# Linear System Theory II (Sp '08)

LSTI:

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

$x$ & $y$ are functions of time

$x(t) = \delta(t) \rightarrow y(t) = h(t)$ $\underbrace{\text{'impulse response'}}_{\text{one per system}}$

How do we get $y(t)$ if $x(t)$ & $h(t)$ are known?

$$y(t) = \underset{\text{"convolution"}}{x(t) * h(t)} \equiv \int_{-\infty}^{\infty} d\tau \; x(\tau)\, h(t-\tau)$$

$$= \int_{-\infty}^{\infty} d\tau \; x(t-\tau)\, h(\tau)$$

$$= h(t) * x(t)$$

"Convolution is commutative"

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LST II:

- → Introduce random signals: noise, described with probability distributions
- → Incorporate into $y(t) = x(t) * h(t)$
  How to calculate properties of the output when the input is a random signal.

Application: noise elimination: e.g. precise target location

# Ch 1: Introduction to Probability

**Random experiment:** example: rolling a dice, outcome is uncertain

(Matlab: use "rand" function)

**Event:** "getting a four"; "getting a three or a four"

**Relative-frequency probability:**

| Roll a dice | # times getting a four | Rel-frequency probability for this event |
|---|---|---|
| 120 | 27 | 27/120 → 22.5 % |
| 1200 | 185 | 15.4 % |
| 12 000 | 1972 | 16.4 % |
| 1 200 000 | 200 000 | $\frac{1}{6}$ → 16.67 % |

Meaningful experiment (Sufficient # of samples) → 1 200 000

$\frac{1}{6} = 0.1667$ : probability (expected theoretical value) of getting one number out of six possibilities.

A = "Event of getting a 3 or a 4" (A composite event)

Rel-freq probability for event A = $Pr(A) = \frac{2}{6} = \frac{1}{3}$

$$Pr(E) = \frac{n}{N} \leq 1$$

n: # times E happens or # of single events in E
N: # all possible single events

Certain event: $\Omega$ will have $Pr(\Omega) = \frac{6}{6} = 1$

↓

In the experiment of rolling a dice, the certain event is that getting either a 1, a 2, a 3, 4, 5, or 6.

$\Omega = \{1, 2, 3, 4, 5, 6\}$ : composed of all possible single events.

Impossible event: $\emptyset$ will have $Pr(\emptyset) = \frac{0}{6} = 0$

↓

In the experiment of rolling a dice, the impossible event is "getting a number that is not between 1 and 6"

MatLab experiment: use attached code, then place data cursor

| # times we roll a dice | # times we get a 4 (*) | $Pr(A)$ |
|---|---|---|
| 120 | 22 | $\frac{22}{120} = 0.1833$ or 18.3% |
| 1 200 | 201 | |
| 12 000 | 2046.. | |
| 1 200 000 | 199700 | |

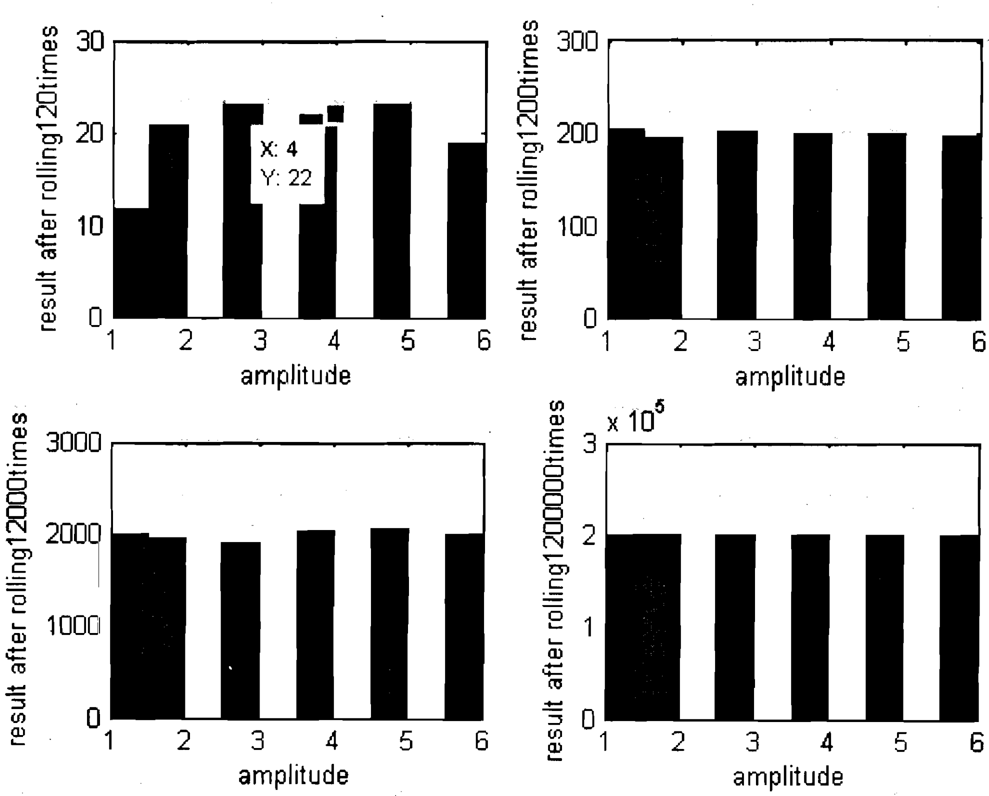
on the upper right corner of each bar that corresponds to $X = 4$ of the four subplots (runs)

(*) The uncertain outcome in a random experiment shows in the different number of times we get a four

| | Scott | Tenzil | Zhuo | Chen |
|---|---|---|---|---|
| 12000 | 2030 | 1970 | 2046 | 2026 |
| 1 200 000 | 200 000 | 200 200 | 199700 | 199700 |

```
% Rolling a dice experiment (a random experiment)
% Jan 29, 2008


nplot=4; %number of plots, one per run
nx=2; % number of subplots
ny=2;
nroll =[120 1200 12000 1200000]; %number of times to roll in each of
the four runs
for i=1:nplot
    subplot(nx,ny,i)
    y1=ones(1,nroll(i)) + floor(6*rand(1,nroll(i)));
    hist(y1)
    axis([1 6 0 nroll(i)/4]);
    xlabel('amplitude')
    ylabel(strcat('result after rolling', num2str(nroll(i)), 'times'))
end
```



## Joint probability:

In a box of 1000 resistors* we have:
* Each resistor is classified by two labels: resistance ($\Omega$) & power (W)

| P \ R | $1\Omega$ | $10\Omega$ | $100\Omega$ | $1000\Omega$ | Total |
|---|---|---|---|---|---|
| 1W | 50 | 300 | 90 | 0 | 440 |
| 2W | 50 | 50 | 0 | 100 | 200 |
| 5W | 0 | 150 | 60 | 150 | 360 |
| Total | 100 | 500 | 150 | 250 | 1000 |

If we cover the box and pick one resistor, this is a random experiment

Event A ≡ "getting a resistor with $R = 10\Omega$"; $P_r(A) = \frac{500}{1000} = 0.5$

Event B ≡ " " " " " $P = 5W$"; $P_r(B) = \frac{360}{1000} = 0.36$

Event A&B = "getting a resistor with $R = 10\Omega$ and $P = 5W$"

$$P_r(A, B) = \frac{150}{1000} = 0.15$$

→ Is there any connection between this joint probability and the single $P_r(A)$ or $P_r(B)$ ?

Conditional probability: of A $\underbrace{\text{given B happend}}_{\text{condition}}$ :

$$P_r(A|\underbrace{B}_{\text{condition}}) = \frac{150}{360} = 0.417$$ Proba. of getting a $10\Omega$ given it is a 5W

$$P_r(B|A) = \frac{150}{500} = 0.3$$ Prob. of getting a 5W given that it is a $10\Omega$

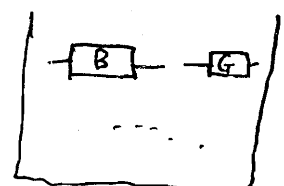
Can we obtain $Pr(A,B)$ from $Pr(B)$ & $Pr(A|B)$ ?

Yes:
$$\boxed{Pr(A,B) = Pr(A|B) \cdot Pr(B)}$$
$$0.15 = 0.417 \times 0.36 \quad \checkmark$$

or
$$\boxed{Pr(A,B) = Pr(B|A) \times Pr(A)}$$
$$\underbrace{0.15}_{\text{joint probability}} = \underbrace{0.3}_{\text{conditional probability}} \times \underbrace{0.5}_{\text{marginal probability}} \quad \checkmark$$

Ex. 1-4.1:

a) Box of 50 diods $\begin{cases} 10 \text{ bad} \\ 40 \text{ good} \end{cases}$




50 diods

A diode is picked at random, event A = "getting a bad diode"

$$Pr(A) = \frac{10}{50} = 0.2$$

b) If first diode drawn is good, what is the prob. the second diode drawn will be good?

$\boxed{\begin{matrix} 10 \text{ bad} \\ 40 \text{ good} \end{matrix}}$ After 1st draw: $\boxed{\begin{matrix} 10 \text{ bad} \\ 39 \text{ good} \end{matrix}}$ $\rightarrow Pr(\text{good}) = \frac{39}{49} = 0.795$ (less than initial prob. of 0.8)

$\hookrightarrow Pr(\text{good}) = \frac{40}{50} = 0.8$

c) If two diodes are drawn from the box what is the probability that they are both good?

| 10 bad |
|---|
| 40 good |

Two approaches:

1) Relative-frequency approach:

$$P_r(\text{getting 2 good diodes}) = \frac{\#\text{ combinations of 2 out of 40 good diodes}}{\#\text{ combinations of 2 out of 50 diodes}}$$

**Combinatorics**: $\binom{n}{a} = \dfrac{n!}{a!(n-a)!}$ = # combinations of a elements out of a universe of n elements

$n! = n(n-1)(n-2)\dots 1$
"n factorial"

$$= \frac{\binom{40}{2}}{\binom{40}{2} + \binom{10}{2} + 40\cdot 10} \quad \begin{matrix} \leftarrow (\text{~~only~~ one type}) \\ \leftarrow (\text{two types of diode}) \end{matrix} = \frac{\frac{40!}{2!\,38!}}{\frac{40!}{2!\,38!} + \frac{10!}{2!\,8!} + 400}$$

$\binom{40}{2}$ ↓ two good; $\binom{10}{2}$ ↓ two bad; $40\cdot 10$ ↓ one good & one bad.

$$= \frac{\frac{40\times 39}{2}}{\frac{40\times 39}{2} + \frac{10\times 9}{2} + 400} = \frac{780}{1225} = \frac{156}{245} = 0.6367$$

2) Joint-probability approach:

Event A = "getting a good diode"

$$Pr(\text{getting two good diodes}) = \underbrace{Pr(A, A)}_{\substack{\text{prob of getting}\\ \text{two good diodes}}} = \underbrace{Pr(A|A)}_{\substack{\text{Prob. of getting a good diode}\\ \text{given the 1st pick was good.}}} \cdot \underbrace{Pr(A)}_{\substack{\text{Prob. of getting}\\ \text{a good one}}}$$

$$\rightarrow Pr(A, A) = \frac{39}{49} \cdot \frac{40}{50} = \frac{39 \times 4}{49 \times 5} = \frac{156}{245} = 0.6367$$

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<u>Review on elementary set theory</u>:

<u>Set</u>: a collection of elements ; $A = \{\alpha_1, \alpha_2, \dots \alpha_n\}$

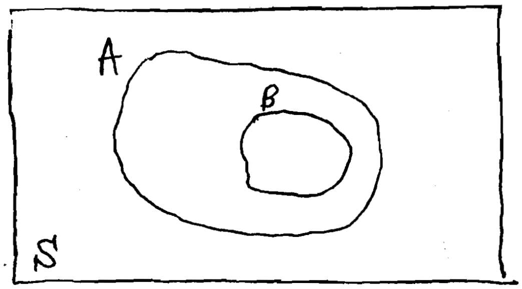
<u>Subset</u>: any set B whose elements are also elements of A. We say B is a subset of A or $B \subset A$ ("B is contained/included in A")

<u>Space or largest set</u>: S. If S has N elements, it has $2^N$ subsets

$S = \{x, y, z\} \rightarrow 2^3 = 8$ subsets:

$\{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}, \{x,y,z\}, \{\emptyset\}$

## Venn diagrams:

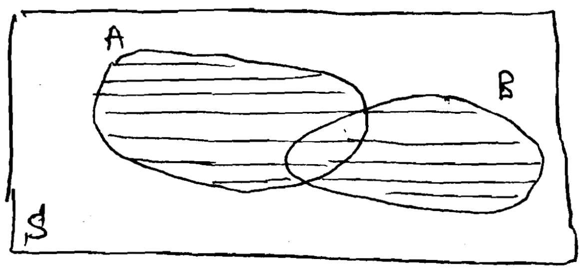



$B \subset A$

**Equality of two sets:** $A = B$ iFF (if and only if) $A \subset B$ and $B \subset A$

## Set operations:

(1) **Sum or Union:** $A \cup B$ contains elements of $A$ and elements of $B$ (also elements that belong to both $A$ and $B$)




$A \cup B$ is described by shaded area

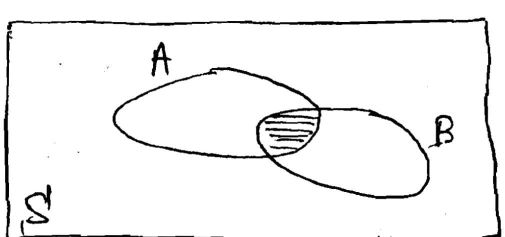
**Properties:**

$A \cup B = B \cup A$ (commutative)

$A \cup \phi = A$ ; $A \cup A = A$ ; $A \cup S = S$

$(A \cup B) \cup C = A \cup (B \cup C)$ (associative)

(2) **Product or Intersection:** $A \cap B$ ("A" intersecting "B") contains elements that are common to **both** $A$ and $B$.




Shaded area is $A \cap B$

<u>Properties</u>:

$$A \cap B = B \cap A \quad (\text{commutative})$$

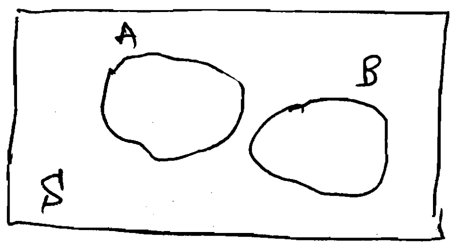
$$A \cap S = A \quad ; \quad A \cap \phi = \phi$$

$$A \cap A = A \quad ;$$

$$(A \cap B) \cap C = A \cap (B \cap C) \quad (\text{associative})$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

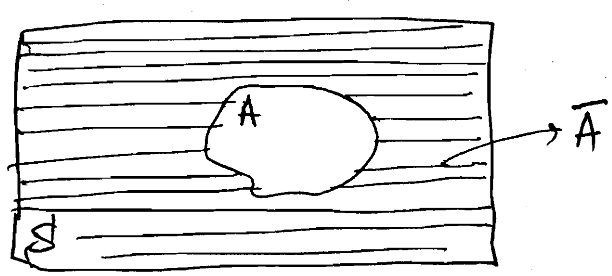
(distributive)




$A \cap B = \phi$

"A and B are mutually exclusive or disjoint"

③ <u>Complement</u>: $\overline{A}$ is the complement of A : contains elements that belong to S but not to A




<u>Properties</u>:

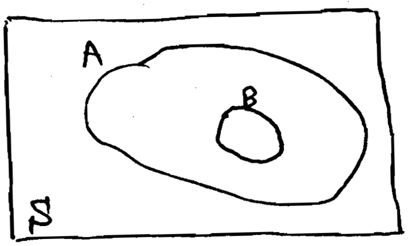
$$\overline{\phi} = S \quad ; \quad \overline{S} = \phi$$

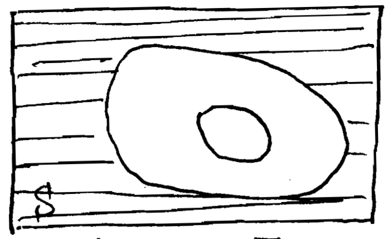
$$\overline{\overline{A}} = A \quad ;$$

$$A \cup \overline{A} = S \quad ;$$

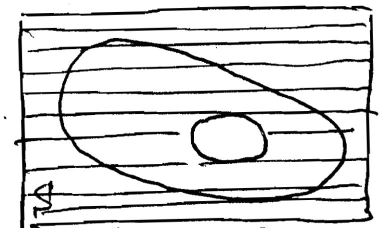
$$A \cap \overline{A} = \phi \quad ;$$

* If $B \subset A \Rightarrow \bar{A} \subset \bar{B}$




Shaded area is $\bar{A}$

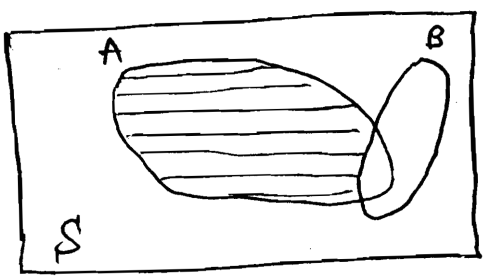



Shaded area $\bar{B}$

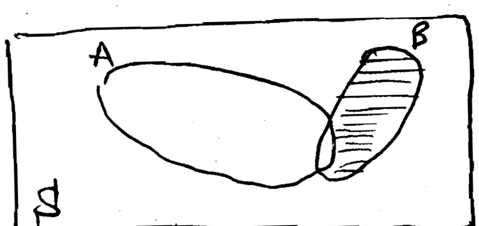
* If $\bar{B} = \bar{A} \Rightarrow B = A$

* $$\left.\begin{aligned} \overline{A \cup B} &= \bar{A} \cap \bar{B} \\ \overline{A \cap B} &= \bar{A} \cup \bar{B} \end{aligned}\right\} \text{De Morgan's Laws}$$

(4) <u>Difference between A and B:</u> $A - B$ elements that belong to A but not to B

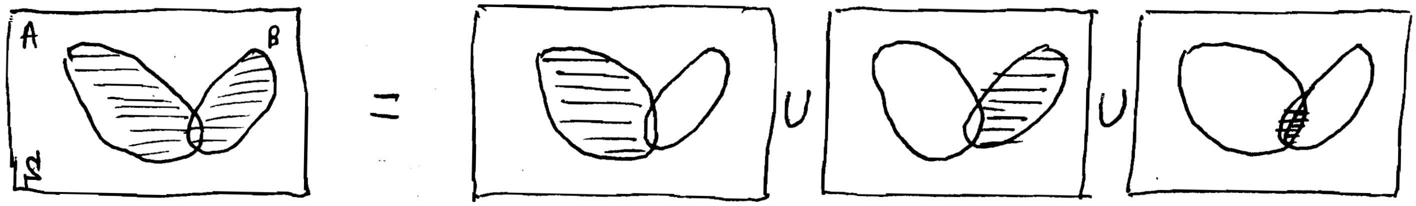



Shaded : $A - B$




Shaded = $B - A$

Write $A \cup B$ in terms of $A - B$; $B - A$; $A \cap B$ :




$$A \cup B = \underbrace{(A-B) \cup (B-A) \cup (A \cap B)}$$

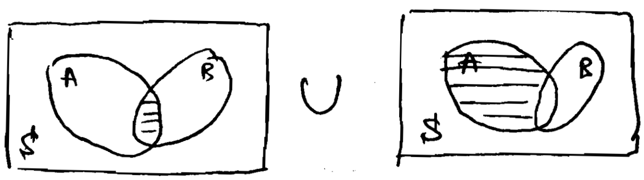
These 3 sets are mutually exclusive or disjoint

Properties:

$$\bar{A} = S - A$$
$$A - B = A \cap \bar{B}$$
$$= A - (A \cap B)$$

Example 1-5-1: $A, B$ subsets of $S$

a) $(A \cap B) \cup (A - B) = A$ ✓




b) $\bar{A} \cap (A - B) = \phi$


Shaded: $\bar{A}$ Shaded $A-B$

c) $(A \cap B) \cap (B \cup A) = A \cap B$

## Corollaries (consequences)

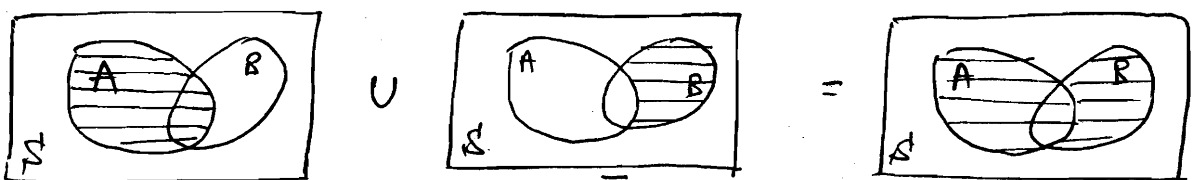
1) $A \cap \bar{A} = \emptyset \rightarrow \left.\begin{array}{l} Pr(\underbrace{A \cup \bar{A}}_{S}) \overset{\text{3rd axiom}}{=} Pr(A) + Pr(\bar{A}) \\ Pr(S) \overset{\text{2nd axiom}}{=} 1 \end{array}\right\}$

$$\Rightarrow \boxed{Pr(\bar{A}) = 1 - Pr(A)}$$

2) If $A \cap B \neq \emptyset \rightarrow Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$

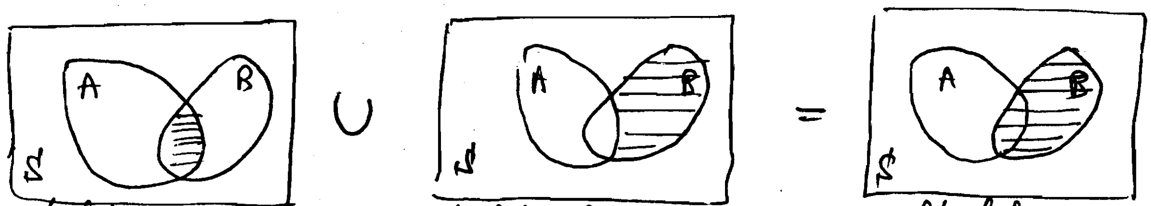
**Proof:** * $A \cup B = A \cup (\bar{A} \cap B)$ ✓ (A and $\bar{A} \cap B$ are disjoint)

In example 1.5.2 : $\bar{A} \cap B = B - A$ $\quad$ ↓ 3rd axiom : $Pr(A \cup B) = Pr(A) + Pr(\bar{A} \cap B)$ (i)




Shaded : A $\qquad$ Shaded : $\bar{A} \cap B = B - A$ $\qquad$ Shaded : $A \cup B$

* $B = (A \cap B) \cup (\underbrace{\bar{A} \cap B}_{B - A})$




Shaded : $A \cap B$ $\qquad$ Shaded : $B - A = \bar{A} \cap B$ $\qquad$ Shaded : B

Observation: $A \cap B$ and $\bar{A} \cap B$ are disjoint → 3rd axiom

$$Pr(B) = Pr(A \cap B) + Pr(\bar{A} \cap B) \quad \text{(ii)}$$

So far we have derived:

$$Pr(A \cup B) = Pr(A) + Pr(\bar{A} \cap B) \quad (i)$$

$$Pr(B) = Pr(A \cap B) + Pr(\bar{A} \cap B) \quad (ii)$$

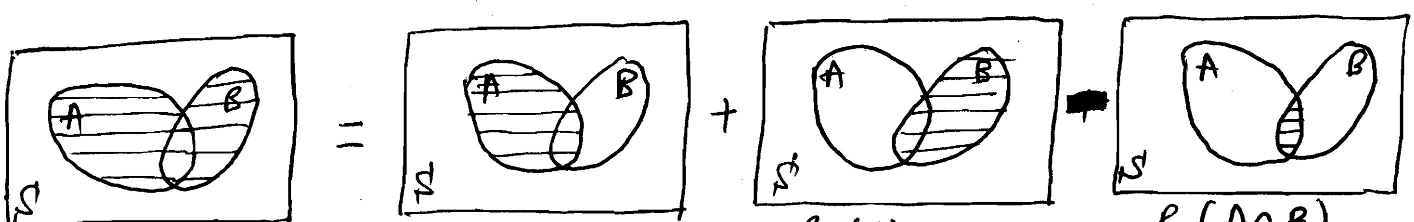
$$\hookrightarrow Pr(\bar{A} \cap B) = Pr(B) - Pr(A \cap B)$$

Now, use this in (i)

$$\rightarrow \boxed{Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)}$$

When $A \cap B \neq \emptyset$

Interpretation of the Venn diagram for this corollary:




$$Pr(A \cup B) = \underbrace{Pr(A) + Pr(B) - Pr(A \cap B)}$$

The overlap area b/w A & B is double-counted, so we need to substract once.

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<u>Random experiment</u>: throwing a dice $S = \{1, 2, 3, 4, 5, 6\}$

$\quad\downarrow$ event of getting a "1"

Dice is not tricked $\rightarrow$ all of the six single events are equally probable $\rightarrow$ $A$ = event of getting 1 $\rightarrow Pr(A) = \frac{1}{6}$

Let's define event $\hat{A}$ of getting a "1" or a "3" $\rightarrow \hat{A} = \{1, 3\}$

Let's define event $\hat{B}$ of getting a "3" or a "5" $\rightarrow \hat{B} = \{3, 5\}$

Using the axiomatic approach to probability, calculate $\Pr(\hat{A} \cup \hat{B})$.

$$\Pr(\hat{A} \cup \hat{B}) = \Pr(\hat{A}) + \Pr(\hat{B}) - \Pr(\hat{A} \wedge \hat{B})$$

$$\Pr(\{1,3,5\}) = \Pr(\{1,3\}) + \Pr(\{3,5\}) - \Pr(\{3\})$$

$$= \frac{2}{6} + \frac{2}{6} - \frac{1}{6} = \frac{3}{6} = 0.5$$

HW1 : (Ch 1) 4.5; 6.1; 7.1, 8.4