2) \( X \) is also ergodic: statistical average is given by the time average requires information on pdf, distribution via the density function

\[
R_X(\tau) = \text{E} \left[ X(t_2)X(t_2) \right] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt \ x(t) x(t+\tau)
\]

(statistical or ensemble average) \( \equiv \) \( \langle x(t) x(t+\tau) \rangle \) (time average)

\[
\equiv \mathcal{R}_X(\tau)
\]

"Only \( \mathcal{R} \) to go with time average"

Properties of the Auto-correlation function:

1) \( R_X(0) = R_X(t_1, t_1) = E \left[ X_1 X_2 \right] = X^2 \)

Auto-correlation function at \( \tau = 0 \) is the moment of second order

2) \( \mathcal{R}_X(\tau) = R_X(t_2, t_2) = R_X(t_2 + T, t_2 + T) \)

\( t_2 - t_1 = \tau \) stationary

\[
\left[ \begin{array}{c} \tau = -t_2 \\ T = t_2 \end{array} \right] \quad = \mathcal{R}_X(t_2 - t_2, 0) = \mathcal{R}_X(-\tau)
\]

Auto-correlation function is symmetric or even in \( \tau \)

3) \( |R_X(\tau)| \leq R_X(0) \quad \tau_{\text{max}} \)

Proof: \( E \left[ (X(t_1) + X(t_2))^2 \right] \geq 0 \)\n
(ensemble average of a non-negative variable is non-negative)
\[
\left\{\begin{array}{l}
\text{If } X(t) = \bar{X} + N(t) \rightarrow R_X(\tau) = \bar{X}^2 + R_N(\tau)
\end{array}\right.
\]

**Proof:**

\[
R_X(\tau) = E\left[ (\bar{X} + N(t_1)) (\bar{X} + N(t_2)) \right] = E\left[ \bar{X}^2 + \bar{X}N(t_1) + \bar{X}N(t_2) + N(t_1)N(t_2) \right] = \bar{X}^2 + R_N(\tau)
\]

**Consequence:**

\[
\lim_{\tau \to \infty} R_X(\tau) = \frac{\bar{X}^2 + R_N(\tau \to \infty)}{0} = \bar{X}^2
\]
Summary: \( X(t) = \bar{X} + N(t) \) →

<table>
<thead>
<tr>
<th>Statistical</th>
<th>Autocorrelation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{X}^2 )</td>
<td>( R_X(0) )</td>
</tr>
<tr>
<td>( \overline{X^2} )</td>
<td>( R_X(\infty) )</td>
</tr>
</tbody>
</table>

There is the importance or usefulness of the autocorrelation function.

We can obtain statistical information such as moment of second order and the mean (ensemble average) w/o knowing the probability distribution (or density function) if we know autocorrelation function at \( t=0 \) & \( t=\infty \).

Furthermore, for ergodic & stationary processes, these autocorrelations are equal to time autocorrelations (using time average).

**Cross-correlation function:**

\( X_1 = X(t_1) ; \quad Y_2 = Y(t_2) ; \quad t_2 = t_1 + \tau \)

\( X_1, Y \) are stationary & ergodic random variables.

\[
R_{XY}(\tau) = E \left[ X_1 Y_2 \right] = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dy_2 \; x_1 y_2 \; f(x_1) f(y_2)
\]

\[
R_{YX}(\tau) = E \left[ Y_1 X_2 \right] = \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dx_2 \; x_2 y_1 \; f(x_2) f(y_1)
\]

\( Y_1 = Y(t_1) ; \quad X_2 = X(t_2) \)

**Example:**

\( X \)

\( Y \)

\[ R_{XY}(\tau) \neq R_{YX}(\tau) \]
Cross correlation functions using time average:

\[ R_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt \, x(t) y(t+\tau) \]

\[ R_{yx}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt \, y(t) x(t+\tau) \]

Properties of Cross Correlation Functions:

1. \( R_{xy}(0) = R_{yx}(0) \)

2. No even symmetry as with auto-correlation function:
   \( R_{xy}(-\tau) \neq R_{xy}(\tau) \)

3. \( |R_{xy}(\tau)| \leq R_{xy}(0) \)

4. If \( X \) & \( Y \) are stat. independent:
   \[ R_{xy}(\tau) = E[X_t Y_{t+\tau}] = E[X_t] E[Y_{t+\tau}] = \bar{X} \bar{Y} \]
   \[ R_{yx}(\tau) = E[Y_t X_{t+\tau}] = E[Y_t] E[X_{t+\tau}] = \bar{Y} \bar{X} = \bar{X} \bar{Y} \]
5) \[ R_{xx}^\prime(\tau) = E \left[ X(t) \dot{X}(t+\tau) \right] = \lim_{\varepsilon \to 0} E \left[ X(t) (X(t+\tau+\varepsilon) - X(t+\tau)) \right] \]

\[
\dot{X} = \frac{dx}{dt} = \lim_{\varepsilon \to 0} \frac{X(t+\tau+\varepsilon) - X(t+\tau)}{\varepsilon}
\]

\[
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ E[X(t)X(t+\tau+\varepsilon)] - E[X(t)X(t+\tau)] \right\}
\]

\[ R_{x}(\tau+\varepsilon) - R_{x}(\tau) \]

\[ \Rightarrow \quad R_{xx}^\prime(\tau) = \frac{dR_{x}}{d\tau} \]

Similarly:

\[ R_{xx}^\prime(\tau) = R_{x}^\prime(\tau) = -\frac{d^2R_{x}}{d\tau^2} \]

One more properly for the Auto-covariance function:

Fourier transform of an auto-covariance function:

\[ F[R_{x}(\tau)] = \int_{-\infty}^{\infty} dc \ R_{x}(c) e^{j\omega c} \]

\[ \int_{-\infty}^{\infty} dc \ R_{x}(c) \cos \omega c \quad \text{even in } c \quad \text{odd in } \omega \]

\[ \int_{-\infty}^{\infty} dc \ R_{x}(c) \sin \omega c \quad \text{odd in } c \quad \text{even in } \omega \]

\[ F[R_{x}(\tau)] = 2\int_{0}^{\infty} dc \ R_{x}(c) \cos 2\pi c \]

The Fourier transform of a correlation function is often used in statistics and signal processing. It can help in identifying the frequency components of a signal or the distribution of the correlation function in the frequency domain. The Fourier transform of the correlation function is also often used in the analysis of stationary stochastic processes.

The graph on the right shows the relationship between the correlation function and its Fourier transform, highlighting how the transform captures the frequency components of the original signal or process.
\[ \text{HW7 (Ch6): (3.2; 5.3; 5.4) due 5/6} \]
\[ \text{HW8 (Ch6): (7.1; 8.2) due 5/6} \]
\[ \text{HW9 (Ch7): (2.2; 3.1; 4.1; 5.4; 6.2; 7.1) due 5/8} \]
\[ \text{HW10 (Ch8): (3.3; 4.1; 4.4; 7.1) due 5/13} \]

**HW7: 5.3**

Find mean & variance for random process with given autocorrelation functions:

a) \[ R_x(t) = 10 e^{-t^2} \]
   \[ \left\{ \begin{array}{l}
   X^2 = R_x(0) = 10 \\
   \overline{X} = \sqrt{R_x(\infty)} = 0
\end{array} \right. \]
   \[ \sigma_x^2 = \overline{X^2} - \overline{X}^2 = 10 \]

b) \[ R_x(t) = 10 e^{-t^2} \cos(2\pi t^2) \]
   \[ \left\{ \begin{array}{l}
   X^2 = R_x(0) = 10 \\
   \overline{X} = \sqrt{R_x(\infty)} = 0
\end{array} \right. \]
   \[ \sigma_x^2 = 10 \]

c) \[ R_x(t) = 10 \frac{t^2 + 8}{t^2 + 4} \]
   \[ \left\{ \begin{array}{l}
   X^2 = R_x(0) = 20 \\
   \overline{X} = \sqrt{R_x(\infty)} = \sqrt{10}
\end{array} \right. \]
   \[ \sigma_x^2 = \overline{X^2} - \overline{X}^2 = 20 - 10 = 10 \]

**HW7: 5.4**

\[ R_x(t) = 10 e^{-2|t|} - 5e^{-4|t|} \]

a) Mean & variance of \( X \):
   \[ \overline{X} = \sqrt{R_x(\infty)} = 0 \]
   \[ \sigma_x^2 = \overline{X^2} - \overline{X}^2 = 5 \]

b) Is this process differentiable? Why?

Property: \( R_{XX}(t) = \frac{dR_X}{dt} \) → if we can find \( R_{XX} \)
→ \( X \) does exist → process described by \( X \) is differentiable.
\[
\frac{dR_x}{dt} = \begin{cases} \frac{d(10e^{-2t} - 5e^{-4t})}{dt} = -20e^{-2t} + 20e^{-4t} & \text{if } t > 0 \\ \frac{d(10e^{-2t} - 5e^{-4t})}{dt} = 20e^{-2t} - 20e^{-4t} & \text{if } t < 0 \end{cases}
\]

\( \Rightarrow R_x(t) \) is differentiable for \( t > 0 \) & \( t < 0 \) \Rightarrow \frac{dR_x}{dt} \text{ exist if}

\( \text{it is continuous at } t = 0 : \)

\[
\frac{dR_x(t=0)}{dt} = \begin{cases} 20 - 20 = 0 & \text{if } t > 0 \\ 20 - 20 = 0 & \text{if } t < 0 \end{cases}
\]

\( \Rightarrow R_{xx}(t) \) does exist \( \Rightarrow X \) in \( \frac{dX}{dt} \) exist \( \Rightarrow X \) is differentiable (also the process it describes)
6) Find autocorrelation function of \( X(t) - Y(t) \)

\[
R_{X-Y}(\tau) = R_X(\tau) + R_Y(\tau) - 2 \bar{X} \bar{Y} = R_X(\tau) + R_Y(\tau)
\]

\[
= 25 e^{-10|\tau|} + 16 \sin \frac{50\tau}{30\pi} + 100\pi \tau
\]

d) Find \( R_{XY}(\tau) \) & \( R_{YX}(\tau) \) (Autocorrelation functions of \( X \) & \( Y \) are not the cross correlation function of \( X \) & \( Y \))

\[
R_{XY}(\tau) = E[XY(z)] = E[X(z)Y(z)]
\]

\[
= E[X_1X_2]E[Y_1Y_2] = R_X(\tau)R_Y(\tau)
\]

\[
R_{YX}(\tau) = E[YX(z)] = E[Y_1X_1]E[X_2X_2] = R_Y(\tau)R_X(\tau)
\]

\[
= R_{YX}(\tau)
\]

c) Find cross correlation functions defined by a) & b):

b) \( X+Y \) & \( X-Y \)

\[
\]

\[
= R_X(\tau) - R_Y(\tau) + \bar{Y} \bar{X} - \bar{X} \bar{Y}
\]

\[
= 25 e^{-10|\tau|} + 100\pi \tau - 16 \sin \frac{50\tau}{30\pi}
\]
Ch 7: Spectral Density

Fourier transform of a random signal $X$: issues:

$$F_X(w) = \int_{-\infty}^{\infty} dt \ x(t) e^{-jwt}$$

- Two problems
  - Integral does not exist for a random signal since it may not go to 0 at $\pm \infty$
  - As it is $F_X(w)$ is another random variable so not useful for signal processing applications: noise filtering etc.

To address these 2 problems: modify the def. of a Fourier transform for a random variable:

1) Limit the integral to a window (of size $T$) of $x(t)$ then will take the limits $T \to \infty$

This can be achieved if we replace $x(t)$ by a $X_T(t) = \begin{cases} x(t) & |t| \leq T \\ 0 & |t| > T \end{cases}$

$$\hat{F}_X(w) = \int_{-\infty}^{\infty} dt \ x_T(t) e^{-jwt}$$

2) To eliminate the randomness, take the ensemble average

new quantity: Spectral density: $S_X(w)$

$$S_X(w) = \lim_{T \to \infty} \frac{E[|\hat{F}_X(w)|^2]}{2T}$$

This will replace a Fourier transform for a random signal.
Properties of the spectral density:

\[
\overline{X^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dw S_X(w)
\]

We can obtain statistical information, such as the moment of second order, by integrating the spectral density \( S_X(w) \) and divide by \( 2\pi \).

Proof: using Parseval theorem:

\[
\int_{-T}^{T} dt x_T^2(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dw |\hat{F}_x(w)|^2
\]

\( x_T(t) \): signal

\( \hat{F}_x(w) \): its Fourier transform.

Let's apply \( \lim_{T \to \infty} E \left\{ \frac{1}{T} x \right\} \) Parseval theorem:

\[
E \left\{ \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} x_T^2(t) \right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dw \lim_{T \to \infty} E \left\{ \frac{1}{2T} |\hat{F}_x(w)|^2 \right\}
\]

\[
\langle x_T^2 \rangle
\]

\[
\langle x^2 \rangle
\]

\[
\overline{X^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dw S_X(w)
\]

\( S_X(w) \)