Let's define event $\hat{B}$ of getting a "3" or a "5" $\rightarrow \hat{B} = \{3, 5\}$

Using the <u>axiomatic approach to probability</u>, calculate $Pr(\hat{A} \cup \hat{B})$:

$$Pr(\hat{A} \cup \hat{B}) = Pr(\hat{A}) + Pr(\hat{B}) - Pr(\hat{A} \cap \hat{B})$$

$$Pr(\{1, 3, 5\}) = Pr(\{1, 3\}) + Pr(\{3, 5\}) - Pr(\{3\})$$

$$= \frac{2}{6} + \frac{2}{6} - \frac{1}{6} = \frac{3}{6} = 0.5$$

<u>HW1: (Ch1) 4.5; 6.1; 7.1, 8.4</u> due 2/14/08.

7.1 Binary communication: 0, 1

a) Probability that a received 0 was transmitted as a 0:

Conditional probability: here <u>condition</u> is "<u>received a 0</u>": $Pr(T_0 | R_0)$

$$\begin{cases} T_0 = \text{"transmitted a 0"} \\ R_0 = \text{"received a 0"} \end{cases}$$

Data: $Pr(T_0) = 0.4$; $Pr(T_1) = 0.6 = 1 - Pr(T_0)$
$Pr(R_1 | T_0) = 0.08$
$Pr(R_0 | T_1) = 0.05$

We can write:
$$\begin{cases} Pr(T_0, R_0) = \boxed{Pr(T_0 | R_0)} \cdot Pr(R_0) \\ Pr(T_0, R_0) = Pr(R_0 | T_0) \cdot Pr(T_0) \end{cases}$$

$$Pr(T_0 | R_0) = \frac{Pr(R_0 | T_0) \cdot Pr(T_0)}{Pr(R_0)} = \frac{0.92 \times 0.4}{Pr(R_0)} = \frac{0.368}{0.398} = 0.925$$

$Pr(R_0 | T_0)$: prob. that a 0 is received given a 0 was transmitted: when a 0 is transmitted, we will receive a 0 (correct) or a 1

$$\rightarrow Pr(R_0 | T_0) = 1 - Pr(R_1 | T_0) = 0.92$$

We need to calculate $Pr(R_0)$ = prob. of receiving a 0

$$Pr(R_0) = Pr(R_0, T_0) + Pr(R_0, T_1)$$

Total probability

$$= Pr(R_0 | T_0) . Pr(T_0) + Pr(R_0 | T_1) . Pr(T_1)$$

$$= 0.92 \times 0.4 + 0.05 \times 0.6$$

$$= 0.398$$

b) Probability that a received 1 (condition) was transmitted as 1

$$Pr(T_1 | R_1) = 0.9468$$

c) 0.062

Example 1.6.1 :

Roulette wheel with 37 slots $\begin{cases} \text{"1"} \rightarrow \text{"36"} : \text{red or black alternately} \\ \text{"0" slot green (37}^{th} \text{ slot)} \end{cases}$

Bets $\begin{cases} \text{1) select a number b/w 1 and 36; if wins: pays 35 : 1} \\ \text{2) select two adjacent numbers; if wins: pays 17 : 1} \end{cases}$

either number wins

Event A = "getting number 1"
Event B = "getting number 2"

a) Find $P_r(A)$ and the return of a \$1 bet on number "1"

$$P_r(A) = \frac{1}{37}$$ (physically the roulette can stop at any of its 37 slots)

Return on a dollar bet $= \underbrace{\text{what would we get if winning}}_{(35+1)} \times \underbrace{P_r(A)}_{\frac{1}{37}}$

$= \$0.973$

Return on betting 1000 times: $1000 \times \$0.973 = \$973$

b) Find $P_r(A \cup B)$ and the probable return on a \$1 bet on $A \cup B$ (1 and 2)

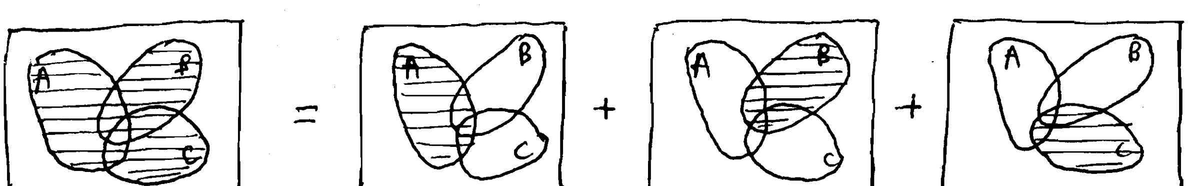
$$P_r(A \cup B) = P_r(\{1, 2\}) = \frac{2}{37}$$

Prob. return on a dollar bet $= \underbrace{\text{what we would get if winning}}_{(17+1)} \times \underbrace{P_r(A \cup B)}_{\frac{2}{37}}$

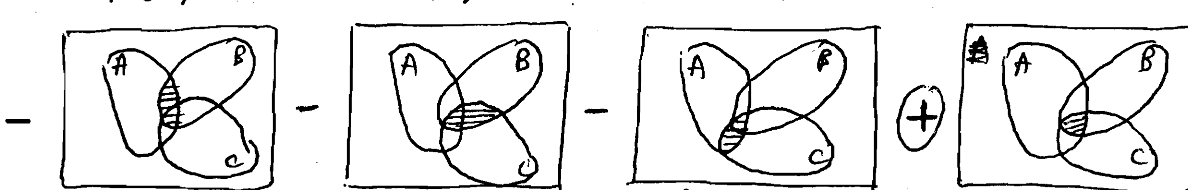
$= \$0.973$

1.6.2: Draw Venn diagram for 3 subsets that are not mutually exclusive, derive:

$$P_r(A \cup B \cup C) = ?$$



Shaded: $A \cup B \cup C$
$P_r(A \cup B \cup C)$ = $P_r(A)$ + $P_r(B)$ + $P_r(C)$



− $P_r(A \cap B)$ − $P_r(B \cap C)$ − $P_r(A \cap C)$ + $P_r(A \cap B \cap C)$

based set theory

Conditional Probability in the Axiomatic Approach:

$$\Pr(A|B) = \frac{\Pr(A,B)}{\Pr(B)} = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$\Pr(A|B)$ → condition

$\Pr(A,B)$ → joint probability

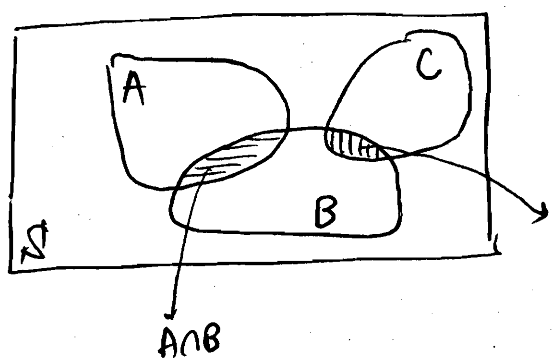
$\Pr(B)$ → marginal probability

3 axioms:

1) $\Pr(A|B) \geq 0$ (non-negative) since it is a ratio of non-neg. probabilities.

2) $\Pr(S|B) = 1$

3) $A \cap C = \phi \rightarrow \Pr(A \cup C | B) = \dfrac{\Pr[(A\cap B) \cup (C \cap B)]}{\Pr(B)}$



C∩B

Since $A \cap C = \phi$ → $(A\cap B)$ and $(C \cap B)$ are disjoint

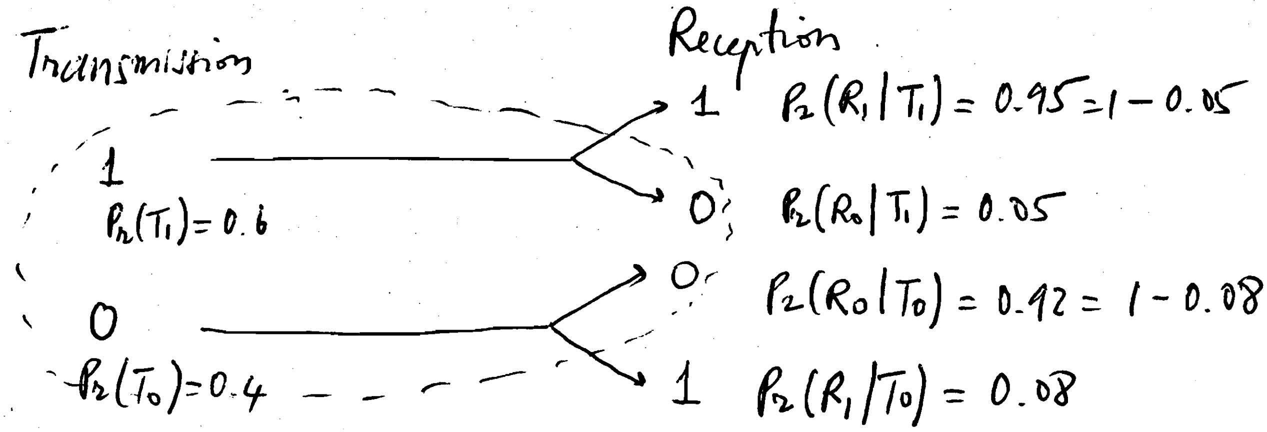
$$= \frac{\Pr(A\cap B)}{\Pr(B)} + \frac{\Pr(C \cap B)}{\Pr(B)}$$

$$= \Pr(A|B) + \Pr(C|B)$$

→ Summary: $\boxed{\Pr(A \cup C|B) = \Pr(A|B) + \Pr(C|B)}$

(3rd axiom for conditional prob.)

7.1



Question a) : What is the probability that a received 0 was transmitted as a 0 : $\boxed{Pr(T_0|R_0)}$

$\neq Pr(R_0|T_0)$

Exercise 1.7-1 (pg 26)

Table 1-3

	1	2	3	4	5	6	Total
10Ω	500	0	200	800	1200	1000	3700
100Ω	300	400	600	200	800	0	2300
1000Ω	200	600	200	600	0	1000	2600
Total	1000	1000	1000	1600	2000	2000	8600

8600 resistors of 3 types in 6 bins

b) $Pr(B_3|R_{10}) = \dfrac{200}{3700}$: given (condition) it's a 10Ω, prob. that it came from bin 3

→ $Pr(R_{10}|B_3) = \dfrac{200}{1000}$: given (condition) it came from bin 3, what is the prob. of getting a 10 Ω

a) → $Pr(R_{1000}|B_4) = \dfrac{600}{1600} = \cancel{0.00375}\ 0.375$

$Pr(B_4|R_{1000}) = \dfrac{600}{2600} = \cancel{0.0012}\ 0.231$

7.1 c) Probability that any symbol (0 or 1) is received in error

$$\to T_0 \; R_1$$
$$\to T_1 \; R_0$$

$$\Pr(R_1 | T_0)\Pr(T_0) + \Pr(R_0 | T_1)\Pr(T_1)$$

$$= 0.08 \times 0.4 + 0.05 \times 0.6$$

$$= 0.062 \text{ or } 6.2\,\%$$

b) $\Pr(T_1 | R_1) = 0.9468$

$$\begin{cases} \Pr(T_1, R_1) = \boxed{\Pr(T_1|R_1)}\Pr(R_1) \\ \quad \shortparallel \\ \Pr(R_1, T_1) = \Pr(R_1|T_1)\Pr(T_1) \end{cases}$$

Bayes' Theorem

$$\to \Pr(T_1 | R_1) = \frac{\Pr(R_1|T_1) \cdot \Pr(T_1)}{\Pr(R_1)} = \frac{0.95 \times 0.6}{\Pr(R_1)}$$

$$\Pr(R_1) = \Pr(R_1, T_0) + \Pr(R_1, T_1)$$

Total prob.

$$= \Pr(R_1|T_0)\Pr(T_0) + \Pr(R_1|T_1)\Pr(T_1)$$

$$= 0.08 \times 0.4 + 0.95 \times 0.6$$

$$= 0.032 + 0.57 = 0.602$$

$$\to \Pr(T_1|R_1) = \frac{0.95 \times 0.6}{0.602} = \frac{0.57}{0.602} = 0.9468$$

8.4 if A is independent of B prove a) A independent of $\bar{B}$
b) $\bar{A}$ independent of $\bar{B}$

Statistical independence :

A & B are stat. independent IFF $\boxed{Pr(A \cap B) = Pr(A) \cdot Pr(B)}$

Consequence:- two events A & B
1) $A \cap B = \phi$ (they are mutually exclusive) → the only way the can be independent is either A or B is ϕ → Two non-empty events that are disjoint can-not be stat. indep.
2) $A \cap B \neq \phi$

8.4 a) To prove A indep. of $\bar{B}$ = $Pr(A \cap \bar{B}) \overset{?}{=} Pr(A) \cdot Pr(\bar{B})$

Starting with the LHS, try to see if we could arrive at the RHS, if yes, A and $\bar{B}$ are independent

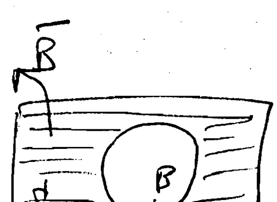
$\bar{B} = S - B$ ($\bar{B}$ is the complement of B)

$$\rightarrow Pr(A \cap \bar{B}) = Pr(A \cap (S - B)) = Pr[A \cap S - A \cap B]$$

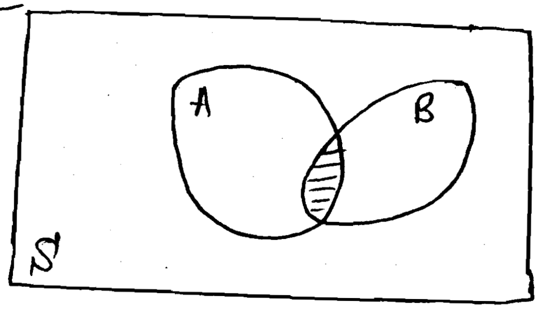
$$= Pr(A) - Pr(A \cap B) = Pr(A) - Pr(A) \cdot Pr(B)$$

A&B independent
$Pr(A \cap B) = Pr(A) Pr(B)$

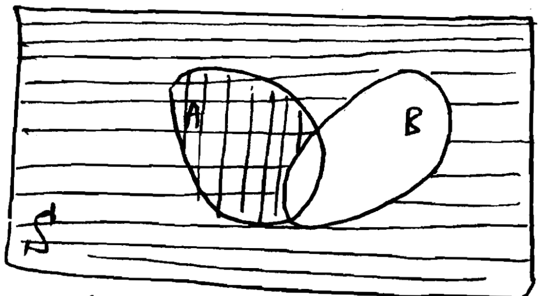
$$= Pr(A) \underbrace{[1 - Pr(B)]}_{Pr(\bar{B})} \rightarrow A \,\&\, \bar{B} \text{ are stat. independent.}$$



Using Venn



$Pr(A \cap B) = Pr(A) Pr(B)$



Shaded: $\bar{B}$
Cross shaded: $A \cap \bar{B}$

$Pr(A \cap \bar{B}) = Pr(A) . Pr(\bar{B})$

4.5 a) 26.7% b) 36.7% c) 20% d) 30%

6.1 a) 50% b) 50% c) 66.7% ; d) 100% ;
e) 83.3% ; f) = 66.7%

8.4 a) A & B independent $\longrightarrow$ A & $\bar{B}$ are independent
b) A & B independent $\longrightarrow$ $\bar{A}$ & $\bar{B}$ are independent

$$\boxed{\underbrace{Pr(\bar{A} \cap \bar{B})}_{LHS} \overset{?}{=} \underbrace{Pr(\bar{A}) Pr(\bar{B})}_{RHS}}$$

$$LHS : Pr(\bar{A} \cap \bar{B}) = Pr((S - A) \cap \bar{B}) = Pr(S \cap \bar{B}) - Pr(A \cap \bar{B})$$
$$= Pr(\bar{B}) - Pr(A) Pr(\bar{B}) = (1 - Pr(A)) Pr(\bar{B})$$
$$= Pr(\bar{A}) Pr(\bar{B}) : RHS$$

Ch 2: Random Variable. Distribution Function. Density Function

→ Random experiment: rolling a die: outcomes are uncertain:
$\{1,2,3,4,5,6\}$: integers ⇒ random variable (discrete)

a box of 10^6 resistors of 100Ω : pick one
→ outcome is uncertain with a range: lots of real numbers → random variable (continuous)

value code → 100Ω; precision code → 1% ; 10%

1% → actual value for R varies (continuously) between $100 - 1\% = 99\Omega$ and $100 + 1\% = 101\Omega$

→ Probability of a random variable:
- Rolling a die : just six values
- Pick a resistor out of 10^6 : → use a probability distribution function (PDF)

PDF: associates a probability to a $\boxed{\text{range}}$ of values of the random variable

- Rolling a die : $\Pr(1) = \frac{1}{6}$; $\Pr(2) = \frac{1}{6}$
- Pick a resistor 10^6 : Not $\Pr(R = 99.1\,\Omega) = $ What is the probability of getting an exact value within $(99\Omega, 101\Omega)$? How many values are there b/w $(99\Omega, 101\Omega)$: ∞ → 0
  But $\Pr(R \le r) =$

$$F(x) \equiv \Pr(X \le x)$$

x ↓ lower case ↓ value; X ↓ upper case ↓ Variable; x ↓ lower case ↓ value

For example: $F(100\Omega) = \Pr(R \le 100\Omega)$
$F(100)$: probability that the random variable R takes a value equal or less than 100Ω

Let's check $F(x)$, a probability distribution function, against the axioms for probability:

1) Is it non-negative? Yes: $\quad 0 \le \Pr(X \le x) \le 1$

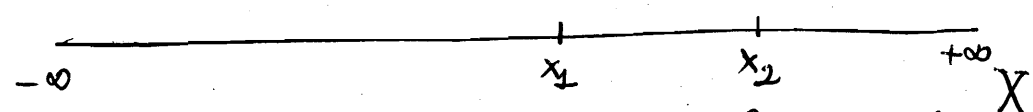
2) Does it satisfy: "Prob. of certain event is 1"? What is the certain event? → It is certain that the random variable will take a value less than ∞.

$$F(\infty) = \Pr(X \le \infty) = 1$$

How do we write "Prob of the impossible event is 0":

$$F(-\infty) = 0$$

3)



Prob. of a range of values is proportional to the length of the range

$$\underbrace{\qquad\qquad\qquad}_{\Pr(X \le x_1)}$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\Pr(X \le x_2)}$$

$$\Pr(x_1 < X \le x_2) = \overset{-\infty \to x_2}{\Pr(X \le x_2)} - \overset{-\infty \to x_1}{\Pr(X \le x_1)}$$

$$= \underbrace{F(x_2) - F(x_1)}$$

x_1 excluded $\qquad$ x_2 included

Discrete random variable:

Random expt = tossing four quarters

Random variable $\bar{X}$ = number of heads $= \{0, 1, 2, 3, 4\}$

↳ Discrete random variable

a) Sketch the PDF for $\bar{X}$:

$$F(0) = P_r(\bar{X} \le 0) = P_r(\bar{X} = 0) = P_r(TTTT) = \frac{1}{16}$$

$$F(1) = P_r(\bar{X} \le 1) = P_r(\bar{X} = 0) + P_r(\bar{X} = 1) = P_r(\text{Four } T\text{'s}) + P_r(\text{One } H) = \frac{1}{16} + \frac{4}{16} = \frac{5}{16}$$

$$F(2) = P_r(\bar{X} \le 2) = P_r(\bar{X} \le 1) + P_r(\bar{X} = 2) = \frac{5}{16} + P_r(\text{Two } H\text{'s}) = \frac{5}{16} + \frac{6}{16} = \frac{11}{16}$$

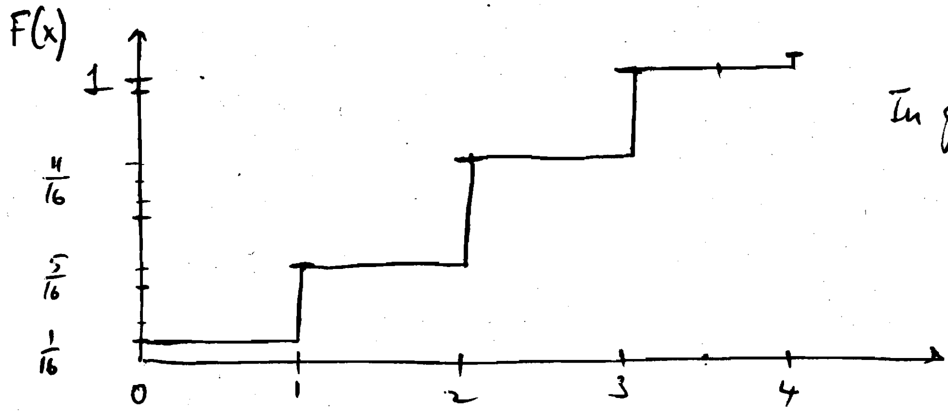
$$F(3) = P_r(\bar{X} \le 3) = P_r(\bar{X} \le 2) + P_r(\bar{X} = 3) = \frac{11}{16} + P_r(\text{Three } H\text{'s}) = \frac{11}{16} + \frac{4}{16} = \frac{15}{16}$$

$$F(4) = P_r(\bar{X} \le 3) + P_r(\bar{X} = 4) = \frac{15}{16} + \frac{1}{16} = 1$$

→ How many possible events for this random expt.?

Combinations of 4 elements when each can be of two types H or T

$$\underset{\text{1st quarter}}{2} \times \underset{\text{2nd}}{2} \times \underset{\text{3rd}}{2} \times \underset{\text{4th}}{2} = 16$$



In general: F(x) is increasing with x

1st	2nd	3rd	4th	
H	H	T	T	6 combinations of Two H's
H	T	H	T	
H	T	T	H	
T	H	H	T	
T	H	T	H	

O	O	O	O	
H	H	H	T	4 combs of three H's (or one T)
H	H	T	H	
H	T	H	H	
T	H	H	H	

b) $F(3.5) = \Pr(X \le 3.5) \underset{\substack{\downarrow \\ \text{discrete random} \\ \text{variable}}}{=} \Pr(X \le 3) = F(3)$

staircase sketch for $F(x)$!

c) $\Pr(X > 2.5) = \Pr(X = 3) + \Pr(X = 4)$

$= \Pr(\text{three H's or one T}) + \Pr(\text{four H's})$

$= \dfrac{4}{16} + \dfrac{1}{16}$

$= \dfrac{5}{16} \underset{\substack{\downarrow \\ \text{coincidence}}}{=} F(1) = \Pr(X \le 1)$

$\Pr(X > 2.5) = \boxed{\Pr(X > 2)} = 1 - \Pr(X \le 2)$

$= \boxed{1 - F(2)}$

$= 1 - \dfrac{11}{16} = \dfrac{5}{16}$

d) $\Pr(0.5 < X \le 3) \underset{\substack{\downarrow \\ \text{3rd Axiom}}}{=} F(3) - F(0.5) = F(3) - F(0) \underset{\substack{\downarrow \\ \text{staircase} \\ \text{sketch.}}}{=} \dfrac{15}{16} - \dfrac{1}{16} = \dfrac{14}{16}$

Continuous Random Variable:

X, prob dist. function $F_X(x) = \begin{cases} 0 & -\infty < x \le 0 \\ 1-e^{-2x} & 0 \le x < \infty \end{cases}$

X ↓ upper case, x ↓ lower case

Reminder: $F_X(x) \equiv Pr(X \le x)$

1) $Pr(X > 0.5) = 1 - Pr(X \le 0.5) = 1 - F_X(0.5)$

$= 1 - (1 - e^{-1}) = \frac{1}{e} = 36.63\%$

2) $Pr(X \le 0.25) = F_X(0.25) = 1 - e^{-0.5} = 39.36\%$

3) $Pr(0.3 < X \le 0.7) = F_X(0.7) - F_X(0.3) = e^{-0.6} - e^{-1.4} = 30.22\%$

(2-2.3)

$$F_X(x) = \begin{cases} A\left[1 - e^{-(x-1)}\right]; & 1 < x < \infty \\ 0 & -\infty < x \le 1 \end{cases}$$

a) Find A such that $F_X(x)$ is a valid probability:

↓ satisfy the 3 axioms of probability

↓

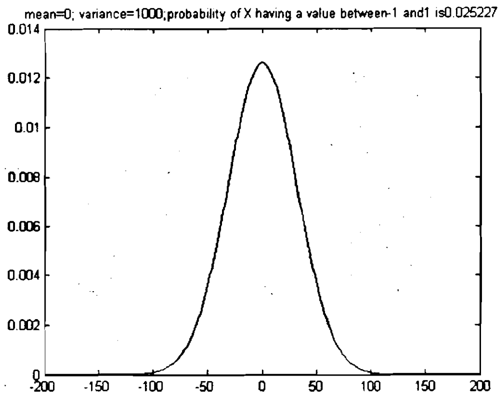
Axiom # 2:

Probab. of certain event is 1

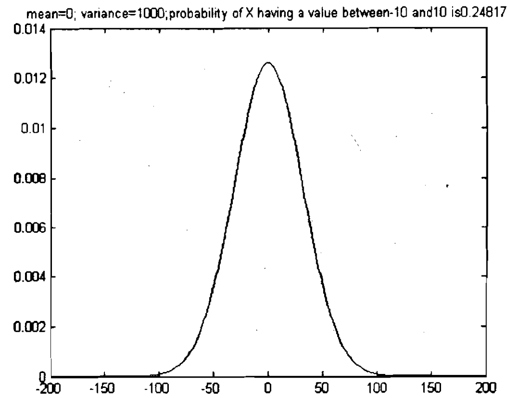
or $\boxed{F_X(\infty) = 1}$

$1 = F_X(\infty) = A\left[1 - \underbrace{e^{-(\infty-1)}}_{0}\right] = A \Rightarrow \boxed{A=1}$ (We used an axiom to get info on a PDF)

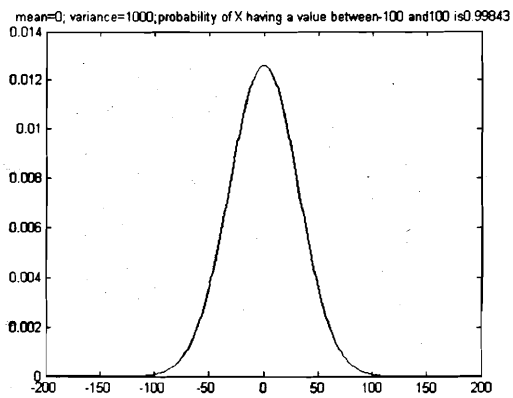
b) $Pr(X \le 2) = F_X(2) = 1\left[1 - e^{-1}\right] = 1 - \frac{1}{e} = 63.2\%$



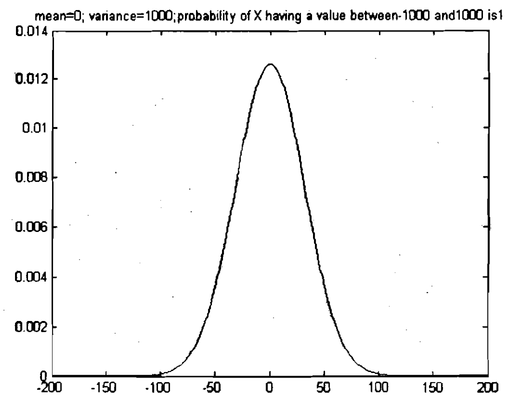
x1=-1; x2=1



x1=-10; x2=10



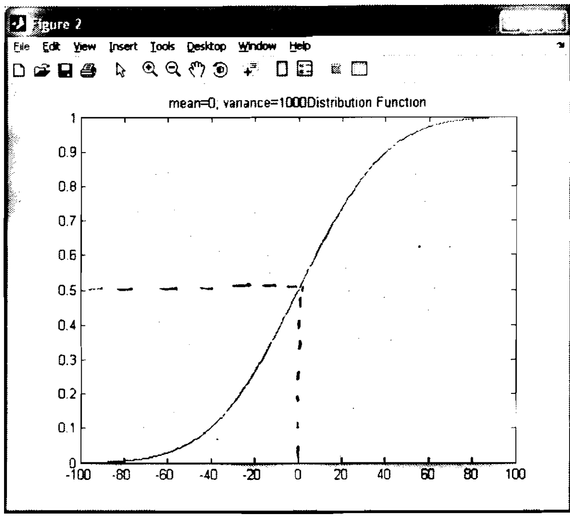
x1=-100; x2=100

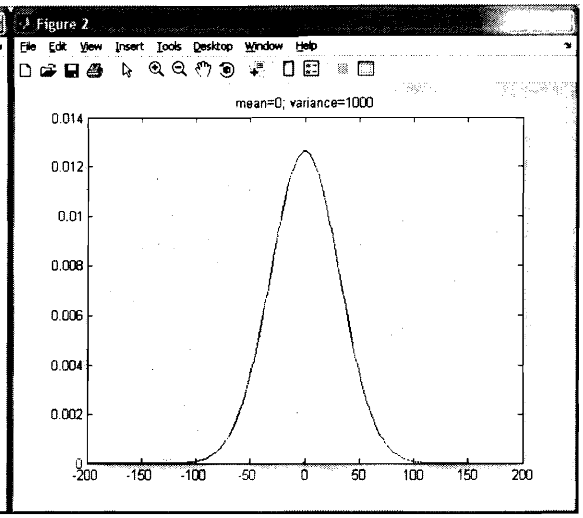


x1=-1000; x2=1000

Before the next class; run the code to get Fdist values for x1=-10000; x2=0; 1; 10; 100; 500; 1000; 2000; 10000

```
% Plotting a Gaussian Density Function
% Providing the probability that X takes a value between x1 & x2
% Feb 29, 2008
% Engin 322
clear all
close all
syms fx Fdist x;
var=1000; %variance
sigma=sqrt(var); % standard deviation
xbar=0; % mean value of the random variable
%Gaussian Density Function
fx=1/(sqrt(2*pi)*sigma)*exp(-(x-xbar)^2/(2*var));
%Interval
x1=-100;
x2=100;
%Distribution Function: by integrating the Density Function
Fdist=eval(int(fx,x,x1,x2))
%Plotting the Density Function
clear fx x;
x=-200:200;
fx=1/(sqrt(2*pi)*sigma)*exp(-(x-xbar).^2/(2*var));
plot(x,fx), title(strcat('mean=',num2str(xbar),';
variance=',num2str(var),';probability of X having a value between
',num2str(x1),' and ', num2str(x2), ' is ', num2str(Fdist)))
```






```
% Plotting a Gaussian Density Function
% Plotting the Distribution Function
% Feb 21, 2008
% Engin 322
```

Note: $F_X(0) = 0.5$

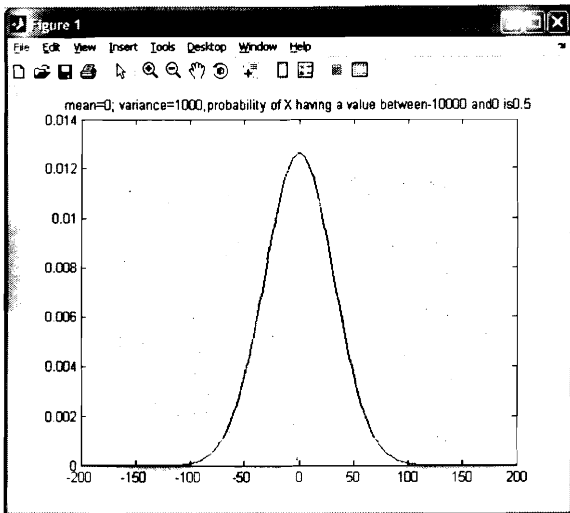
What if $X = 20$ instead of 0?

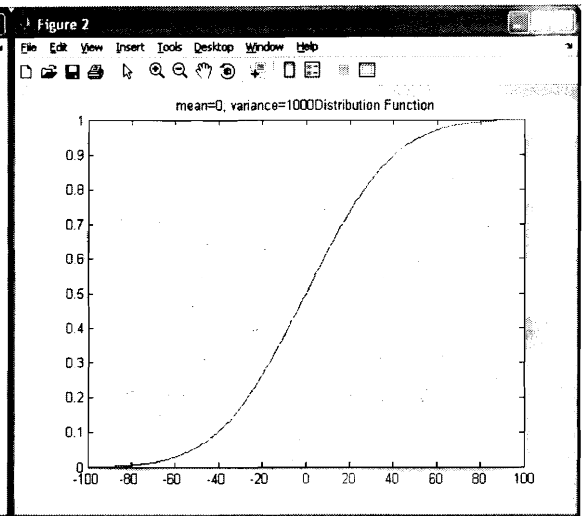
↳ $F_X(0)$ larger or smaller than 0.5?

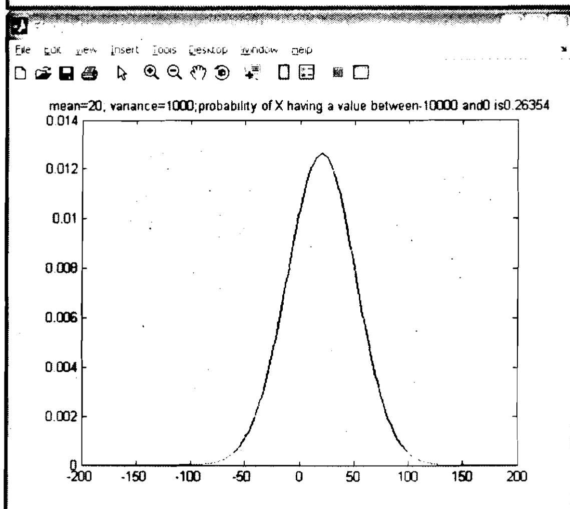
```
clear all
close all
syms fx Fdist x;
var=1000; %variance
sigma=sqrt(var); % standard deviation
xbar=0; % mean value of the random variable

%Gaussian Density Function
fx=1/(sqrt(2*pi)*sigma)*exp(-(x-xbar)^2/(2*var));
%Interval
x1=-10000;
x2=-100:100;

%Distribution Function: by integrating the Density Function
for j=1:201;
Fd(j)=eval(int(fx,x,x1,j-101));
end
%Plotting the Distribution Function
figure(2)
plot(x2,Fd), title(strcat('mean=',num2str(xbar),';
variance=',num2str(var),'Distribution Function'))
```

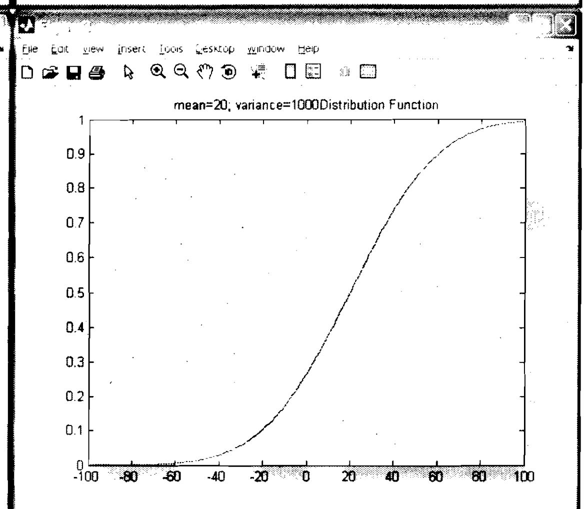
Density Function

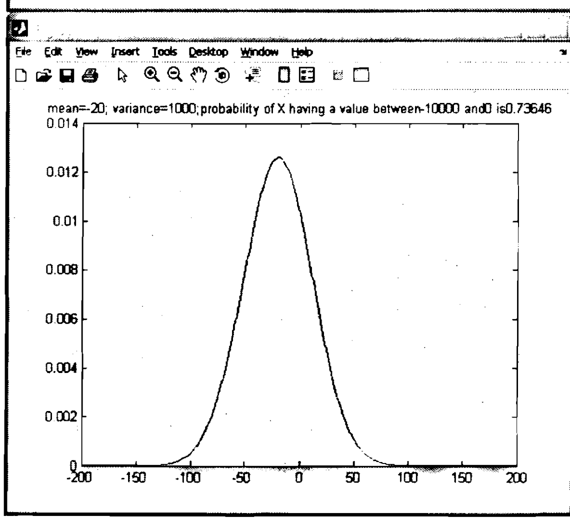
Distribution Function

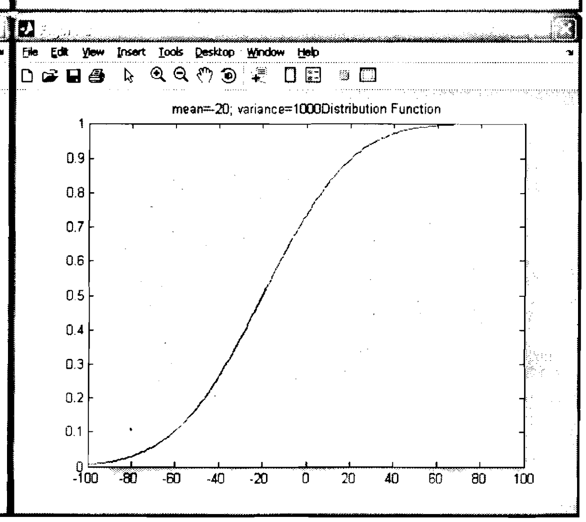












Connection between two Random Variables X & Y:

Linear connection: $Y = AX$

X & Y are two random variables, just related by a linear equation: which would not affect their statistical independence (probability)

↳ Prob. for X taking values in an infinitesimal interval is the same as the prob. for Y taking values in another infinitesimal interval.

$$P_r(x < X \leq x+dx) = P_r(y < Y < y+dy)$$

$$\downarrow \text{3rd axiom.} \qquad\qquad\qquad \downarrow$$

$$\underbrace{F_X(x+dx) - F_X(x)}_{dF_X} \qquad\qquad \underbrace{F_Y(y+dy) - F_Y(y)}_{dF_Y}$$

$$dF_X = \underbrace{\frac{dF_X}{dx}}_{f_X(x)} dx \qquad\qquad \underbrace{\frac{dF_Y}{dy}}_{f_Y(y)} dy$$

$$\rightarrow \quad f_Y(y) = f_X(x) \frac{dx}{dy}$$

↓ Prob. is non-negative:

$$\boxed{f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|}$$

Example: $Y = 2X \rightarrow f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right| = f_X\left(\frac{y}{2}\right) \cdot \frac{1}{2}$

Cubic Connection: $Y = X^3$ $\begin{cases} X: \text{ side of a cube} \\ Y: \text{ volume of the cube} \end{cases}$

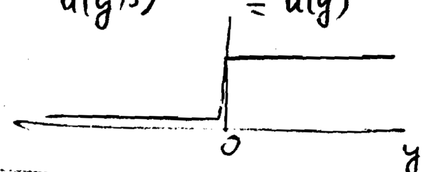
Example: $f_X(x) = e^{-x} u(x)$

$$\boxed{f_Y(y) \overset{?}{=} f_X\left(y^{\frac{1}{3}}\right) \left|\frac{dx}{dy}\right|}$$

$dy = 3x^2 dx \rightarrow \boxed{\frac{dx}{dy} = \frac{1}{3x^2} = \frac{1}{3y^{2/3}}}$

$$= e^{-y^{\frac{1}{3}}} \underbrace{u\left(y^{\frac{1}{3}}\right)}_{\downarrow} \frac{1}{3y^{2/3}} = \boxed{e^{-y^{\frac{1}{3}}} \frac{1}{3y^{2/3}} u(y)}$$

$u(y^{1/3}) = u(y)$



Quadratic connection: $Y = X^2$ $\begin{cases} X: \text{ side of a square} \\ Y: \text{ area of the square.} \end{cases}$

HW2: 2.3; 3.4; 4.4; 5.2 (Ch 2) due next Thursday.

$x = \pm\sqrt{y} \rightarrow \frac{dx}{dy} = \pm \frac{1}{2\sqrt{y}}$: we need to take into consideration both signs:

$$f_Y(y) = f_X(+\sqrt{y}) \left|\frac{1}{+2\sqrt{y}}\right| + f_X(-\sqrt{y}) \left|\frac{1}{-2\sqrt{y}}\right|$$

$$= \frac{1}{2\sqrt{y}} \left[f_X(+\sqrt{y}) + f_X(-\sqrt{y}) \right]$$

3.4 $\quad Y = 3X - 4 \quad ; \quad f_X(x) = e^{-2|x|} \quad ; \quad -\infty < x < \infty$

$$\hookrightarrow f_Y(y) = ? \quad = f_X\left(\frac{y+4}{3}\right) \left|\frac{dx}{dy}\right|$$

$$= \frac{1}{3} e^{-2\left|\frac{y+4}{3}\right|}$$

This is the density function for Y

a) $\Pr(Y<0) = ? \quad F_Y(0) \quad = \int_{-\infty}^{0} f_Y(\tilde{y})\,d\tilde{y}$

$\downarrow$ Dist. Function for Y

$$= \frac{1}{3}\int_{-\infty}^{0} e^{-2\left|\frac{\tilde{y}+4}{3}\right|}\,d\tilde{y} = \frac{1}{3}\int_{-\infty}^{0} e^{-\frac{2}{3}|\tilde{y}+4|}\,d\tilde{y}$$

(In general: $F_Y(y) = \int_{-\infty}^{y} f_Y(\tilde{y})\,d\tilde{y}$)

For any absolute value : $\rightarrow$ there are 2 cases

$$|\tilde{y}+4| = \begin{cases} \tilde{y}+4 & \text{if } \tilde{y}+4 \geq 0 \text{ or } \tilde{y} \geq -4 \\ -\tilde{y}-4 & \text{if } \tilde{y}+4 < 0 \text{ or } \tilde{y} < -4 \end{cases}$$

$$\rightarrow \Pr(Y<0) = \frac{1}{3}\int_{-\infty}^{-4} e^{\frac{2}{3}(\tilde{y}+4)}\,d\tilde{y} + \frac{1}{3}\int_{-4}^{0} e^{-\frac{2}{3}(\tilde{y}+4)}\,d\tilde{y}$$

$$= \frac{1}{3}\left[\frac{e^{\frac{2}{3}(\tilde{y}+4)}}{\frac{2}{3}}\right]_{-\infty}^{-4} + \frac{1}{3}\left[\frac{e^{-\frac{2}{3}(\tilde{y}+4)}}{-\frac{2}{3}}\right]_{-4}^{0}$$

$$= \frac{1}{2}\left\{1 - 0 \quad - \quad \left(e^{-\frac{8}{3}} - 1\right)\right\} = \frac{1}{2}\left\{2 - e^{-\frac{8}{3}}\right\}$$

$$= 0.965$$

(This makes sense since ~~mostly~~ X will be mostly around $0 \to 1$)

b) $\Pr(Y > X) = \Pr(3X - 4 > X) = \cdots$

$$= \frac{1}{2}\left\{ 1 - 0 + \left(e^{-\frac{8}{3}} - 1 \right) \right\} = \frac{1}{2}\left\{ 2 - e^{-\frac{8}{3}} \right\} = 0.965$$

b) $Pr(Y > X) =$

Mean values and moments of Random Variables.

Mean value = also a moment of 1st order

$$\overline{X} \equiv \underbrace{\int_{-\infty}^{\infty} dx \; x f_{X}(x)} = \underbrace{E[x]}$$

$E[x]$: "ensemble average"

$\int_{-\infty}^{\infty} dx \; x f_{X}(x)$: more general version of the wellknown average:

$$\frac{x_1 + x_2 + \cdots x_N}{N} = x_1 \underbrace{\frac{1}{N}} + x_2 \frac{1}{N} + \cdots x_N \frac{1}{N}$$

$\frac{1}{N}$ is now replaced by $f_{X}(x)$

$$\overline{X^n} \equiv \int_{-\infty}^{\infty} dx \; x^n f_{X}(x) \quad : \quad \text{moment of order } n$$

$$\overline{(X - \overline{X})^n} \equiv \int_{-\infty}^{\infty} dx \; (x - \overline{x})^n f_{X}(x) \quad : \quad \text{central moment of order } n$$

central → ("centered about the mean")

$$\overline{(X - \overline{X})^2} = \quad \text{1) central moment of order 2 : or Variance}$$

$$\text{2) } \underbrace{\overline{X^2}}_{\text{moment of order 2}} - \underbrace{\overline{X}^2}_{\text{mean squared}}$$

Observation: averaging is a linear operation $\Rightarrow$ the average of a sum is the sum of the averages:

$$\overline{(X-\overline{X})^2} = \overline{(X^2 - 2X\overline{X} + \overline{X}^2)} = \overline{X^2} \underbrace{- 2\overline{X}^2 + \overline{X}^2} = \overline{X^2} - \overline{X}^2 \checkmark$$

→ Average of X is $\overline{X}$ (a number)

→ Average of $2X\overline{X}$ is $2\overline{X}\,\overline{X} = 2(\overline{X})^2$

→ Average of $\overline{X}$ is $\overline{X}$:

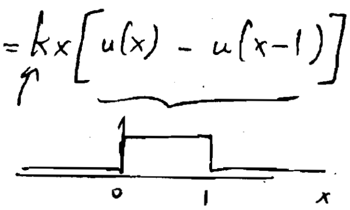
$$\hookrightarrow \int_{-\infty}^{\infty} dx\, \overline{X} f_X(x) = \overline{X} \underbrace{\int_{-\infty}^{\infty} dx\, f_X(x)}_{1 = F(\infty)}$$

→ Note that $\overline{X^2} = \int_{-\infty}^{\infty} dx\, x^2 f_X(x)$ is different than $\overline{X}^2 = \left[\int_{-\infty}^{\infty} dx\, x f_X(x)\right]^2$

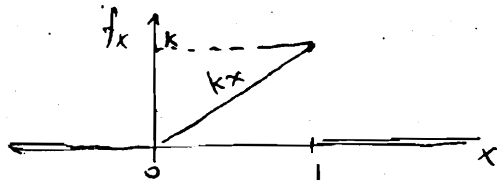
→ We have used "averaging algebra" to prove

$$\overline{(X-\overline{X})^2} = \overline{X^2} - \overline{X}^2$$

Example: X random variable with $f_X(x) = k x \underbrace{[u(x) - u(x-1)]}$



a) What is k?



$$F(\infty) = 1 = \int_{-\infty}^{\infty} dx\, f_X(x) = k \int_0^1 dx\, x = k\left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{k}{2} \rightarrow k = 2$$

are numbers by definition

Mean Values and Moments of Random Variable:

- Mean value is a moment of 1st order:

$$\overline{X} \equiv \underbrace{\int_{-\infty}^{\infty} dx \; x^1 f_X(x)}_{\text{Moment of 1st order}} = E[x]$$

↳ "ensemble average"

Probability density function is included in definition of moments

Mean value & arithmetic average:

↳ $\frac{x_1 + x_2 + x_3 + \cdots x_N}{N} = x_1 \frac{1}{N} + x_2 \frac{1}{N} + \cdots + x_N \frac{1}{N}$

arithmetic average is a moment of 1st order when the density function is $\frac{1}{N}$ (a uniform distribution).

- Moment of order n = $\overline{X^n} \equiv \int_{-\infty}^{\infty} dx \; x^n f_X(x)$

- Central moment of order n: (wrt the mean value):

$$\overline{(X-\overline{X})^n} = \int_{-\infty}^{\infty} dx \; (x-\overline{x})^n f_X(x)$$

- Central moment of order 2 = variance

$$\overline{(X-\overline{X})^2} = \text{variance} = \underbrace{\overline{X^2}}_{\substack{\text{moment}\\ \text{of 2nd}\\ \text{order}}} - \underbrace{\overline{X}^2}_{\substack{\text{mean}\\ \text{squared}}}$$

?

$$\overline{(X-\overline{X})^2} \underset{\substack{\downarrow \\ \text{expanded} \\ \text{binomial}}}{=} \overline{(X^2 - 2X\overline{X} + \overline{X}^2)} \underset{\substack{\downarrow \\ \text{the mean value} \\ \text{is a linear operation}}}{=} \overline{X^2} - 2\,\overline{X\overline{X}} + \overline{\overline{X}^2}$$

$$\overline{X_1 + X_2} = \int_{-\infty}^{\infty} dx\,(x_1 + x_2)\, f_X(x)$$

$$= \int_{-\infty}^{\infty} dx\, x_1 f_X(x) + \int_{-\infty}^{\infty} dx\, x_2\, f_X(x)$$

$$= \overline{X_1} + \overline{X_2}$$

I can apply the average to each of the 3 terms

$$* \quad \overline{X\overline{X}} = \overline{X}\,\overline{X} = \overline{X}^2$$

$\downarrow$ variable, $\downarrow$ Number

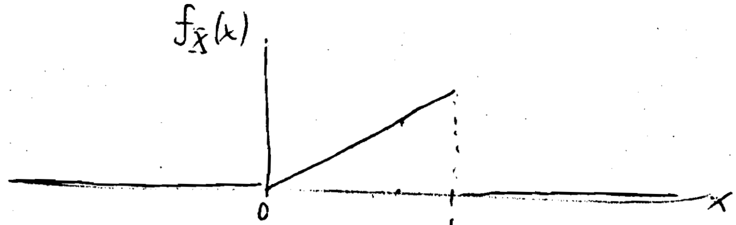
$$* \quad \overline{\overline{X}^2} = \overline{X}^2 \qquad (\text{for example } \overline{3} = 3)$$

number

$$\Rightarrow \boxed{\overline{(X-\overline{X})^2} = \overline{X^2} - 2\overline{X}^2 + \overline{X}^2 = \overline{X^2} - \overline{X}^2} \rightarrow \text{Averaging algebra}$$

<u>Example:</u> X a random variable ; $f_X(x) = kx\,[\underbrace{u(x) - u(x-1)}]$

This says: the Density Function for X is linear b/w 0 & 1 and 0 otherwise



a) what is k? (the slope of $f_X(x)$) $\rightarrow$ Use 2^{nd} Axiom: $F(\infty) = 1$

$$1 = F(\infty) = \int_{-\infty}^{\infty} dx\, f_X(x) = \int_0^1 dx\, kx = k\int_0^1 dx\, x = k\left[\frac{x^2}{2}\right]_0^1 = \frac{k}{2}$$

$$\rightarrow \boxed{k=2}$$

b) What is the mean value for X?

$$\overline{X} = \int_{-\infty}^{\infty} dx\, x f_X(x) = 2\int_0^1 dx\, x\, x = 2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3}$$

c) What is the moment of order 2 for X?

$$\overline{X^2} = 2\int_0^1 dx\, x^2 x = 2\left[\frac{x^4}{4}\right]_0^1 = \frac{1}{2}$$

d) What is the variance for X?

$$\text{Variance}(X) = \overline{(X-\overline{X})^2} = \overline{X^2} - \overline{X}^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2$$

$$= \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

e) What is the central moment of order 4?

$$\overline{(X-\overline{X})^4} = \int_0^1 dx\, \underbrace{(x-\overline{X})^4}_{\text{expand the binomial}} 2x = 2\int_0^1 dx\, x^5 - 8\overline{X}\int_0^1 dx\, x^4 + 12\overline{X}^2\int_0^1 dx\, x^3$$

$$-8\overline{X}^3\int_0^1 dx\, x^2 + 2\overline{X}^4\int_0^1 dx\, x$$

$$= 2\frac{1}{6} - 8\frac{2}{3}\frac{1}{5} + 12\frac{4}{9}\frac{1}{4} - 8\frac{8}{27}\frac{1}{3} + 2\frac{16}{81}\frac{1}{2}$$

$$= 0.0074$$

Triangle of Tartaglia

1

1 1

1 2 1 → 2nd order

1 3 3 1 → 3rd order

1 4 6 4 1 → 4th order

1 5 10 10 5 1

$$\rightarrow (x-\overline{x})^4 = x^4 - 4x^3\overline{x} + 6x^2\overline{x}^2 - 4x\overline{x}^3 + \overline{x}^4$$

Gaussian Random Variable

$\downarrow$

X

Prob. Density Function : $f_X(x) \equiv \dfrac{1}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{(x-\bar{X})^2}{2\sigma_X^2}}$, $(-\infty < x < \infty)$

Notation : σ_X = standard deviation = $\sqrt{\text{variance}}$

σ_X^2 = variance of X

$\bar{X}$ = mean value or moment of order 1 of X.

$\rightarrow$ Given 2 numbers : mean value, and central moment of order 2 or variance

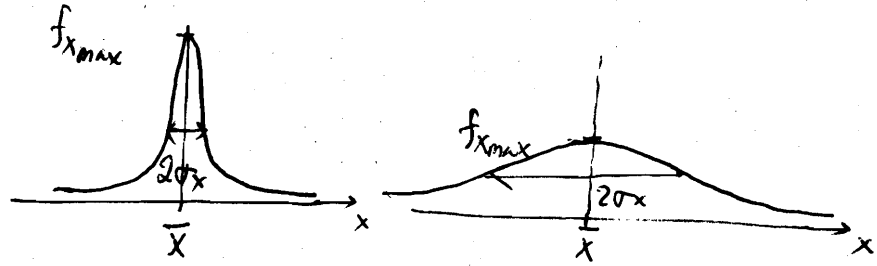
$\hookrightarrow$ a Gaussian Density Function is determined uniquely.

Properties:

1) $f_X(x)$ is maximum when $X = \bar{X}$ $\rightarrow$ $f_{X_{max}}(x) = \dfrac{1}{\sqrt{2\pi}\,\underbrace{\sigma_X}_{\text{width}}}$

$\rightarrow$ Max & width are related :
wider distribution $\rightarrow$ lower max.

width : larger σ_X $\rightarrow$ wider distribution



2) GRV very good to model the noise
3) if X & Y are GRV's $\rightarrow$ $aX + bY$ is also a GRV
4) The one with complete statistical analysis results
5) Distribution Function : $F(x) \equiv \Pr(X < x) = \int_{-\infty}^{x} du\, f_X(u)$

$$= \int_{-\infty}^{x} du \frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{(u-\bar{X})^2}{2\sigma_X^2}} \equiv \Phi\left(\frac{x-\bar{X}}{\sigma_X}\right)$$

$$\boxed{\int_0^\infty dx\, x^n e^{-r^2x^2} = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2r^{n+1}}}$$

Gamma function $\Gamma(\alpha) = \begin{cases} \Gamma(\frac{1}{2}) = \sqrt{\pi} \\ \Gamma(1) = 1 \\ \Gamma(\alpha+1) = \alpha\Gamma(\alpha) \end{cases}$

$\Gamma(2) = 1 = 1\,\Gamma(1)$; $\Gamma(3) = 2$; $\Gamma(4) = 6$; $\Gamma(5) = 24$; ...

$F(x) = \underset{\text{"Phi"}}{\Phi}\left(\frac{x-\bar{X}}{\sigma_x}\right)$;
$\begin{cases} \Phi(x) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{x} e^{-\frac{u^2}{2}}\,du \\ Q(x) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{x}^{\infty} e^{-\frac{u^2}{2}}\,du \end{cases}$

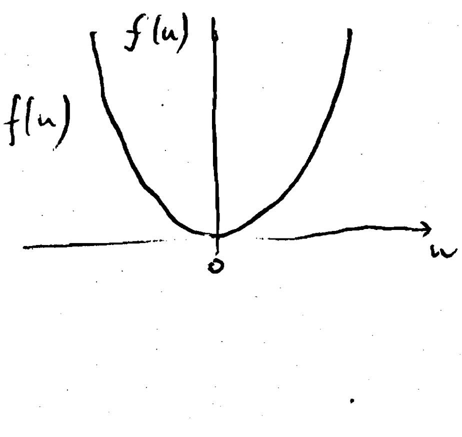
$\to 1 = \Phi(\infty) = \Phi(x) + Q(x)$

$$\Phi(\infty) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underbrace{e^{-\frac{u^2}{2}}}_{\text{w.r.t } u}\,du = \frac{2}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{u^2}{2}}\,du \overset{\substack{n=0 \\ r^2=\frac{1}{2}}}{=} \frac{\cancel{2}}{\sqrt{2\pi}}\frac{\Gamma(\frac{1}{2})}{\cancel{2}\sqrt{\frac{1}{2}}} = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$$

this is an even function

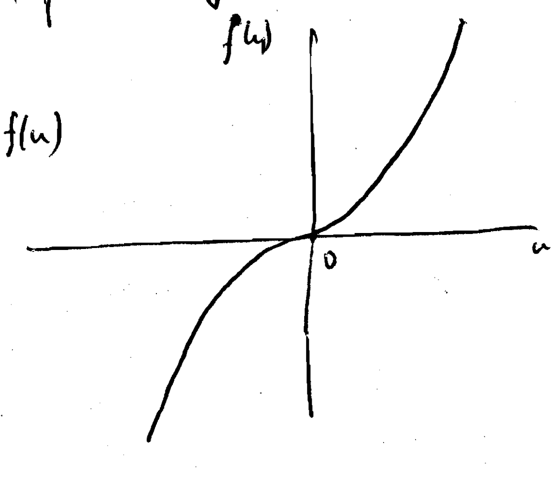
Symmetric
Even function of u.

$f(-u) = f(u)$



Antisymmetric
Odd function of u :

$f(-u) = -f(u)$



In summary: $1 = \Phi(x) + Q(x) \rightarrow \Phi(x) = 1 - Q(x)$

HW2: Answers

2.3	a) 1	b) 0.6321	c) 0.3679	d) 0.8647
3.4	a) $f_Y(y) = \frac{1}{3} e^{-2\left\|\frac{y+4}{3}\right\|}$;	b) 0.9653	c) 9.158×10^{-3}	
4.4	a) 12.5 ;	b) 93.75	c) 0.00977	
5.2	a) 1875	b) 26875	c) 0	d) 1750 (if $\bar{X}=10$; $\sigma_X^2 = 25$)
	a) 768	b) 3793	c) 0	d) 365 (if $\bar{X}=5$; $\sigma_X^2 = 16$)

$\sigma_X^2 = \mathrm{var}(X)$

3.4 c)

$$P_r(Y > X) = P_r(3X - 4 > X) = P_r(X > 2)$$

$$= 1 - \underbrace{P_r(X \le 2)}_{F_X(2)} = 1 - \int_{-\infty}^{2} dx\, e^{-2|x|}$$

$$= \int_2^{\infty} dx\, e^{-2|x|} = \int_2^{\infty} dx\, e^{-2x}$$

in $(2, \infty)$: $|x| = x$

$$= \left[\frac{e^{-2x}}{-2}\right]_2^{\infty} = \frac{0 - e^{-4}}{-2} = \frac{e^{-4}}{2} = 0.00915$$

or $$P_r\left(Y > \frac{Y+4}{3}\right) = P_r(3Y > Y + 4) = P_r(Y > 2)$$

$$= \int_2^{\infty} dy\, \frac{1}{3} e^{-2\left|\frac{y+4}{3}\right|} = \frac{1}{3}\int_2^{\infty} dy\, e^{-\frac{2}{3}(y+4)}$$

$(2, \infty) \rightarrow \left|\frac{y+4}{3}\right| = \frac{y+4}{3}$

$$= \frac{1}{3}\left[\frac{e^{-\frac{2}{3}(y+4)}}{-\frac{2}{3}}\right]_2^{\infty} = -\frac{1}{2}\left[0 - e^{-4}\right] = 0.00915$$

5.2 Given a Gaussian density function, specified by mean value $\bar{X}$ and variance σ_X^2, in principle a central moment is easier to calculate than a moment, due to the central nature of the Gaussian function. For example:

$$\overline{(X-\bar{X})^3} = \frac{1}{\sqrt{2\pi}\,\sigma_X}\int_{-\infty}^{\infty} dx\,(X-\bar{X})^3 e^{-\frac{(X-\bar{X})^2}{2\sigma_X^2}} = \frac{1}{\sqrt{2\pi}\,\sigma_X}\int_{-\infty}^{\infty} dy\, y^3 e^{-\frac{y^2}{2\sigma_X^2}}$$

change of variable
$y \equiv X-\bar{X}$

$= 0$ (Any central moment of odd order is zero)

y^3 is an odd function

$$\overline{(X-\bar{X})^4} = \frac{1}{\sqrt{2\pi}\,\sigma_X}\int_{-\infty}^{\infty} dx\,(X-\bar{X})^4 e^{-\frac{(X-\bar{X})^2}{2\sigma_X^2}} = \frac{1}{\sqrt{2\pi}\,\sigma_X}\int_{-\infty}^{\infty} dy\, y^4 e^{-\frac{y^2}{2\sigma_X^2}}$$

$$= \frac{2}{\sqrt{2\pi}\,\sigma_X}\,\frac{\Gamma\left(\frac{5}{2}\right)}{2\left(\frac{1}{\sqrt{2}\,\sigma_X}\right)^5} = 3\sigma_X^4$$

$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\,\frac{1}{2}\sqrt{\pi}$

$$\overline{(X-\bar{X})^6} = \frac{2}{\sqrt{2\pi}\,\sigma_X}\,\frac{\Gamma\left(\frac{7}{2}\right)}{2\left(\frac{1}{\sqrt{2}\,\sigma_X}\right)^7} = 15\sigma_X^6$$

$\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\,\frac{3}{2}\,\frac{1}{2}\sqrt{\pi}$

Corollary:

$$\overline{(X-\bar{X})^2} = \sigma_X^2$$
$$\overline{(X-\bar{X})^4} = 3\sigma_X^4 \rightarrow \left\{ \overline{(X-\bar{X})^n} = \begin{cases} 1\cdot 3\cdot 5\cdots(n-1)\,\sigma_X^n & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \right.$$
$$\overline{(X-\bar{X})^6} = 15\sigma_X^6$$

After a change of variable, moments can be calculated in several integrals. Moments can be also calculated using the "averaging algebra".

$$\sigma_X^2 = \overline{(X-\bar{X})^2} = \overline{X^2} - \bar{X}^2$$

Moments using averaging algebra:

i) $\overline{X^3} = 3\sigma_X^2 \bar{X} + \bar{X}^3$

Proof: $\overline{(X-\bar{X})^3} = \overline{X^3} - 3\overline{X^2}\,\bar{X} + 3\bar{X}\bar{X}^2 - \bar{X}^3 = 0$

$= \overline{X^3} - 3(\underbrace{\overline{X^2} - \bar{X}^2}_{\sigma_X^2})\bar{X} - \bar{X}^3 = 0$

$\Rightarrow \overline{X^3} = 3\sigma_X^2 \bar{X} + \bar{X}^3$

ii) $\overline{X^4} = 3\sigma_X^4 + 6\sigma_X^2 \bar{X}^2 + \bar{X}^4$

Proof: $\overline{(X-\bar{X})^4} = \overline{X^4} - 4\overline{X^3}\bar{X} + 6\overline{X^2}\,\bar{X}^2 \underbrace{- 4\bar{X}^4 + \bar{X}^4}_{-3\bar{X}^4}$

$= \overline{X^4} - 12\sigma_X^2\bar{X}^2 - 4\bar{X}^4 + 6\overline{X^2}\,\bar{X}^2 - 3\bar{X}^4$

$= \overline{X^4} - 12\sigma_X^2\bar{X}^2 + 6\overline{X^2}\,\bar{X}^2 - 7\bar{X}^4$

$= \overline{X^4} - 12\sigma_X^2\bar{X}^2 + 6(\underbrace{\overline{X^2} - \bar{X}^2}_{\sigma_X^2})\bar{X}^2 - \bar{X}^4$

$= \overline{X^4} - 6\sigma_X^2\bar{X}^2 - \bar{X}^4$

$\Rightarrow \overline{X^4} = 3\sigma_X^4 + 6\sigma_X^2\bar{X}^2 + \bar{X}^4$

Using these results:

GRV: $\bar{X} = 5$; $\sigma_X^2 = 16$

5.2 a) $\overline{(X-\bar{X})^4} = 3\sigma_X^4 = 3\times16^2 = 768$

b) $\overline{X^4} = 3\sigma_X^4 + 6\sigma_X^2\bar{X}^2 + \bar{X}^4 = 768 + 6\times16\times5^2 + 5^4 = 3793$

c) $\overline{(X-\bar{X})^3} = 0$

d) $\overline{X^3} = 3\sigma_X^2\bar{X} + \bar{X}^3 = 3\times16\times5 + 5^3 = 365$

Check these results with the Matlab code provided.