Chi-Square or χ-square or χ^2 distribution (cont.)

$\chi^2 = Y_1^2 + Y_2^2 + \dots + Y_n^2$ where Y_i are Gaussian Random Variables with zero mean ($\overline{Y_i} = 0$) and unit variance ($\sigma_{Y_i}^2 = 1$)

$$f(\chi^2) = \begin{cases} \dfrac{(\chi^2)^{\frac{n}{2}-1}}{2^{\frac{n}{2}}\,\Gamma\left(\frac{n}{2}\right)}\, e^{-\frac{\chi^2}{2}} & \chi^2 \geq 0 \\ 0 & \chi^2 < 0 \end{cases}$$

1) $$\overline{\chi^2} \equiv \int_0^\infty d\chi^2\, \chi^2 f(\chi^2) = \int_0^\infty d\chi^2\, \chi^2 \frac{(\chi^2)^{\frac{n}{2}-1}}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} e^{-\frac{\chi^2}{2}}$$

$$= \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \underbrace{\int_0^\infty d(\chi^2)\, \chi^n\, e^{-\frac{\chi^2}{2}}}_{\text{change of variable: } y \equiv \chi^2} = \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \underbrace{\int_0^\infty dy\, y^{\frac{n}{2}} e^{-\frac{y}{2}}}_{n \to \frac{n}{2},\ a \to \frac{1}{2}}$$

$$\boxed{\int_0^\infty dx\, x^n e^{-ax} = \frac{\Gamma(n+1)}{a^{n+1}}\,; \quad \Gamma(n+1) = n\Gamma(n)}$$

$$= \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \frac{\Gamma\left(\frac{n}{2}+1\right)}{\left(\frac{1}{2}\right)^{\frac{n}{2}+1}} = \frac{1}{\underbrace{2^{\frac{n}{2}}}\cancel{\Gamma\left(\frac{n}{2}\right)}} \frac{\frac{n}{\cancel{2}}\cancel{\Gamma\left(\frac{n}{2}\right)}}{\underbrace{\left(\frac{1}{2}\right)^{\frac{n}{2}}}\cdot\frac{1}{\cancel{2}}} = n$$

1

2) $$\overline{\chi^4} = \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \int_0^\infty d(\chi^2)\, \underbrace{\chi^4 \left(\chi^2\right)^{\frac{n}{2}-1}}_{\chi^{n+2}} e^{-\frac{\chi^2}{2}} \underset{y \equiv \chi^2}{=} \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \underbrace{\int_0^\infty dy\, y^{\frac{n+2}{2}} e^{-\frac{y}{2}}}_{\substack{n \to \frac{n+2}{2} \\ a = \frac{1}{2}}}$$

$$= \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \frac{\Gamma\left(\frac{n}{2}+1+1\right)}{\left(\frac{1}{2}\right)^{\frac{n}{2}+2}} = \frac{1}{\underbrace{\cancel{2^{\frac{n}{2}}}\Gamma\left(\frac{n}{2}\right) \cancel{\left(\frac{1}{2}\right)^{\frac{n}{2}}}}_{1}\cdot\frac{1}{2^2}} \left(\frac{n}{2}+1\right)\Gamma\left(\frac{n}{2}+1\right)$$

$$= \frac{\left(\frac{n}{2}+1\right) \frac{n}{2}\, \cancel{\Gamma\left(\frac{n}{2}\right)}}{\cancel{\Gamma\left(\frac{n}{2}\right)}\, \frac{1}{2^2}} = \boxed{(n+2)\,n}$$

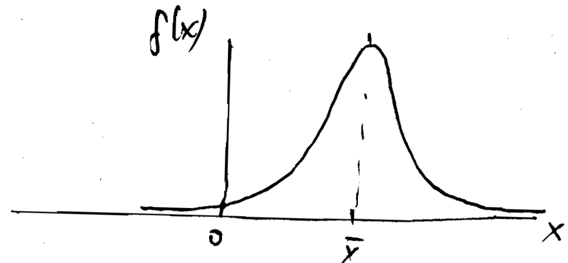
3) $$\boxed{\sigma_{\chi^2}^2 = \overline{\chi^4} - \overline{\chi^2}^{\,2} = n(n+2) - n^2 = 2n}$$

Most probable or prefered value in the different distributions:

1) Gaussian distribution:

↳ Mean: $\bar{X}$; variance is σ_X^2

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{(x-\bar{X})^2}{2\sigma_X^2}}$$



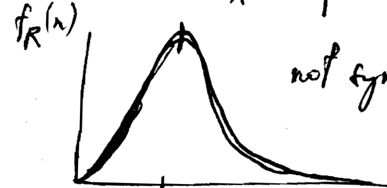
Symmetric wrt. $\bar{X}$

→ Most probable value (max. probability)

$$\frac{\partial f}{\partial x} = 0 \to \boxed{x_{max} = \bar{X}}$$

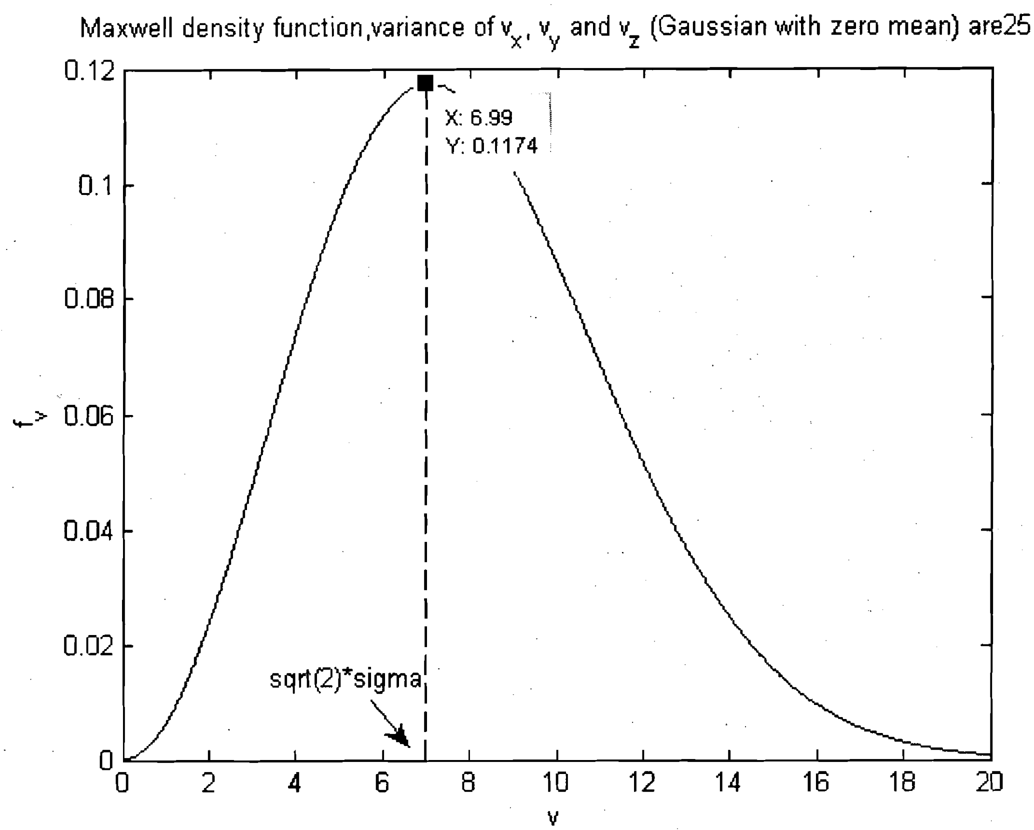
2) Rayleigh distribution: $R = \sqrt{X^2 + Y^2}$: X, Y are Gaussian with zero means ($\bar{X} = \bar{Y} = 0$) and standard dev. $\sigma = \sigma_X = \sigma_Y$

$$f_R(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}$$



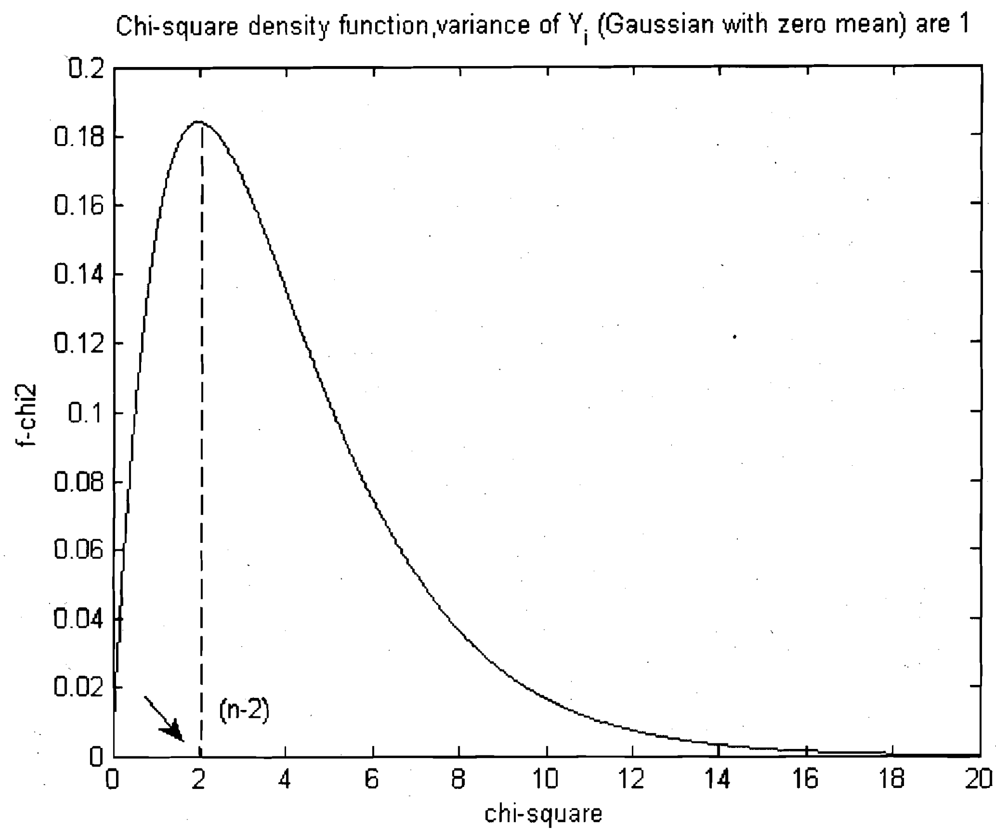
not symmetric.

Most probable value: $\frac{\partial f_R}{\partial r} = 0 \to \boxed{r_{max} = \sigma}$




```
%sketch of Maxwell density function

va=0:0.01:20;
var=25;
sigma=sqrt(var);
for i=1:2001
    v=i/100;
    fv(i)=sqrt(2/pi)*(v)^2/sigma^3*exp(-v^2/(2*var));
end
figure(1), plot(va,fv); title(strcat('Maxwell density function, ',
'variance of v_x, v_y and v_z (Gaussian with zero mean) are',
num2str(var)))
xlabel('v'), ylabel('f_v')
```




```
%sketch of Chi_square density function

chia=0:0.01:20;
n=4;
for i=1:2001
    chi2=i/100;
    fchi2(i)=chi2^(n/2-1)/(2^(n/2)*gamma(n/2))*exp(-chi2/2);
end
figure(1), plot(chia,fchi2); title(strcat('Chi-square density function,
', 'variance of Y_i (Gaussian with zero mean) are 1'))
xlabel('chi-square'), ylabel('f-chi2')
```

Ch3 : Several Random Variables :

We will look at 2 random variable, extension to 3 or more is straightforward.

2D Distribution Functions : $F(x,y) \equiv Pr(X \le x, Y \le y)$

Three-axioms check :

1) $0 \le F(x,y) \le 1$ still true since $F(x,y)$ is defined as a probability.

2) $F(-\infty, 0) \equiv Pr(X \le -\infty, Y \le 0) = 0$

More generally: $F(-\infty, y) = F(x, -\infty) = F(-\infty, -\infty) = 0$ ✓

involving impossible events : no number can be less than $-\infty$

Certain event: $F(+\infty, +\infty) = 1$ ✓

$\hookrightarrow Pr(X < +\infty, Y < +\infty)$

Also: $F(x, +\infty) \equiv Pr(X \le x, \underbrace{Y < +\infty}) = F_X(x)$ "Marginal Distribution Function in X"

$F(+\infty, y) \equiv Pr(X < +\infty, Y \le y) = F_Y(y)$ "Marginal Distribution Function in Y"

3) $Pr(x_1 < X \le x_2, y_1 < Y \le y_2) = F(x_2, y_2) - F(x_1, y_1)$

2D Density Functions :

$$f(x,y) \equiv \frac{\partial^2 F}{\partial x \, \partial y}$$

$\longrightarrow$ partial derivatives since now we have more than one variable

$$f(x,y)\,dx\,dy = \Pr\left(x < X \leq x+dx,\ y < Y \leq y+dy \right)$$

Properties of the 2D density function:

1) Always positive : $f(x,y) \geq 0$: since $F(x,y)$ is always increasing in both x & y, Similar to the 1D property!
1D: larger value $x \to$ larger interval $\to$ larger probability $F(x)$
2D: larger values x & $y \to$ larger area $\to$ larger probability $F(x,y)$

2) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\, \underbrace{f(x,y)}_{\frac{\partial^2 F}{\partial x \partial y}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cancel{dx\,dy}\, \frac{\partial^2 F}{\cancel{\partial x \partial y}} = \underbrace{F(\infty,\infty)}_{1} - \underbrace{F(-\infty,-\infty)}_{0}$$

$$= 1$$

3) $$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} du\,dv\, f(u,v)$$ (similar to the 1D property)

4) $$\int_{-\infty}^{\infty} dy\, f(x,y) = f_X(x)$$: Marginal density function in x.

$$\int_{-\infty}^{\infty} dx\, f(x,y) = f_Y(y)$$: " " " " y.

5) $$\Pr\left(x_1 < X \leq x_2,\ y_1 < Y \leq y_2\right) = \int_{x_1}^{x_2}\int_{y_1}^{y_2} dx\,dy\, f(x,y)$$

$$= F(x_2,y_2) - F(x_1,y_1)$$

2D Uniform Distribution: $X \subset [x_1, x_2]$; $Y \subset [y_1, y_2]$

$f(x,y) \rightarrow$ density function : any point within this rectangular region will have equal probability.

$$f(x,y) = \begin{cases} \dfrac{1}{(x_2-x_1)(y_2-y_1)} & \text{if } x_1 < X \leq x_2 \ \& \ y_1 < Y \leq y_2 \\ 0 & \text{otherwise} \end{cases}$$

→ What is the marginal density function in x?

$$f_X(x) = \int_{-\infty}^{\infty} dy\, f(x,y) = \int_{y_1}^{y_2} dy \underbrace{\frac{1}{(x_2-x_1)(y_2-y_1)}} = \frac{1}{(x_2-x_1)\cancel{(y_2-y_1)}} \cancel{(y_2-y_1)}$$

$$= \frac{1}{x_2-x_1}$$ (same as the 1D uniform density function for x)

→ what is the marginal density function in y?

$$f_Y(y) = \frac{1}{y_2-y_1}$$

→ $f(x,y) = f_X(x) \cdot f_Y(y)$ when $x_1 < X \leq x_2$; $y_1 < Y \leq y_2$

Use the 2D density function to define the correlation b/w two random variables: X & Y:

Correlation b/w X & Y → $\overline{XY} = E[XY] = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\; xy\, f(x,y)$

(direct extension of $\overline{X} = \int_{-\infty}^{\infty} dx\; x f(x)$)

Let's find what is the correlation b/w two uniform random variable X & Y:

$$f(x,y) = \begin{cases} \dfrac{1}{(x_2-x_1)(y_2-y_1)} & \text{when } x_1 < X \le x_2,\ y_1 < Y \le y_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\overline{XY} = E[XY] = \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} dy \; xy \underbrace{\frac{1}{(x_2-x_1)(y_2-y_1)}}$$

$$= \frac{1}{(x_2-x_1)(y_2-y_1)} \underbrace{\int_{x_1}^{x_2} dx\, x}_{\frac{x_2^2-x_1^2}{2}} \cdot \underbrace{\int_{y_1}^{y_2} dy\, y}_{\frac{y_2^2-y_1^2}{2}}$$

$$= \underbrace{\frac{x_1+x_2}{2}}_{\overline{X}} \cdot \underbrace{\frac{y_1+y_2}{2}}_{\overline{Y}}$$

$$\overline{X} \equiv \int_{x_1}^{x_2} dx\, x f_X(x) = \frac{1}{x_2-x_1} \int_{x_1}^{x_2} dx\; x = \frac{x_2^2-x_1^2}{2(x_2-x_1)} = \frac{x_1+x_2}{2}$$

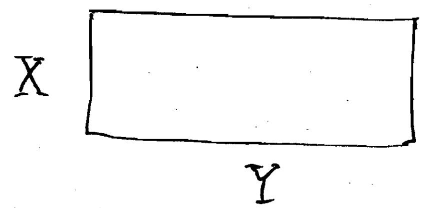
For two uniform random variable X & Y: their correlation is just the product of their averages:

$$\underbrace{\overline{XY} = \overline{X} \cdot \overline{Y}}_{X \,\&\, Y \text{ are statistically independent}} \quad (\underline{\text{Not a general result!}})$$

HW4 (ch 3): 1.2; 2.1; 3.3; 4.4.

Exercise 3.1.1 (pg 124)

Semiconductor substrate:



X & Y are Gaussian random variables: $\overline{X} = 1\,\text{cm}$; $\overline{Y} = 2\,\text{cm}$
and $\sigma_X = \sigma_Y = \sigma = 0.1\,\text{cm}$

a) Prob. that both dimensions are larger than their mean values by 0.05 cm:

$$\Pr(X \geq 1.05\,\text{cm},\ Y \geq 2.05\,\text{cm}) = [1 - F_X(1.05)][1 - F_Y(2.05)]$$
$$= Q\left(\frac{1.05 - 1}{0.1}\right)\ Q\left(\frac{2.05 - 2}{0.1}\right)$$
$$= Q(0.5)\quad Q(0.5)$$
$$= [Q(0.5)]^2 \underset{\substack{\downarrow \\ \text{Appendix E}}}{=} 0.3085^2 = 0.09517$$

Recall: $F(x) = \Pr(X \leq x)$
$$\underset{\downarrow}{=} 1 - Q\left(\frac{x - \overline{X}}{\sigma_X}\right)$$
~~F is Gaussian~~

b) Prob. that larger dimension (Y) is greater than its mean value by 0.05 cm, and smaller dimension X is less than its mean value by 0.05 cm:

$$\Pr(X \leq 0.95\,\text{cm},\ Y \geq 2.05\,\text{cm}) = F_X(0.95)[1 - F_Y(2.05)]$$
$$= \left[1 - Q\left(\frac{0.95 - 1}{0.1}\right)\right]\left[Q\left(\frac{2.05 - 2}{0.1}\right)\right]$$
$$= [1 - Q(-0.5)]\,[Q(0.5)]$$

Recall: $Q(-y) = 1 - Q(y)$ (Appendix E)
$$= [Q(0.5)]^2 = 0.09517.$$

Same answer! Since X is a Gaussian which is symmetric wrt to the mean value

3.1.2 Two random variable: X & Y

$$f(x,y) = \begin{cases} kxy & 0 \le x \le 1 \ \& \ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Determine k such that $f(x,y)$ is valid prob. density function.

Property #2:
$$1 = \int_0^1 dx \int_0^1 dy \; kxy = k \underbrace{\int_0^1 dx\, x}_{\left[\frac{x^2}{2}\right]_0^1} \underbrace{\int_0^1 dy\, y}_{\left[\frac{y^2}{2}\right]_0^1}$$

$$1 = \frac{k}{4} \rightarrow \boxed{k = 4}$$

b) $F(x,y) =$

Do it as a homework (HW4)

...

..

$$\underbrace{F(x,y)}_{\text{2D distribution function}} = \int_{-\infty}^{x} \int_{-\infty}^{y} d\lambda \, d\rho \; \underbrace{f(\lambda,\rho)}_{\text{2D density function}} = 4 \int_0^x \int_0^y d\lambda \, d\rho \; \lambda\rho$$

$$= 4 \int_0^x d\lambda \, \lambda \int_0^y d\rho \, \rho$$

$$= x^2 y^2 \qquad \begin{cases} 0 \le x \le 1 \\ \& \\ 0 \le y \le 1 \end{cases}$$

c) $P_r\left(X \le \frac{1}{2}, Y > \frac{1}{2}\right) =$

Recall: $F(x,y) \equiv P_r(X \le x, Y \le y)$

$\rightarrow P_r\left(0 \le X \le \frac{1}{2}, \frac{1}{2} < Y \le 1\right)$

$$= 4 \int_0^{\frac{1}{2}} d\lambda \, \lambda \int_{\frac{1}{2}}^{1} d\rho \, \rho = 4 \, \frac{1}{8} \left(\frac{1 - \frac{1}{4}}{2}\right)$$

$$= \frac{3}{16}$$

d) $\rightarrow$

Conditional Probability involving 2 random variables:

We know: $$Pr(A|B) = \frac{Pr(A,B)}{Pr(B)} \quad \begin{matrix}(\text{joint}) \\ (\text{marginal})\end{matrix}$$

$\underbrace{}_{\text{conditional}}$

With 2 random variables: we can apply a condition on the second random variable.

a) $B = \{Y \leq y\}$ "B is the event that the random variable Y takes a value less or equal than y"

$$\Rightarrow Pr(B) = Pr(Y \leq y) \equiv \underbrace{F_Y(y)}_{\text{Marginal Prob. Distribution Function in } Y}$$

$A = \{X \leq x\}$ "A is the event that the random variable X takes a value less or equal than x"

$$\Rightarrow Pr(A) = Pr(X \leq x) \equiv \underbrace{F_X(x)}_{\text{Marginal Prob. Distrib. Function in } X}$$

$$\boxed{F_X(x \mid Y \leq y) = \frac{Pr(X \leq x,\ Y \leq y)}{Pr(Y \leq y)} = \frac{F(x,y)}{F_Y(y)}}$$

b) Let's define B in a different way: $B = \{y_1 < Y \leq y_2\}$

$$F_X(x \mid y_1 < Y \leq y_2) = \frac{Pr(X \leq x,\ y_1 < Y \leq y_2)}{Pr(y_1 < Y \leq y_2)} = \frac{F(x,y_2) - F(x,y_1)}{F_Y(y_2) - F_Y(y_1)}$$

c) We can write the event $B = \{Y = y\}$ as $\lim_{\Delta y \to 0} \{y < Y \leq y + \Delta y\}$

Now the second alternative can use the results in part b) where

$y_1 \to y$; $y_2 \to y + \Delta y$

$$\underbrace{F(x \mid Y = y)}_{\text{Marg. Distrib. in } x} = \lim_{\Delta y \to 0} \frac{F(x, y+\Delta y) - F(x, y)}{F_Y(y+\Delta y) - F_Y(y)}$$

$$= \lim_{\Delta y \to 0} \frac{\dfrac{F(x, y+\Delta y) - F(x,y)}{\Delta y}}{\dfrac{F_Y(y+\Delta y) - F_Y(y)}{\Delta y}} = \frac{\dfrac{\partial F(x,y)}{\partial y}}{\dfrac{\partial F_Y(y)}{\partial y}}$$

* Total derivative : $\dfrac{dF_Y(y)}{dy}$
* Partial derivative : $\dfrac{\partial F(x,y)}{\partial y}$

Recall: (prob 3.1.2)

$$F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} d\lambda \, d\rho \; f(\lambda, \rho) \to \frac{\partial F}{\partial y} = \int_{-\infty}^{x} d\lambda \; f(\lambda, \rho)$$

$$\underbrace{F_Y(y)}_{\text{Marginal distribution in } Y} = \int_{-\infty}^{y} du \; \underbrace{f_Y(u)}_{\text{Marginal density in } Y} \to \frac{dF_Y}{dy} = f_Y(y)$$

$$F(x \mid Y = y) = \frac{\int_{-\infty}^{x} d\lambda \; f(\lambda, \rho)}{f_Y(y)}$$

d) $$\frac{\partial}{\partial x} F(x \mid Y = y) = \frac{\partial}{\partial x} \cdot \frac{\int_{-\infty}^{x} d\lambda \; f(\lambda, \rho)}{f_Y(y)}$$

$$\boxed{f(x \mid Y = y) = \frac{f(x,y)}{f_Y(y)}}$$

also : $$\boxed{f(y \mid X = x) = \frac{f(x,y)}{f_X(x)}}$$

Switching $x \leftrightarrow y$

$$f(y|X=x) \underset{\text{Notation}}{=} f(y|x) \rightarrow \boxed{f(y|x) = \frac{f(x,y)}{f_X(x)}}$$

Bayes Theorem.

<u>Application of Bayes Theorem in Signal Processing:</u>

→ <u>Random noise elimination:</u>

Assume we are measuring X, noise (random) will come along with the measurements N, we are recording:

$$Y = X + N$$

Our goal is to eliminate the random noise: getting X from the noisy Y → get $f_X(x|y)$ (prob. density function in X) and maximize it

<u>Bayes Theorem:</u> $$f_X(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f_Y(y|X=x)\, f_X(x)}{f_Y(y)}$$

Observations: 1) $f_Y(y|X=x) = f_N(n) = f_N(y-x)$

noisy signal $Y = X + N$

2) $$f_Y(y) = \int_{-\infty}^{\infty} dx\, f(x,y) = \int_{-\infty}^{\infty} dx\, \underbrace{f_Y(y|X=x)}_{f_N(y-x)} f_X(x)$$

$$f_Y(y|X=x) = \frac{f(x,y)}{f_X(x)}$$

$$= \int_{-\infty}^{\infty} dx\, f_N(y-x)\, f_X(x)$$

$$\rightarrow f_X(x|y) = \frac{f_N(y-x)\, f_X(x)}{\int_{-\infty}^{\infty} dx\, f_N(y-x)\, f_X(x)} \underset{\text{Calculus}}{=} \frac{\partial}{\partial x} \ln\left[\int_{-\infty}^{\infty} dx\, f_N(y-x)\, f_X(x)\right]$$

Now that we have $f_X(x|y)$, to get x, will need to maximize:

$$\frac{\partial f_X(x|y)}{\partial x} = 0$$ to find what x is most probable given $Y = y$.

Example: Assume we would like to measure X where

$$f_X(x) = \begin{cases} b e^{-bx} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Assume X contaminated with broad band noise (Gaussian noise)

$$f_N(n) = \frac{1}{\sqrt{2\pi}\,\sigma_N} e^{-\frac{(n-\bar{N})^2}{2\sigma_N^2}} \quad (\bar{N} = 0 \text{ for a noise})$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_N} e^{-\frac{n^2}{2\sigma_N^2}}$$

Want to retrieve X from $Y = X + N$ → Random Noise Elimination using method described earlier:

Y ↓ contaminated measurements, X ↓ goal, N ↓ noise

$$\frac{\partial f_X(x|y)}{\partial x} = 0 = \frac{\partial}{\partial x}\left\{ \frac{f_N(y-x) f_X(x)}{\int_{-\infty}^{\infty} dx\, f_N(y-x) f_X(x)} \right\}$$ → does not depend on x!

$$\leftrightarrow \quad 0 = \frac{\partial}{\partial x}\left[f_N(y-x) f_X(x) \right]$$

$$0 = \frac{\partial}{\partial x}\left[\frac{1}{\sqrt{2\pi}\,\sigma_N} e^{-\frac{(y-x)^2}{2\sigma_N^2}} \cdot \underbrace{b e^{-bx}}_{x \geq 0} \right]$$

$$\leftrightarrow \quad 0 = \frac{\partial}{\partial x}\left[e^{-\frac{(y-x)^2}{2\sigma_N^2}} \cdot e^{-bx} \right] = \left(\frac{y-x}{\sigma_N^2} - b \right) \underbrace{\left[e^{-\frac{(y-x)^2}{2\sigma_N^2}} e^{-bx} \right]}_{\text{bring to LHS dividing} \to \text{done}}$$

$$\leftrightarrow \quad \frac{y-x}{\sigma_N^2} - b = 0 \quad \text{or} \quad \boxed{x = y - b\sigma_N^2}$$

$\Rightarrow$ Use $x = y - b\sigma_N^2$ to eliminate random noise from Y.

Statistical Independence between two Random Variables:

Definition of Stat. independence:

X & Y are stat independent if $f(x,y) = f_X(x) \cdot f_Y(y)$
(joint density function in x & y can be factored out into the marginal density in x & marginal density in y)

Consequence:

1) $E[XY] = E[X] \cdot E[Y]$

ensemble average : or correlation b/w X & Y

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dy\, xy f(x,y) = \underbrace{\int_{-\infty}^{\infty} dx\, x f_X(x)}_{E[X]=\bar{X}} \underbrace{\int_{-\infty}^{\infty} dy\, y\, f_Y(y)}_{E[Y]=\bar{Y}}$$

2) $$\underbrace{f_X(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f_X(x) \cdot \cancel{f_Y(y)}}{\cancel{f_Y(y)}} = f_X(x)}_{\text{Makes sense since } X \,\&\, Y \text{ are stat. independent}}$$

3) $f_Y(y|x) = f_Y(y)$ (similarly).

Correlation b/w two random variable X & Y:

$E[(X-\bar{X})(Y-\bar{Y})]$: covariance between X & Y

$$\rho \equiv E[\underbrace{\frac{X-\bar{X}}{\sigma_x}}_{\text{zeta}} \underbrace{\frac{Y-\bar{Y}}{\sigma_y}}_{\text{eta}}]$$: normalized covariance b/w X & Y

$\xi \quad \eta \quad \rightarrow \quad \rho = \overline{\xi\eta}$

$$\bar{\zeta} = \overline{\frac{X-\bar{X}}{\sigma_x}} = \frac{\bar{X}-\bar{X}}{\sigma_x} = 0$$

HW 4: ch3 1.2, 2.1, 3.3; 4.4 → due 4/8

$$\bar{\eta} = \overline{\frac{Y-\bar{Y}}{\sigma_y}} = \frac{\bar{Y}-\bar{Y}}{\sigma_y} = 0$$

$$\sigma_\zeta^2 = \overline{(\zeta-\bar{\zeta})^2} = \overline{\zeta^2} = \frac{\overline{(X-\bar{X})^2}}{\cancel{\sigma_x^2}} \frac{\cancel{\sigma_x^2}}{} \cdot \frac{1}{\cancel{\sigma_x^2}} = 1$$

$$\zeta \equiv \frac{X-\bar{X}}{\sigma_x}$$

→ ζ (zeta) has zero mean & unit variance
→ Also η (eta)

} η & ζ are random variables with zero mean & unit variance

$\boxed{\text{Consequence}}$ for $\rho = \overline{\zeta\eta}$ "normalized covariance between X & Y"

$$0 \le \overline{(\zeta\pm\eta)^2} = \overline{(\zeta^2+\eta^2\pm 2\zeta\eta)} = \underbrace{\overline{\zeta^2}}_{1} + \underbrace{\overline{\eta^2}}_{1} \pm 2\underbrace{\overline{\zeta\eta}}_{\equiv\rho}$$

non-negative since we have a squared.

$$= 2 \pm 2\rho = 2(1\pm\rho)$$

→ $0 \le 2(1\pm\rho)$: i.e. $(1\pm\rho)$ is non-negative.

$$\Rightarrow \begin{cases} 1+\rho \geq 0 \to \rho \geq -1 \\ 1-\rho \geq 0 \to \rho \leq 1 \end{cases} \to \boxed{-1 \le \rho \le 1}$$

"Normalized Covariance between X & Y will be between −1 and 1"

What if X & Y are stat. independent ? $f(x,y) = f_X(x) f_Y(y)$

↳ average of xy or $\xi\eta$ can be done separately:

$$\rho = \overline{\xi\eta} = \underbrace{\overline{\xi}}_{0} \cdot \underbrace{\overline{\eta}}_{0} = 0$$

→ For two stat. independent random variables, their normalized covariance is 0

3.2.1 (HW 4 due 4/8)

→ From notes on Chapter 3:

$$f_X(x|y) = \frac{f_N(y-x) \cdot f_X(x)}{\int_{-\infty}^{\infty} dx \; f_N(y-x) \cdot f_X(x)}$$

→ From the info provided in this problem: Noise → uniform distribution:

↓ in general: $f(x) = \begin{cases} \frac{1}{x_2 - x_1} & ; \; x_1 < x \le x_2 \\ 0 & ; \; \text{otherwise} \end{cases}$ | $\overline{X} = \frac{x_1 + x_2}{2}$, $\sigma_X^2 = \frac{(x_2 - x_1)^2}{12}$ } Results for uniform distrib.

Noise has mean 0, variance 12 → $\begin{cases} \frac{n_1 + n_2}{2} = 0 \text{ or } n_2 = -n_1 \\ \frac{(n_2 - n_1)^2}{12} = 12 \rightarrow n_2 - n_1 = 12 \end{cases}$

$$\boxed{n_2 = 6 \; ; \; n_1 = -6}$$

$$\boxed{f_N(n) = \begin{cases} \frac{1}{12} & -6 < n \le 6 \\ 0 & \text{otherwise} \end{cases}}$$

→ From the info provided in this problem: X → Rayleigh distribution with $\overline{X} = 10$

In general: $f_R(r) = \begin{cases} \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} & r \ge 0 \\ 0 & r < 0 \end{cases}$ } $\overline{R} = \sigma\sqrt{\frac{\pi}{2}}$, $\sigma_R^2 = 0.429\sigma^2$ } Results for Rayleigh distribution.

X (Rayleigh) with $\bar{X} = 10$:

$$\bar{X} = 10 = \sigma\sqrt{\frac{\pi}{2}} \rightarrow \sigma = \frac{10\sqrt{2}}{\sqrt{\pi}}$$

$$\boxed{f_{\bar{X}}(x) = \begin{cases} \frac{x\pi}{200} e^{-\frac{x^2}{\frac{400}{\pi}}} & x \geq 0 \\ 0 & x < 0 \end{cases}}$$

$$\Rightarrow \quad f(x|y) = \frac{f_N(y-x) \frac{x\pi}{200} e^{-\frac{x^2}{\frac{400}{\pi}}}}{\int_{-\infty}^{\infty} dx\, f_N(y-x) \frac{x\pi}{200} e^{-\frac{x^2}{\frac{400}{\pi}}}} = \frac{\frac{x\pi}{2400} e^{-\frac{x^2}{\frac{400}{\pi}}}}{\frac{\pi}{2400}\boxed{\int_{y-6}^{y+6} dx\, x e^{-\frac{x^2}{\frac{400}{\pi}}}}}$$

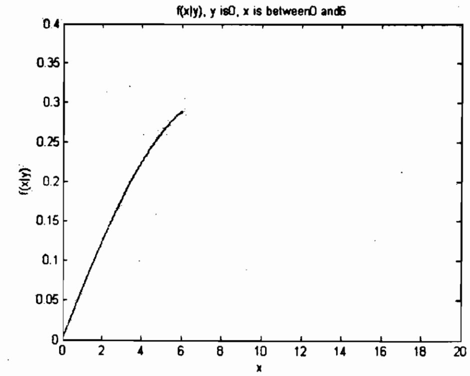
Notes:
1) $f_N(y-x)$ is $\frac{1}{12}$ if $\boxed{-6 < y-x \leq 6}$
2) $f_{\bar{X}}(x) = \frac{x\pi}{200} e^{-\frac{x^2}{\frac{400}{\pi}}}$ if $x \geq 0$

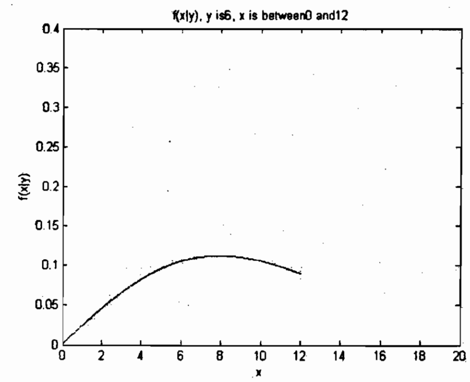
$$\left.\begin{aligned} y-x \leq 6 &\rightarrow y-6 \leq x \\ y-x > -6 &\rightarrow y+6 > x \end{aligned}\right\} y-6 \leq x < y+6$$

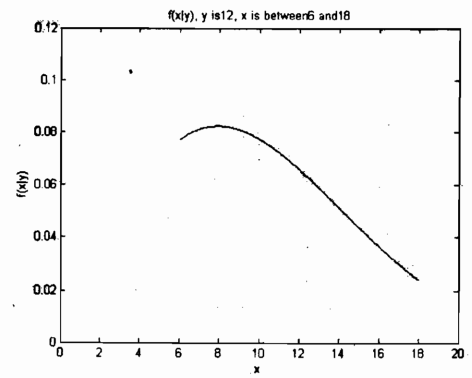
$$\boxed{\textbf{Math result:} \quad \int dx\, x e^{-ax^2} = -\frac{e^{-ax^2}}{2a}}$$

So:
$$\int_{y-6}^{y+6} dx\, x e^{-\frac{x^2}{\frac{400}{\pi}}} = \begin{cases} \frac{200}{\pi}\left[1 - e^{-\frac{(y+6)^2\pi}{400}}\right] & \text{if } 0 \leq y \leq 6 \text{ (when lower limit is reset to 0)} \\ \frac{200}{\pi}\left[e^{-\frac{(y-6)^2\pi}{400}} - e^{-\frac{(y+6)^2\pi}{400}}\right] & \text{if } 6 < y \text{ (when the lower limit is } y-6) \end{cases}$$

$$\Rightarrow \quad f(x|y) = \begin{cases} = \dfrac{x e^{-\frac{x^2}{\frac{400}{\pi}}}}{\frac{200}{\pi}\left[1 - e^{-\frac{(y+6)^2\pi}{400}}\right]} & 0 \leq y \leq 6 \\ = \dfrac{x e^{-\frac{x^2}{\frac{400}{\pi}}}}{\frac{200}{\pi}\left[e^{-\frac{(y-6)^2\pi}{400}} - e^{-\frac{(y+6)^2\pi}{400}}\right]} & 6 < y \end{cases}$$







b) If $y=12$ (measured) what is the best estimate of the true value of x? or what x would make $f(x|y=12)$ maximum?

$$\frac{\partial f(x|12)}{\partial x} = 0$$

Since after replacing $y=12$, denominator in $\underset{(y>6)}{f(x|y)}$ is a number

→ we only need look at the denominator:

$$\frac{\partial}{\partial x}\left(x e^{-\frac{x^2}{\frac{400}{\pi}}}\right) = 0 = e^{-\frac{x^2}{\frac{400}{\pi}}}\left(1 - \frac{2x^2}{\frac{400}{\pi}}\right)$$

$$\Rightarrow \quad 2x^2 = \frac{400}{\pi} \quad \text{or} \quad x = \sqrt{\frac{200}{\pi}} = 7.98$$

3.3 $\quad P(xy>0) = 0.733$

4.4

a) $\sigma_W^2 = 3$

b) $\rho_{XW} = \frac{\sqrt{3}}{2}$

c) $\rho_{W(Y+Z)} = \frac{\sqrt{3}}{2}$

Ch4. Sampling Theory on Random Variables.

Goals: distinguish between $\begin{cases} \text{mean} & \& \text{ sample mean} \\ \text{variance} & \& \text{ sample variance} \end{cases}$

→ Population (of resistors, transistors, etc.) with N elements
(very large number) → $\begin{cases} \text{Mean (of pop.)} : & \bar{X} \equiv \int_{-\infty}^{\infty} dx\, x f(x) \\ \text{Variance} : & \sigma_X^2 \equiv \overline{X^2} - \bar{X}^2 \end{cases}$

↳ In quality control: take samples of this large population, and test the samples.

→ Samples: with n elements (a manageable number)

To reflect the whole population behavior, samples are picked randomly ⟶ $\begin{cases} \text{Sample mean} : & \hat{X} \equiv \frac{1}{n} \sum_{i=1}^{n} x_i \\ \text{Sample variance} : & S^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \hat{X})^2 \end{cases}$

these are random variable themselves, since the samples are picked randomly.

→ Difference: $\underbrace{\begin{matrix} \bar{X} \text{ (mean)} \\ \sigma_X^2 \text{ (variance)} \end{matrix}}_{\text{numbers}}$ $\underbrace{\begin{matrix} \hat{X} \text{ (sample mean)} \\ S^2 \text{ (sample variance)} \end{matrix}}_{\text{random variable}}$

Can define
$$\boxed{\begin{matrix} \text{Mean of sample mean} \\ \text{Mean of sample variance} \\ \text{Variance of sample mean} \\ \text{Variance of sample variance} \end{matrix}}$$
← → each sample of n elements will have its own mean and variance → random samples of n elements each will have random sample means and variances

→ Mean of sample mean: $\overline{\hat{X}} = \overline{\frac{1}{n}\sum_{i=1}^{n} x_i} = \frac{1}{n}\sum_{i=1}^{n} \overline{x_i} = \overline{X}$

↳ is the mean of the population.

Mean of an element in a sample
$\parallel$
mean of population: $\overline{X}$

Example: population 10^7 resistors: each resistor bears a resistance X, which is a random variable whose mean is $\overline{X}$. That resistor could be part of a sample → $\overline{x_i} = \overline{X}$.

→ Variance of the sample mean: $var(\hat{X}) = \overline{(\hat{X})^2} - \overline{\hat{X}}^2$

$$= \overline{(\hat{X})^2} - \overline{X}^2$$

$$= \frac{\sigma_X^2}{n}\left(\frac{N-n}{N-1}\right)$$

Normally $N \to \infty$ (very large population): $\boxed{var(\hat{X}) = \frac{\sigma_X^2}{n}}$

larger samples (larger n) have smaller variance between their means.

Expt #1:
Collect 20 samples of 5 elements each.
Calculate the mean for each sample
→ 20 sample means.

↓
variance of these 20 $\hat{X}$'s

$>$

Expt #2
→ Collect 20 sample of 1000 elements each.
→ Calculate mean for each sample
→ 20 sample means.

variance of these 20 $\hat{X}$'s