6.4 c) Power supply provides up to 40W

$Pr(\underline{X} > 40W) = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^5 = 0.00977$

total power required

when all 5 loads draw power at the same time

H or T

H H H H H $\rightarrow \left(\frac{1}{2}\right)^5$

HW3 : 6.4 ; 7.1 ; 8.2 ; 9.3 (Ch. 2) due 3/11

2.5.1 Gaussian Random voltage: $\underline{X}$: $\begin{cases} \bar{X} = 5 & \text{(mean)} \\ \sigma_X^2 = 16 & \text{(Variance)} \end{cases}$

a) $Pr(\underline{X} > 0) = 1 - Pr(\underline{X} \le 0)$

$= 1 - F(0) = 1 - \Phi\left(\frac{\cancel{0} - \bar{X}}{\sigma_X}\right) = 1 - \left[1 - \Phi\left(\frac{\bar{X}}{\sigma_X}\right)\right]$

$F(x) = \Phi\left(\frac{x - \bar{X}}{\sigma_X}\right)$

$= \Phi\left(\frac{\bar{X}}{\sigma_X}\right) = \Phi\left(\frac{5}{4}\right)$

$= \Phi(1.25) = 0.8944$ App D pg 432

<u>Property of Φ:</u> $\boxed{\Phi(-y) = 1 - \Phi(y)}$; $\Phi(y) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{u^2}{2}} du$

<u>Proof:</u> $\Phi(-y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-y} e^{-\frac{u^2}{2}} du = \frac{1}{\sqrt{2\pi}} \left\{ \underbrace{\int_{-\infty}^{\infty} \underbrace{e^{-\frac{u^2}{2}}}_{\text{even}} du}_{2\int_0^\infty e^{-\frac{u^2}{2}} du} - \int_{-y}^{\infty} e^{-\frac{u^2}{2}} du \right\}$

$\frac{\cancel{2}\,\Gamma\left(\frac{1}{2}\right)}{\cancel{2}\left(\frac{1}{\sqrt{2}}\right)} = \sqrt{2\pi}$

$= 1 - \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-y}^{\infty} e^{-\frac{u^2}{2}} du}_{\text{change of dummy } u,\ u \to -u} = 1 - \frac{1}{\sqrt{2\pi}} \int_{y}^{-\infty} e^{-\frac{u^2}{2}} (-du)$

$= 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{u^2}{2}} du$

$= 1 - \Phi(y)$ ✓

Now, find $Pr(\bar{X} > 0)$ using $Q(x)$ → App. E pg 434.

Recall: $\boxed{1 = \Phi(y) + Q(y)}$

$$Pr(\bar{X}>0) = 1 - F(0) = \Phi\left(\frac{\bar{X}}{\sigma_X}\right) = \Phi(1.25) = 1 - Q(1.25)$$

$$= 1 - 0.1056 = 0.8944$$

App. E pg 434

Related Density Functions (to the Gaussian Density Function)

1) Power Distribution: $P = I^2 R$

R: constant, fixed

I = Gaussian Random Variable

→ What is the Density Function for the power P?

Recall: $Y = X^2 \rightarrow f_Y(y) = \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right]$

Now: $f_I(i) = \frac{1}{\sqrt{2\pi}\,\sigma_I} e^{-\frac{(i-\bar{I})^2}{2\sigma_I^2}}$ Mean = $\bar{I}$, Variance is σ_I^2

$$f_P\left(\frac{P}{R}\right) = \frac{1}{2\sqrt{\frac{P}{R}}}\left[f_I\left(\sqrt{\frac{P}{R}}\right) + f_I\left(-\sqrt{\frac{P}{R}}\right)\right]$$

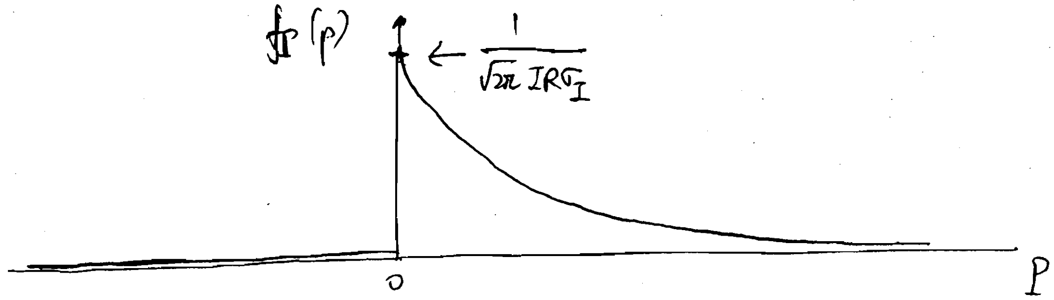
$$f_P(p) = \frac{1}{R} f\left(\frac{P}{R}\right) = \frac{1}{2\sqrt{PR}}\left[f_I\left(\sqrt{\frac{P}{R}}\right) + f_I\left(-\sqrt{\frac{P}{R}}\right)\right]$$

linear connection: $Y = aX$

$\boxed{\text{Assume } \bar{I} = 0}$ → $f_P(p) = \frac{1}{\cancel{2}\sqrt{PR}}\left[\cancel{2}\frac{1}{\sqrt{2\pi}\,\sigma_I} e^{-\frac{\left(\sqrt{\frac{P}{R}}\right)^2}{2\sigma_I^2}}\right]$

→ Density function for the Power

$$\boxed{f_P(p) = \begin{cases} \frac{1}{\sqrt{2\pi}\sqrt{PR}\,\sigma_I} e^{-\frac{P}{2R\sigma_I^2}} & : p \geq 0 \\ 0 & p < 0 \end{cases}}$$



What are $\overline{P}$ (mean) & σ_P^2 (variance) ?

$\boxed{\text{Mean}}$ $$\overline{P} = \int_0^\infty dp\; p\, f_P(p) = \frac{1}{\sqrt{2\pi}\,R\sigma_I}\int_0^\infty dp\, \frac{p}{\sqrt{\frac{p}{R}}}\, e^{-\frac{p}{2R\sigma_I^2}} = \frac{1}{\sqrt{2\pi R}\,\sigma_I}\int_0^\infty dp\, p^{1/2} e^{-\frac{p}{2R\sigma_I^2}}$$

$\boxed{f_P(p) = 0 \text{ when } p < 0}$

$$\boxed{\int_0^\infty dx\, x^n e^{-ax} = \frac{\Gamma(n+1)}{a^{n+1}} \underset{\text{if } n \text{ integer}}{=} \frac{n!}{a^{n+1}}}$$

In our integral for $\overline{P}$ $\begin{cases} n = \frac{1}{2} \\ a = \frac{1}{2R\sigma_I^2} \end{cases}$ $\rightarrow \overline{P} = \frac{1}{\sqrt{2\pi R}\,\sigma_I} \cdot \frac{\Gamma\left(\frac{1}{2}+1\right)}{\left(\frac{1}{2R\sigma_I^2}\right)^{3/2}} = \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\sqrt{2\pi R}\,\sigma_I \frac{1}{\left(\sqrt{2R}\right)^3 \sigma_I^3}}$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\rightarrow \overline{P} = \frac{\cancel{2}R\sigma_I^2\, \frac{\cancel{\sqrt{\pi}}}{\cancel{2}}}{\cancel{\sqrt{\pi}}} \rightarrow \boxed{\overline{P} = R\sigma_I^2}$$ (we assumed $\overline{I} = 0$)

Makes sense ?

$\overline{P} = R\,\overline{(I-\overline{I})^2} = R\,\overline{I^2} = \overline{RI^2}$

R is constant.

Yes, since $P = I^2 R \rightarrow \overline{P} = \overline{I^2 R}$

$\boxed{\text{Variance}}$ $$\sigma_P^2 = \overline{(P-\overline{P})^2} = \overline{P^2} - \overline{P}^2 = \overline{P^2} - R^2\sigma_I^4$$

averaging algebra

Need to find now

$\boxed{\Gamma(\alpha+1) = \alpha\Gamma(\alpha) \text{ property of } \Gamma \text{ functions}}$

use definition: $$\overline{P^2} = \int_0^\infty dp\; p^2 f_P(p) = \frac{1}{\sqrt{2\pi R}\,\sigma_I}\int_0^\infty dp\; p^{3/2} e^{-\frac{p}{2R\sigma_I^2}} = \frac{1}{\sqrt{2\pi R}\,\sigma_I}\frac{\Gamma\left(\frac{3}{2}+1\right)}{\left(\frac{1}{2R\sigma_I^2}\right)^{\frac{5}{2}}}$$

$$\Rightarrow \overline{P^2} = \frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{\frac{1}{(2R)^2}\sqrt{\pi}\,\frac{1}{\sigma_I^4}} = \frac{\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\frac{\sqrt{\pi}}{(2R)^2\sigma_I^4}} = \frac{\frac{3}{4}\cancel{\sqrt{\pi}}\,(2R)^2\sigma_I^4}{\cancel{\sqrt{\pi}}} = \boxed{3R^2\sigma_I^4}$$

$$\rightarrow \boxed{\sigma_P^2 = \overline{P^2} - \overline{P}^2 = 3R^2\sigma_I^4 - R^2\sigma_I^4 = 2R^2\sigma_I^4}$$

Variance for Power P when $\overline{I} = 0$

---

2) Rayleigh Distribution:

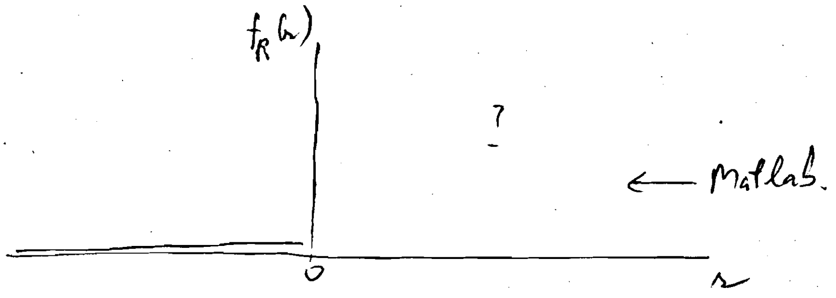
Hypothenuse or length of a 2D vector (X, Y), where X & Y are Gaussian random variables with zero means and same variance: $\sigma_x^2 = \sigma_y^2 = \sigma^2$, would follow the Rayleigh distribution, that means: $R = \sqrt{X^2+Y^2}$, follows Rayleigh distribution:

Recall: $Z = \sqrt{W}$, if W is Gaussian R. variable $\rightarrow f_Z(z) = f_W(w)\left|\frac{dw}{dz}\right|$

$$\frac{dz}{dw} = -\frac{1}{2w^{1/2}} = -\frac{1}{2z} \rightarrow \left|\frac{dw}{dz}\right| = 2z \rightarrow f_Z(z) = f_W(z^2)\,2z$$

$$\rightarrow f_R(r) = f_{X^2+Y^2}(r^2)\,2r = \begin{cases} \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} & r \geq 0 \\ 0 & r < 0 \end{cases}$$

$\leftarrow$ added for common sense



Not symmetrical like the Gaussian dens. funct. for X & Y.

$$\overline{R} = \int_0^\infty dr\, r\, f_R(r) = \frac{1}{\sigma^2}\int_0^\infty dr\, r^2 e^{-\frac{r^2}{2\sigma^2}} = \frac{1}{\sigma^2}\frac{\Gamma\left(\frac{3}{2}\right)}{2\left(\frac{1}{2\sigma^2}\right)^{\frac{3}{2}}} = \frac{\sigma\frac{\sqrt{\pi}}{2}}{2^{-\frac{1}{2}}}$$

$n = 2$, $\lambda^2 = \frac{1}{2\sigma^2}$

Recall: $\int_0^\infty dx\, x^n e^{-\lambda^2 x^2} = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2\lambda^{n+1}}$

$$\boxed{\overline{R} = \sigma\sqrt{\frac{\pi}{2}}}$$

$$\overline{R^2} = \int_0^\infty dr \; r^2 f_R(r) = \frac{1}{\sigma^2}\int_0^\infty dr \; r^3 e^{-\frac{r^2}{2\sigma^2}} = \frac{1}{\sigma^2}\frac{\Gamma(2)}{2\left(\frac{1}{2\sigma^2}\right)^{\frac{4}{2}}}$$

$$\left\{\begin{array}{l} n=3 \\ r^2 = \frac{1}{2\sigma^2}\end{array}\right.$$

$$= \frac{2}{\frac{\sigma^2}{\sigma^4}\, 2\, \frac{2}{2^2}} = 2\sigma^2$$

→ Variance: $\sigma_R^2 = \overline{(R-\bar{R})^2} = \overline{R^2} - \bar{R}^2 = 2\sigma^2 - \left(\sigma\sqrt{\frac{\pi}{2}}\right)^2$

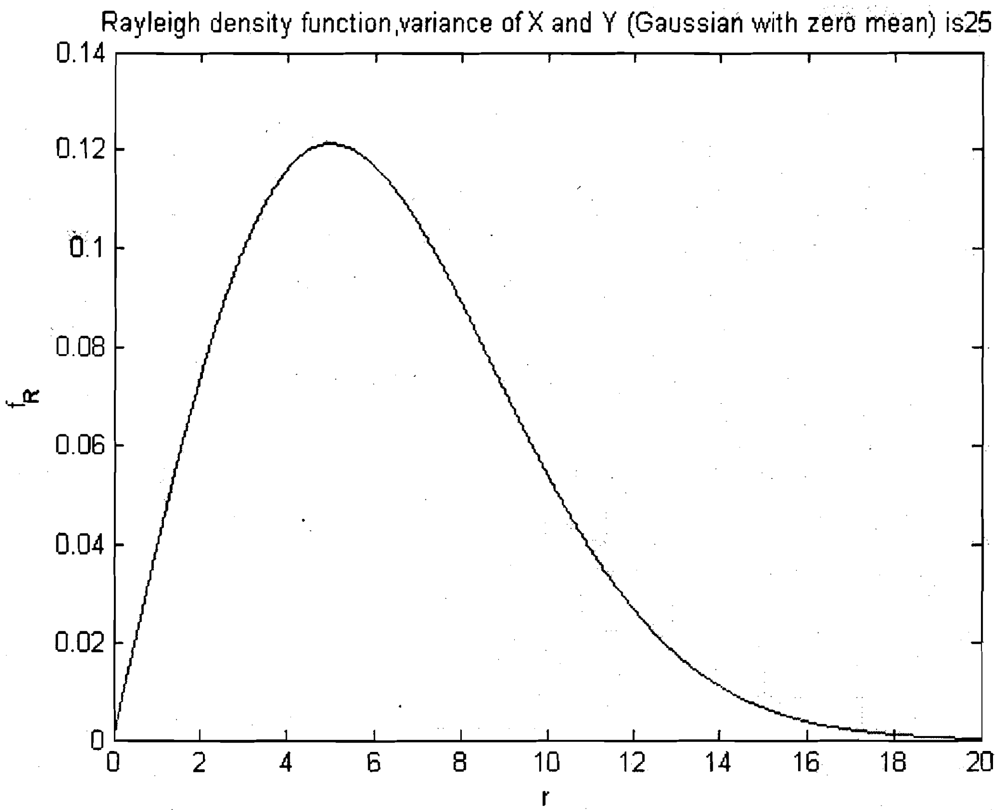
$$= \left(2 - \frac{\pi}{2}\right)\sigma^2 = 0.429\,\sigma^2$$

$$\boxed{\sigma_R^2 = 0.429\,\sigma^2}$$

Rayleigh

variance of Gaussian for $\bar{X}$ & $\bar{Y}$. (mean is 0)

Engin 322
March 6, 2008




```
%Plot of Rayleigh density function

ra=0:0.01:20;
var=25;
for i=1:2001
    r=i/100;
    fR(i)=(r)/var*exp(-r^2/(2*var));
end
figure(1), plot(ra,fR); title(strcat('Rayleigh density function, ',
'variance of X and Y (Gaussian with zero mean) is', num2str(var)))
xlabel('r'), ylabel('f_R')
```

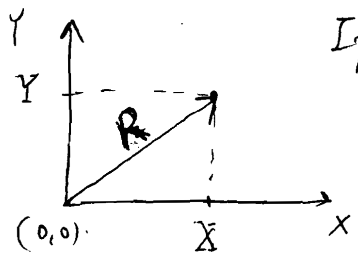
6.4 $\rightarrow$ $X \rightarrow v_x$; $Y = v_y$ Gaussian Random Variable.
zero mean ; $\sigma_x = \sigma_y = \sigma = 4\,ft/s$

a) Most probable speed $v = \sqrt{v_x^2 + v_y^2}$
peak of the Rayleigh density function
or max.

b) Mean value of speed: $\bar{v}$ (we derived formula for mean value of a Rayleigh dens. function)

c) $$P_r(v > 10\,ft/s) = 1 - \underbrace{P_r(v \leq 10\,ft/s)}_{F_v(10\,ft/s)}$$

2D planes



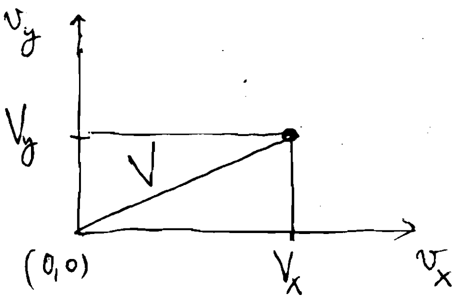
$R = \sqrt{X^2 + Y^2}$

If X, Y are Gaussian Random Variables with zero means and standard deviation $\sigma_x = \sigma_y \equiv \sigma$
$\rightarrow$ **R** is a random variable described by Rayleigh distribution:

Recall
$$\begin{cases} f_R(r) = \dfrac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} & r \geq 0 \\ \bar{R} = \sigma\sqrt{\dfrac{\pi}{2}} \\ \sigma_R^2 = 0.429\,\sigma^2 \end{cases}$$

Notation:
- $\rightarrow$ Capital letters $\rightarrow$ Random variables
- $\rightarrow$ Lower-case letters $\rightarrow$ values or argument of density functions or distribution functions

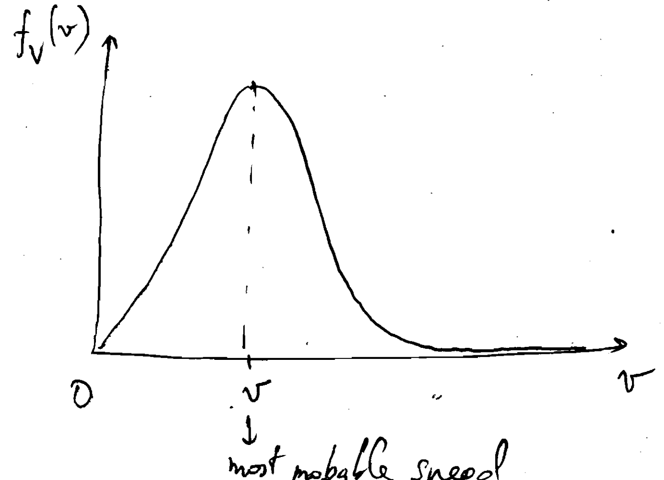
Another situation we can use Rayleigh distribution is problem (6.4)



If V_x, V_y are Gaussian Random Variables with zero means and standard deviation $\sigma_{V_x} = \sigma_{V_y} = \sigma$

$\rightarrow$ $V = \sqrt{V_x^2 + V_y^2}$ is the speed, is a random variable described by the Rayleigh distribution:

$$\begin{cases} f_V(v) = \dfrac{v}{\sigma^2} e^{-\frac{v^2}{2\sigma^2}} & v \geq 0 \\ \overline{V} = \sigma\sqrt{\dfrac{\pi}{2}} \\ \sigma_V^2 = 0.429\,\sigma^2 \end{cases}$$



(6.4) a) ?

Since the Rayleigh dist. is NOT symmetric, the most probable value is NOT the mean value (which happens in the symmetric Gaussian distribution)

$$f(v) = \frac{v}{\sigma^2} e^{-\frac{v^2}{2\sigma^2}}$$ (σ is known: it's the standard deviation of V_x and V_y)

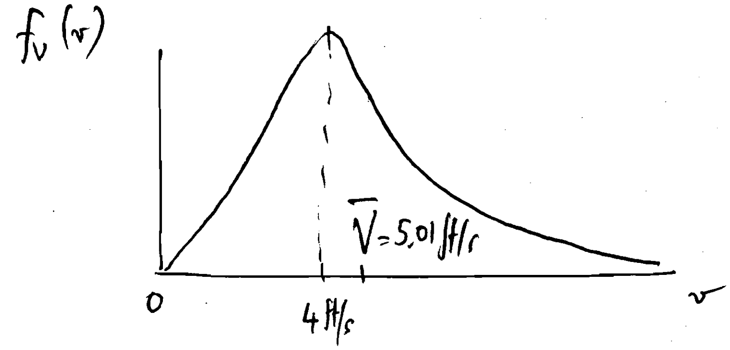
$$\frac{df_V}{dv} = 0 = \frac{1}{\sigma^2} e^{-\frac{v^2}{2\sigma^2}} - \frac{2v^2}{\sigma^2} e^{-\frac{v^2}{2\sigma^2}} = \left(1 - \frac{\cancel{2}v^2}{\cancel{2}\sigma^2}\right) \underbrace{\frac{e^{-\frac{v^2}{2\sigma^2}}}{\sigma^2}}_{\text{Non-zero}}$$

$$\Rightarrow 1 - \frac{\cancel{2}v^2}{\cancel{2}\sigma^2} = 0 \rightarrow v_{\text{most prob.}} = \sigma \text{ ft/s} = 4 \text{ ft/s}$$

$$\frac{v^2}{\sigma^2} = 1 \Rightarrow \frac{v}{\sigma} = 1$$

b) Mean value of speed

$$\overline{V} = \sigma\sqrt{\frac{\pi}{2}} = 4\sqrt{\frac{\pi}{2}} = 5.01 \text{ ft/s}$$



↑
Most probable value.

c) $$P_r(v > 10 \text{ ft/s}) = 1 - P_r(v \leq 10 \text{ ft/s})$$

$$= 1 - F_v(10)$$

$$= 1 - \int_{-\infty}^{10} dv\, f_v(v)$$

$$\boxed{f_v(v) = 0 \quad v \leq 0} \longleftarrow\ = 1 - \frac{1}{\sigma^2}\int_0^{10} v\, e^{-\frac{v^2}{2\sigma^2}}\, dv$$

Change of variable: $t = \frac{v^2}{2\sigma^2} \rightarrow dt = \frac{2v\, dv}{2\sigma^2}$

$$P_r(v > 10 \text{ ft/s}) = 1 - \int_0^{\frac{100}{2\sigma^2}} dt\; e^{-t}$$

$$= 1 - \left[\frac{e^{-t}}{-1}\right]_{t=0}^{t=\frac{100}{2\sigma^2}} = 1 - \left[\frac{e^{-\frac{100}{2\sigma^2}} - 1}{-1}\right]$$

$$= e^{-\frac{100}{2\sigma^2}} = e^{-\frac{100}{32}} = 0.0439$$

or 4.39 %

In general: <u>Rayleigh distribution</u>:

$$f_v(v) = \begin{cases} \frac{v}{\sigma^2} e^{-\frac{v^2}{2\sigma^2}} & v \geq 0 \\ 0 & v < 0 \end{cases} \rightarrow F_v(v) = \int_0^{v} \frac{u}{\sigma^2} e^{-\frac{u^2}{2\sigma^2}}\, du = \int_0^{\frac{v^2}{2\sigma^2}} dt\, e^{-t}$$

$$\boxed{F_v(v) = 1 - e^{-\frac{v^2}{2\sigma^2}}} \qquad \boxed{t = \frac{u^2}{2\sigma^2}}$$

3) Maxwell Distribution: 3D counterpart of Rayleigh ~~...~~ distrib.

Gas molecules have speed ~~...~~ components v_x, v_y, v_z that are Gaussian random variables with zero means and variance $\sigma_{v_x}^2 = \sigma_{v_y}^2 = \sigma_{v_z}^2 = \sigma^2$ $\longrightarrow$ Total speed of the molecules is a random variable described by the Maxwell distribution:

$$f_v(v) = \begin{cases} \sqrt{\frac{2}{\pi}} \frac{v^2}{\sigma^3} e^{-\frac{v^2}{2\sigma^2}} & v \geq 0 \\ 0 & v < 0 \end{cases}$$

(speed can't be negative)

1) $$\overline{v} = \int_0^\infty dv\, v f_v(v) = \sqrt{\frac{2}{\pi}} \int_0^\infty dv \frac{v^3}{\sigma^3} e^{-\frac{v^2}{2\sigma^2}} = \sqrt{\frac{2}{\pi}} \frac{1}{\sigma^3} \frac{\Gamma(2)}{2\left(\frac{1}{\sqrt{2\sigma^2}}\right)^4}$$

$$\boxed{\overline{v} = \sqrt{\frac{8}{\pi}}\, \sigma}$$

2) $$\overline{v^2} = \int_0^\infty dv\, v^2 f_v(v) = \sqrt{\frac{2}{\pi}} \frac{1}{\sigma^3} \int_0^\infty dv\, v^4 e^{-\frac{v^2}{2\sigma^2}} = \sqrt{\frac{2}{\pi}} \frac{1}{\sigma^3} \frac{\Gamma\left(\frac{5}{2}\right)}{2\left(\frac{1}{\sqrt{2\sigma^2}}\right)^5}$$

$$\boxed{\overline{v^2} = 3\sigma^2}$$

3) $$\boxed{\sigma_v^2 = \overline{v^2} - \overline{v}^2 = 3\sigma^2 - \frac{8}{\pi}\sigma^2 = \left(3 - \frac{8}{\pi}\right)\sigma^2 = 0.453\sigma^2}$$

4) Chi-Square Distribution: $\chi^2 = Y_1^2 + Y_2^2 + \dots Y_n^2$

Y_i = Gaussian random variables with zero means and unit variance $\rightarrow$ χ^2 follows the Chi-Square distribution:

Conditional Probability Distribution and Density Function:

$$F(x|M) = Pr(X \le x|M) = \frac{Pr\{X \le x, M\}}{Pr(M)}$$

$\{X \le x, M\}$ is joint event of all outcomes ξ (xi or 'x' in Greek alphabet) such that $X(\xi) \le x$ and $\xi \subset M$

$F(x|M)$ is a valid probability distribution, if we check the three axioms:

1) $0 \le F(x|M) \le 1$ ✓ Since $Pr(M) \ge Pr\{X \le x, M\}$

2) $F(-\infty|M) = Pr(X \le -\infty|M) = 0$ ✓
$F(\infty|M) = Pr(X \le \infty|M) = 1$ ✓

3) $Pr(x_1 < X \le x_2 | M) = F(x_2|M) - F(x_1|M) \ge 0$ ✓

HW3 : due 3/13 . 6.4; 7.1; 8.2; 9.3 (EX1 : HW1 → 3 & Ch1-2 on 3/25)

6.4 a) $v_{\text{most probable}} = 4$ ft/s b) $\bar{V} = 5.01$ ft/s c) $Pr(v > 10 \text{ ft/s}) = 4.39\%$
Rayleigh distribution.

7.1 a) $f_X(x) = \begin{cases} \frac{1}{2\pi\sqrt{1-x^2}} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$ b) $E[X] = 0$ c) $\overline{X^2} = \frac{1}{2}$, $\sigma_X^2 = \frac{1}{2}$

d) $Pr(X > 0.5) = \frac{1}{3}$

8.2 a) $f(x|M) = \frac{2}{\pi\sqrt{1-x^2}}$: $-1 \le x \le 1$

b) $E[X|M] = \frac{2}{\pi}$

9.3 a) $f(v_0) = \begin{cases} \frac{1}{\sqrt{2\pi}\,\sigma_{v_0}} e^{-\frac{(v_0 - \bar{v}_0)^2}{2\sigma_{v_0}^2}} & -B < v_0 < B \\ \Phi(-A) & v_0 = \pm B \\ 0 & \text{otherwise} \end{cases}$

b) $E[v_0] = 2.55$

HW3 : 7.1 Θ : Random variable, uniformly distributed, b/w 0 & 2π

$\underline{X}$: another random variable : $\underline{X} = \cos\Theta$

a) Find $f_{\underline{X}}(x)$:

Recall: "Connection b/w two random variables": $\underline{X}$ & $\underline{Y}$:

$$f_Y(y) = f_{\underline{X}}(x) \left| \frac{dx}{dy} \right|$$

In this problem: $X \to \Theta$, $Y \to \underline{X}$ $\Big\} \to \boxed{f_{\underline{X}}(x) = f_\Theta(\theta) \left| \frac{d\theta}{dx} \right|}$

$$X = \cos\Theta \to \frac{dx}{d\theta} = -\sin\Theta = -\sqrt{1-\cos^2\Theta} = -\sqrt{1-x^2}$$

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi - 0} = \frac{1}{2\pi} & 0 \le \theta \le 2\pi \\ 0 & \text{otherwise} \end{cases}$$

$$f_{\underline{X}}(x) = \begin{cases} \frac{1}{2\pi}\left(\sqrt{1-x^2}\right)^{-1} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

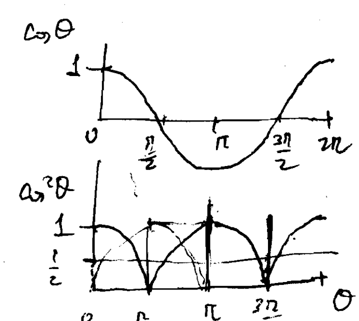
b) $\bar{X} = E[X] = \begin{cases} \int_{-1}^{1} dx \; x \underbrace{\frac{1}{2\pi}\frac{1}{\sqrt{1-x^2}}}_{f_X(x)} & \text{: using standard definition of the average using } f_X(x) \\ \text{or } \int_0^{2\pi} d\theta \; \cos\theta \underbrace{\frac{1}{2\pi}}_{f_\Theta(\theta)} & \text{: using the connection with } \Theta \text{ and } f_\Theta(\theta) \end{cases}$

$\underbrace{\qquad\qquad}_{=0}$

$\to \bar{X} = E[X] = 0$

c) $\sigma_X^2 = \overline{X^2} - \bar{X}^2 = \overline{X^2}$

$\overline{X^2} = E[X^2] = \begin{cases} \int_{-1}^{1} dx \; x^2 \frac{1}{2\pi}\frac{1}{\sqrt{1-x^2}} \\ \text{or } \int_0^{2\pi} d\theta \; \cos^2\theta \; \frac{1}{2\pi} = \frac{1}{2} \end{cases}$



$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \cos^2\theta \;=\;$$ integration by parts: $\int u\,dv = uv - \int v\,du$

$$u \equiv \cos\theta \longrightarrow du = -\sin\theta\, d\theta$$

$$dv \equiv \cos\theta\, d\theta \longrightarrow v = \sin\theta$$

$$\rightarrow \frac{1}{2\pi}\Big[\ \underbrace{\cos\theta\sin\theta\Big]_0^{2\pi}}_{0} + \int_0^{2\pi} d\theta \sin^2\theta\ \Big]$$

$$= \frac{1}{2\pi}\Big[\int_0^{2\pi} d\theta\,(1-\cos^2\theta)\Big] = \frac{1}{2\pi}\Big[\int_0^{2\pi} d\theta - \int_0^{2\pi} d\theta \cos^2\theta\Big]$$

$$\Rightarrow \frac{2}{2\pi}\int_0^{2\pi} d\theta \cos^2\theta = 1 \rightarrow \boxed{\frac{1}{2\pi}\int_0^{2\pi} d\theta \cos^2\theta = \frac{1}{2}}$$

$$\Rightarrow E[X^2] = \overline{X^2} = \frac{1}{2} \quad\Rightarrow\quad \sigma_X^2 = \overline{X^2} = \frac{1}{2}$$

d) $$P_r(X>0.5) = 1 - P_r(X \le 0.5) = 1 - F_X(0.5) = 1 - \int_{-1}^{0.5} dx\, f_X(x)$$

$$= 1 - \int_{-1}^{0.5} dx\, \frac{1}{2\pi}\frac{1}{\sqrt{1-x^2}} = \int_{0.5}^{1} dx\, \frac{1}{2\pi}\frac{1}{\sqrt{1-x^2}}$$

$$\boxed{\text{In general: } \int \frac{dx}{\sqrt{1-x^2}} \underset{\substack{\downarrow \\ x \equiv \sin\theta \\ dx = \cos\theta\, d\theta}}{=} \int d\theta}$$

$$= \frac{1}{2\pi}\int_{\sin^{-1}0.5}^{\sin^{-1}1} d\theta = \frac{1}{2\pi}\Big[\frac{\pi}{2} - \frac{\pi}{6}\Big]$$

$$= \frac{1}{2\pi}\cdot\frac{4\pi}{12} = \frac{1}{6}$$

8.2 c) For X in 7.1, $f(x|M)$? where M is the event of $0 \le \theta \le \frac{\pi}{2}$.

$$F(x \mid 0 \le \theta \le \tfrac{\pi}{2}) = \Pr(X < x \mid 0 \le \theta \le \tfrac{\pi}{2}) \underset{\substack{\downarrow \\ X=\cos\theta}}{=} \frac{\Pr(X \le x,\ 0 \le X \le 1)}{\Pr(0 \le \theta \le \frac{\pi}{2})}$$

θ : uniform dist. b/w 0 & 2π $\rightarrow$ $\Pr(0 \le \theta \le \frac{\pi}{2}) = \frac{1}{4}$

$$\Rightarrow F(x \mid 0 \le \theta \le \tfrac{\pi}{2}) = 4 \Pr(X \le x,\ 0 \le X \le 1) = 4F(x) \text{ if } 0 \le x \le 1$$

derivative wrt x

$$f(x \mid M) = 4 f_X(x) = \frac{4}{2\pi}\frac{1}{\sqrt{1-x^2}} = \frac{2}{\pi\sqrt{1-x^2}} \text{ if } 0 \le x \le 1$$

b) $$E[X|M] = E[\cos\theta \mid 0 \le \theta \le \tfrac{\pi}{2}] = \int_0^{\frac{\pi}{2}} d\theta \cos\theta \underbrace{f(\theta|M)}_{\frac{4}{2\pi}}$$

use connection with θ $\rightarrow$ simpler integral

$\hookrightarrow$ As $f(x|M) = 4 f_X(x) = \frac{4}{2\pi\sqrt{1-x^2}}$

$f(\theta|M) = 4 f_\theta(\theta) = \frac{4}{2\pi}$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\theta \cos\theta = \frac{2}{\pi}\left[\sin\theta\right]_0^{\frac{\pi}{2}} = \frac{2}{\pi}$$

Alternatively:

$$E[X|M] = \int_0^1 dx\ x\, f(x|M) = \int_0^1 dx\ x \frac{2}{\pi\sqrt{1-x^2}} = \frac{2}{\pi}\int_0^1 dx \frac{x}{\sqrt{1-x^2}}$$

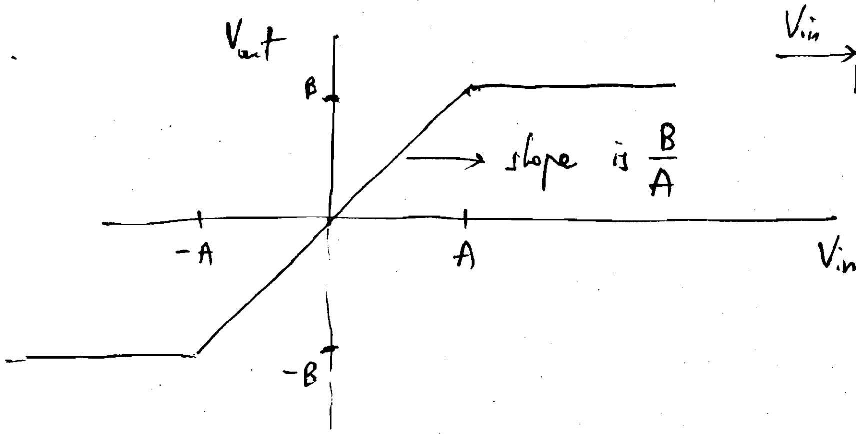
$$= \frac{2}{\pi}\left[-\cos\theta\right]_0^{\frac{\pi}{2}} = \frac{2}{\pi}$$

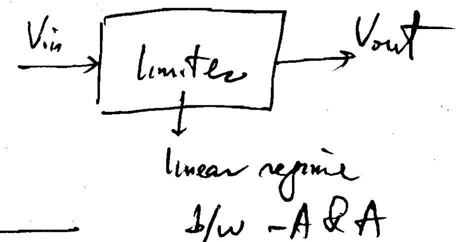
$$\boxed{\text{In general: } \int \frac{x\,dx}{\sqrt{1-x^2}} \underset{\substack{\downarrow \\ x=\sin\theta}}{=} \int \sin\theta\, d\theta}$$

9.3

$$V_{out} = \begin{cases} -B & V_{in} < -A \\ \frac{B}{A} V_{in} & -A < V_{in} < A \\ B & V_{in} > A \end{cases}$$

limiter.





a) V_{in} = GRV V; mean $\overline{V}$, variance σ_V^2

write $f_{V_{out}}$: $f_{V_0}(v_0) = \begin{cases} \frac{1}{\sqrt{2\pi}\,\sigma_{v_0}} e^{-\frac{(v_0 - \overline{v}_0)^2}{2\sigma_{v_0}^2}} & -B < v_0 < B \\ f_{V_{in}}(\pm A) & v_0 = \pm B \\ 0 & \text{otherwise.} \end{cases}$

$v_0 = +B \rightarrow v_{in} = A \rightarrow f_{V_{in}}(A)$

$v_0 = -B \rightarrow v_{in} = -A \rightarrow f_{V_{in}}(-A)$

b) $E[v_0] = 2.55$