}_{XY}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} dt\ x(t)\, y(t+\tau)$$

$$\mathcal{R}_{YX}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} dt\ y(t)\, x(t+\tau)$$

Properties of Cross-Correlation Functions:

1) $R_{XY}(0)$ $\qquad R_{YX}(0)$

$$\left.\begin{matrix} E[X(t_1)\, Y(t_1)] = E[Y(t_1)\, .\, X(t_1)] \end{matrix}\right\} R_{XY}(0) = R_{YX}(0)$$

Cross-Correlations b/w X & Y and b/w Y & X are the same only at $\tau = 0$

2) No even symmetry as with autocorrelation function:

$$R_{XY}(-\tau) \neq R_{XY}(\tau)$$

However: $R_{XY}(-\tau) = R_{YX}(\tau)$

3) $|R_{XY}(\tau)| \not\leq R_{XY}(0)$

However: $|R_{XY}(\tau)| \leq \sqrt{R_X(0)\, R_Y(0)}$

4) If X & Y are stat. independent:

$R_{XY}(\tau) = E[X_1 Y_2] \stackrel{!}{=} E[X_1]\, E[Y_2] = \bar{X}\bar{Y}$

X & Y are stat. independent

$R_{YX}(\tau) = E[Y_1 X_2] = E[Y_1]\, E[X_2] = \bar{Y}\bar{X} = \bar{X}\bar{Y}$

$\Big\}\ R_{XY}(\tau) = R_{YX}(\tau)$

5) $R_{X\dot{X}}(\tau) = E[X(t)\,\dot{X}(t+\tau)] = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}E\left[X(t)\left(X(t+\tau+\varepsilon) - X(t+\tau)\right)\right]$

$$\boxed{\dot{X} \equiv \frac{dX}{dt}} \boxed{= \lim_{\varepsilon\to 0}\frac{X(t+\tau+\varepsilon) - X(t+\tau)}{\varepsilon}}$$

$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big\{\underbrace{E[X(t)X(t+\tau+\varepsilon)]}_{R_X(\tau+\varepsilon)} - \underbrace{E[X(t)X(t+\tau)]}_{R_X(\tau)}\Big\}$$

$$= \frac{dR_X}{d\tau} \quad\Rightarrow\quad \boxed{R_{X\dot{X}}(\tau) = \frac{dR_X}{d\tau}}$$

Similarly:

$$\boxed{R_{\dot{X}\dot{X}}(\tau) = R_{\dot{X}}(\tau) = -\frac{d^2R_X}{d\tau^2}}$$

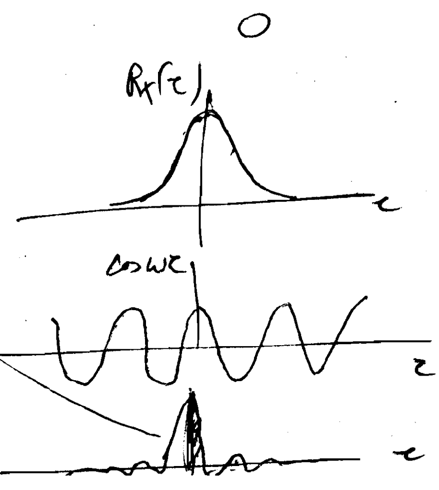
One more property for the Auto-correlation Function:

Fourier transform of an auto-correlation function:

$$F[R_X(\tau)] = \int_{-\infty}^{\infty} d\tau\, R_X(\tau)\, e^{-j\omega\tau}$$

$$= \int_{-\infty}^{\infty} d\tau\, \underbrace{\underbrace{R_X(\tau)}_{\text{even in }\tau}\ \underbrace{\cos\omega\tau}_{\text{even in }\tau}}_{\text{even}} \; - j \underbrace{\int_{-\infty}^{\infty} d\tau\, \underbrace{\underbrace{R_X(\tau)}_{\text{even in }\tau}\ \underbrace{\sin\omega\tau}_{\text{odd in }\tau}}_{\text{odd in }\tau}}_{0}$$

$$\boxed{F[R_X(\tau)] = 2\int_0^{\infty} d\tau\, R_X(\tau)\cos\omega\tau}$$



are under curve

$\boxed{F[R_X(\tau)] \geq 0}$ if tall main peak

or $R_X(\tau)$ not too wide → no flat tops

and sharp vertical sides or discontinuities.

HW7 (Ch6): 3.2 ; 5.3 ; 5.4 due 5/6
HW8 (Ch6): 7.1 ; 8.2 due 5/6

→ HW9 (Ch7): 2.2 ; 3.1 ; 4.1 ; 5.4 ; 6.2 ; 7.1 due 5/8
→ HW10 (Ch8): 3.3 ; 4.1 ; 4.4 ; 7.1 due 5/13

HW7: 5.3

Find mean & variance for random process with given autocorrelation functions:

a) $R_X(\tau) = 10 e^{-\tau^2}$ $\left\{ \begin{array}{l} \overline{X^2} = R_X(0) = 10 \\ \overline{X} = \sqrt{R_X(\infty)} = 0 \end{array} \right\} \rightarrow \sigma_X^2 = \overline{X^2} - \overline{X}^2 = 10$

b) $R_X(\tau) = 10 e^{-\tau^2} \cos(2\pi\tau^2)$ $\left\{ \begin{array}{l} \overline{X^2} = R_X(0) = 10 \\ \overline{X} = \sqrt{R_X(\infty)} = 0 \end{array} \right\}$ $\sigma_X^2 = 10$

c) $R_X(\tau) = 10 \dfrac{\tau^2 + 8}{\tau^2 + 4}$ $\left\{ \begin{array}{l} \overline{X^2} = R_X(0) = 20 \\ \overline{X} = \sqrt{R_X(\infty)} = \sqrt{10} \end{array} \right\}$ $\sigma_X^2 = \overline{X^2} - \overline{X}^2 = 20 - 10 = 10$

HW7: 5.4

$$R_X(\tau) = 10 e^{-2|\tau|} - 5 e^{-4|\tau|}$$

a) Mean & variance of X:

$$\left. \begin{array}{l} \overline{X} = \sqrt{R_X(\infty)} = 0 \\ \overline{X^2} = R_X(0) = 5 \end{array} \right\} \sigma_X^2 = \overline{X^2} - \overline{X}^2 = 5$$

b) Is this process differentiable? Why?

Property: $R_{X\dot{X}}(\tau) = \dfrac{dR_X}{d\tau}$ → if we can find $R_{X\dot{X}}$ → $\dot{X}$ does exist → process described by X is differentiable:

$$\frac{dR_X}{d\tau} = \begin{cases} \tau \geq 0 \rightarrow |\tau| = \tau \rightarrow \dfrac{d(10e^{-2\tau} - 5e^{-4\tau})}{d\tau} = -20e^{-2\tau} + 20e^{-4\tau} \\ \tau < 0 \rightarrow |\tau| = -\tau \rightarrow \dfrac{d(10e^{2\tau} - 5e^{4\tau})}{d\tau} = 20e^{2\tau} - 20e^{4\tau} \end{cases}$$

$\rightarrow R_X(\tau)$ is differentiable for $\tau \geq 0$ & $\tau < 0 \rightarrow \frac{dR_X}{d\tau}$ exist if it is continuous at $\tau = 0$:

$$\frac{dR_X}{d\tau}(\tau=0) \begin{cases} \tau \geq 0 : -20 + 20 = 0 \\ \tau < 0 : 20 - 20 = 0 \end{cases} \checkmark$$

$\Rightarrow R_{X\dot{X}}(\tau)$ does exist $\Rightarrow \dot{X}$ or $\frac{dX}{dt}$ exist $\rightarrow \underline{X}$ is differentiable (also the process it describes)

HW8: 7.1 Two $\underline{\underline{\text{independent}}}$ stationary random processes $X(t)$ & $Y(t)$ with

$$\begin{cases} R_X(\tau) = 25e^{-10|\tau|} \cos(100\pi\tau) \\ R_Y(\tau) = 16 \dfrac{\sin(50\pi\tau)}{50\pi\tau} \end{cases}$$

a) Find autocorrelation function of $X(t) + Y(t)$:

$$R_{X+Y}(\tau) = E[(X_1+Y_1)(X_2+Y_2)] = E[X_1X_2] + E[Y_1Y_2] + E[X_1Y_2] + E[X_2Y_1]$$

$$= R_X(\tau) + R_Y(\tau) + \underbrace{E[X_1]}_{\bar{X}}\underbrace{E[Y_2]}_{\bar{Y}} + \underbrace{E[X_2]}_{\bar{X}}\underbrace{E[Y_1]}_{\bar{Y}}$$

$$= R_X(\tau) + R_Y(\tau) + 2\bar{X}\bar{Y} = R_X(\tau) + R_Y(\tau) + 2\underbrace{\sqrt{R_X(\infty)}}_{0}\underbrace{\sqrt{R_Y(\infty)}}_{0}$$

$$= 25e^{-10|\tau|}\cos 100\pi\tau + 16\frac{\sin(50\pi\tau)}{50\pi\tau}$$

b) Find autocorrelation function of $X(t) - Y(t)$

$$R_{X-Y}(\tau) = R_X(\tau) + R_Y(\tau) - \underbrace{2\bar{X}\bar{Y}}_{0} = R_X(\tau) + R_Y(\tau)$$

$$= 25 e^{-10|\tau|} \cos 100\pi\tau + 16 \frac{\sin 50\pi\tau}{50\pi\tau}$$

d) Find $R_{XY}(\tau)$ & $R_{YX}(\tau)$ (Autocorrelation functions of XY & YX, not the cross correlation functions of X & Y)

$$R_{XY}(\tau) = E[X_1 Y_1 \; X_2 Y_2]$$

$$\underset{\substack{X \& Y \\ \text{are stat.} \\ \text{indep.}}}{=} E[X_1 X_2]\, E[Y_1 Y_2]$$

$$= R_X(\tau) \cdot R_Y(\tau)$$

$$R_{YX}(\tau) = E[Y_1 X_1 \; Y_2 X_2] \overset{\text{stat indep.}}{=} E[Y_1 Y_2]\, E[X_1 X_2]$$

$$= R_Y(\tau) \cdot R_X(\tau)$$

$$= R_{XY}(\tau).$$

c) Find cross correlation functions defined by a) & b): b/w $X+Y$ & $X-Y$

$$R_{(X+Y)(X-Y)} = E[(X_1+Y_1)(X_2-Y_2)] = E[X_1X_2] - E[Y_1Y_2] + E[Y_1X_2] - E[X_1Y_2]$$

$$= R_X(\tau) - R_Y(\tau) + \underbrace{\bar{Y}\cdot\bar{X} - \bar{X}\cdot\bar{Y}}_{0}$$

$$= 25 e^{-10|\tau|} \cos 100\pi\tau - 16 \frac{\sin 50\pi\tau}{50\pi\tau}$$

Ch 7: Spectral Density.

Fourier transform of a random signal X : issues :

$$F_X(\omega) = \int_{-\infty}^{\infty} dt \;\; x(t)\, e^{-j\omega t}$$

↳ Two problems

1) integral does not exist for a random signal since it may not go to 0 at ∞

2) as it is $F_X(\omega)$ is another random variable → not useful for signal processing applications : noise filtering etc.

To address these 2 problems: modify the def. of a Fourier transform for a random variable:

1) Limit the integral to a window (of size T) of $x(t)$ then will take the limit $T \to \infty$

This can be achieved if we replace $x(t)$ by a

$$x_T(t) = \begin{cases} x(t) & |t| \le T \\ 0 & |t| > T \end{cases}$$

$$\to \boxed{\hat{F}_X(\omega) = \int_{-\infty}^{\infty} dt \;\; x_T(t)\, e^{-j\omega t}}$$

This integral always exist b/c of the finite size of the imposed window of size T

2) To eliminate the randomness, take the ensemble average:

→ new quantity: Spectral density : $S_X(\omega)$

$$\boxed{S_X(\omega) = \lim_{T \to \infty} \frac{E\{|\hat{F}_X(\omega)|^2\}}{2T}}$$

This will replace a Fourier transform for a random signal.

Properties of the Spectral density:

$$\boxed{\overline{X^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, S_X(\omega)}$$

We can obtain a statistical information such as the moment of second order by integrating the spectral density $S_X(\omega)$ and divide by 2π.

Proof: using Parseval theorem: $\int_{-T}^{T} dt\ x_T^2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, |\hat{F}_X(\omega)|^2$

$$\begin{cases} x_T(t) : \text{signal} \\ \hat{F}_X(\omega) : \text{its Fourier transform.} \end{cases}$$

Let's apply $\lim_{T\to\infty} E\left\{\frac{1}{2T} \times \text{Parseval Theorem}\right\}$:

$$\underbrace{E\Big\{ \underbrace{\lim_{T\to\infty} \underbrace{\frac{1}{2T}\int_{-T}^{T} dt\ x_T^2(t)}_{\langle x_T^2\rangle}}_{\langle x^2 \rangle} \Big\}}_{\overline{X^2}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \underbrace{\lim_{T\to\infty} E\left\{\frac{|\hat{F}_X(\omega)|^2}{2T}\right\}}_{S_X(\omega)}$$

$$\overline{X^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, S_X(\omega) \quad \checkmark$$