Linear System Theory II (sp'08)

LST I:

\[ x(t) \rightarrow \text{System} \rightarrow y(t) \]

\(x, y\) are functions of time

\[ x(t) = \delta(t) \rightarrow y(t) = h(t) \]

'impulse response'

How do we get \(y(t)\) if \(x(t)\) & \(h(t)\) are known?

\[ y(t) = x(t) \ast h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) \, d\tau \]

"convolution",

\[ = \int_{-\infty}^{\infty} x(t-\tau) h(\tau) \, d\tau \]

= \( h(t) \ast x(t) \)

"Convolution is commutative"

LST II:

Introduce random signals: noise, described with probability distributions

\[ \rightarrow \text{Incorporate into } y(t) = x(t) \ast h(t) \]

How to calculate properties of the output when the input is a random signal.

Application: noise elimination: e.g. precise target location
Chapter 1: Introduction to Probability

Random experiment: example: rolling a die, outcome is uncertain
(Matlab use "rand" function)

Event: "getting a four", "getting a three or a four"

Relative frequency probability:

<table>
<thead>
<tr>
<th>Roll a die</th>
<th># times getting a four</th>
<th>Rel. frequency probability for this event</th>
</tr>
</thead>
<tbody>
<tr>
<td>120</td>
<td>27</td>
<td>27/120 → 22.5%</td>
</tr>
<tr>
<td>1200</td>
<td>185</td>
<td>185/1200 → 15.4%</td>
</tr>
<tr>
<td>12000</td>
<td>1972</td>
<td>1972/12000 → 16.4%</td>
</tr>
<tr>
<td>1200000</td>
<td>200000</td>
<td>200000/1200000 → 16.67%</td>
</tr>
</tbody>
</table>

Meaningful experiment
(Sufficient # of samples)

\[ \frac{1}{6} = 0.1667 : \text{probability (expected theoretical value) of getting one number out of six possibilities.} \]

A = "Event of getting a 3 or a 4" (A composite event)

Rel. freq. probability for event A = \[ P_2(A) = \frac{2}{6} = \frac{1}{3} \]

\[ P(E) = \frac{n}{N} \leq 1 \]

n: # time E happens or # of single events in E
N: # all possible single events
Certain event: \( \Omega \) will have \( P(\Omega) = \frac{6}{6} = 1 \)

In the experiment of rolling a die, the certain event is that getting either a 1, 2, 3, 4, 5, or 6.
\( \Omega = \{1, 2, 3, 4, 5, 6\} \): composed of all possible single events.

Impossible event: \( \phi \) will have \( P(\phi) = \frac{0}{6} = 0 \)

In the experiment of rolling a die, the impossible event is "getting a number that is not between 2 and 6"

Matlab experiment: use attached code; than place data answers.

<table>
<thead>
<tr>
<th># times we roll a die</th>
<th># times we get a 4</th>
<th>( P(A) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>120</td>
<td>22</td>
<td>0.1833 18.3%</td>
</tr>
<tr>
<td>1200</td>
<td>2046</td>
<td></td>
</tr>
<tr>
<td>1,200,000</td>
<td>199700</td>
<td></td>
</tr>
</tbody>
</table>

on the upper right corner of each bar that corresponds to \( X=4 \) of the four subplots (runs)

\( \times \) The uncertain outcome in a random experiment shows in the different number of times we get a four

<table>
<thead>
<tr>
<th>Scott</th>
<th>Tengdi</th>
<th>Zhou</th>
<th>Chen</th>
</tr>
</thead>
<tbody>
<tr>
<td>12000</td>
<td>2030</td>
<td>1970</td>
<td>2046</td>
</tr>
<tr>
<td>1,200000</td>
<td>200000</td>
<td>200200</td>
<td>199700</td>
</tr>
</tbody>
</table>
% Rolling a dice experiment (a random experiment)
% Jan 29, 2008

nplot=4; %number of plots, one per run
nx=2; % number of subplots
ny=2;
nroll = [120 1200 12000 120000]; %number of times to roll in each of
the four runs
for i=1:nplot
    subplot(nx,ny,i)
    y1=ones(1,nroll(i)) + floor(6*rand(1,nroll(i)));
    hist(y1)
    axis([1 6 0 nroll(i)/4]);
    xlabel('amplitude')
    ylabel(strcat('result after rolling', num2str(nroll(i)),'times'))
end
Joint probability:

In a box of 1000 resistors*, we have:

* Each resistor is classified by two labels: resistance (Ω) & power (W)

<table>
<thead>
<tr>
<th>P</th>
<th>R</th>
<th>1Ω</th>
<th>10Ω</th>
<th>100Ω</th>
<th>1000Ω</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1W</td>
<td>50</td>
<td>300</td>
<td>90</td>
<td>0</td>
<td>440</td>
<td></td>
</tr>
<tr>
<td>2W</td>
<td>50</td>
<td>50</td>
<td>0</td>
<td>100</td>
<td>200</td>
<td></td>
</tr>
<tr>
<td>5W</td>
<td>0</td>
<td>150</td>
<td>60</td>
<td>150</td>
<td>360</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
<td>500</td>
<td>150</td>
<td>250</td>
<td>1000</td>
<td></td>
</tr>
</tbody>
</table>

If we open the box and pick one resistor, this is a random experiment.

Event A = "getting a resistor with $R = 10\,\Omega$"; $P(A) = \frac{500}{1000} = 0.5$

Event B = "getting a resistor with $P = 5W"; P(B) = \frac{360}{1000} = 0.36$

Event A&B = "getting a resistor with $R = 10\,\Omega$ and $P = 5W"

$P(A, B) = \frac{150}{1000} = 0.15$

Is there any connection between this joint probability and the single $P(A)$ or $P(B)$?

Conditional probability: of A given B happened =

$P(A|B) = \frac{150}{360} = 0.417$ Prob. of getting a 10Ω given it is a 5W

$P(B|A) = \frac{150}{500} = 0.3$ Prob. of getting a 5W given that it is a 10Ω
Can we obtain $P(A \cap B)$ from $P(B)$ & $P(A | B)$?

**Yes:**

\[
P(A \cap B) = P(A | B) \times P(B)
\]

\[
0.15 = 0.417 \times 0.36
\]

\[\checkmark\]

**Or**

\[
P(A \cap B) = P(B | A) \times P(A)
\]

\[
0.15 = 0.3 \times 0.5
\]

\[\checkmark\]

joint probability, conditional probability, marginal probability

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**Ex. 1-4.1:**

a) Box of 50 diodes | 10 bad

\[
\begin{array}{c}
40 \text{ good} \\
\end{array}
\]

A diode is picked at random, event $A =$ "getting a bad diode."

\[
P(A) = \frac{10}{50} = 0.2
\]

b) If first diode drawn is good, what is the prob. the second diode drawn will be good?

\[
\begin{array}{c|c}
10 \text{ bad} \\
40 \text{ good} \\
\end{array} \quad \text{After 1st draw:} \quad \begin{array}{c|c}
10 \text{ bad} \\
39 \text{ good} \\
\end{array} \quad \Rightarrow P(\text{good}) = \frac{39}{49} = 0.795
\]

(less than initial prob. of 0.8)
c) If two diodes are drawn from the box what is the probability that they are both good?

Two approaches:

1) Relative frequency approach:

\[ p(\text{getting 2 good diodes}) = \frac{\text{# combinations of 2 out of 40 good diode}}{\text{# combinations of 2 out of 50 diode}} \]

Combinatorics: \( \binom{n}{a} = \frac{n!}{a!(n-a)!} = \text{# combinations of } a \text{ elements out of a universe of } n \text{ elements} \)

\[ n! = n(n-1)(n-2)\ldots4 \]

"n factorial"

\[ \binom{40}{2} \quad \text{(only one type)} \quad \frac{40!}{2!38!} \]

\[ \binom{40}{2} + \binom{10}{2} + 40 \cdot 10 \quad \text{(two types of diode)} \quad \frac{40!}{2!38!} + \frac{10!}{2!8!} + 400 \]

two good; two bad; one good & one bad.

\[ \frac{40 \times 39}{2} = \frac{780}{1225} = \frac{156}{245} = 0.6367 \]

2) Joint probability approach:

\[ \text{Event } A = \text{ "getting a good diode"} \]
\[ p_s ( \text{getting two good dice} ) = p_s (A, A) = p_s (A|A) \cdot p_s (A) \]

\[ \text{Prob. of getting a good die} \]
\[ \text{given the 1st pick was good.} \]

\[ \text{Prob. of getting two good dice} \]

\[ \rightarrow p_s (A, A) = \frac{39}{49} \cdot \frac{40}{50} = \frac{39 \times 4}{49 \times 5} = \frac{156}{245} = 0.6367 \]

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**Review on elementary set theory**

- **Set**: a collection of elements \( A = \{x_1, x_2, \ldots, x_n\} \)
- **Subset**: any set \( B \) whose elements are also elements of \( A \). We say \( B \) is a subset of \( A \) \( a \subset B \subset A \) ("\( B \) is contained/included in \( A \)")

**Space or largest set**: \( S \). If \( S \) has \( N \) elements, it has \( 2^N \) subsets.

\[ S = \{x, y, z\} \rightarrow 2^3 = 8 \text{ subsets:} \]

\[ \{x\}, \{y\}, \{z\}, \{xy, \{x, z\}, \{y, z\}, \{xyz, \{x, y, z\}\}, \{x\}, \{y\}, \{z\} \]
Venn diagrams:

Equality of two sets: \[ A = B \iff (\text{if and only if}) \text{ } A \subseteq B \text{ and } B \subseteq A \]

Set operations:

1. **Union**:

   \[ A \cup B \text{ contains elements of } A \text{ and } \text{elements of } B \text{ (also elements that belong to both } A \text{ and } B) \]

   \[ A \cup B \text{ is defined by shaded area} \]

   Properties:
   - \[ A \cup B = B \cup A \text{ (commutative)} \]
   - \[ A \cup \emptyset = A \]
   - \[ A \cup S = S \]
   - \[ (A \cup B) \cup C = A \cup (B \cup C) \text{ (associative)} \]

2. **Intersection**:

   \[ A \cap B \text{ ("A" intersecting "B")} \]

   contains elements that are common to both \( A \) and \( B \).

   Shaded area is \( A \cap B \)
Properties:

\[ A \cap B = B \cap A \] (commutative)
\[ A \cap S = A ; \quad A \cap \phi = \phi \]
\[ A \cap A = A ; \]
\[ (A \cap B) \cap C = A \cap (B \cap C) \] (associative)
\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \] (distributive)

A and B are mutually exclusive or disjoint.

3. Complement: \( \overline{A} \) is the complement of \( A \) that contains elements that belong to \( S \) but not to \( A \).

Properties:

\[ \overline{\phi} = S ; \quad \overline{S} = \phi \]
\[ \overline{\overline{A}} = A ; \]
\[ A \cup \overline{A} = S ; \]
\[ A \cap \overline{A} = \phi ; \]
\* \textbf{If } B \subseteq A \Rightarrow \overline{A} \cap \overline{B} = \overline{B} \\

\* \text{Shaded area is } \overline{A} \cap \overline{B} \\
\text{Shaded area } \overline{B} \\

\* \text{If } \overline{B} = \overline{A} \Rightarrow B = A \\

\* \frac{A \cup B}{A \cap B} = \frac{\overline{A} \cap \overline{B}}{\overline{A} \cup \overline{B}} \text{ } \begin{aligned} \text{De Morgan's Laws} \end{aligned} \\

\* \text{Difference between } A \text{ and } B: A - B \text{ elements that belong to } A \text{ but not to } B \\
\text{Shaded: } A - B \\
\text{Shaded: } B - A \\
\text{Write } A \cup B \text{ in terms of } A - B; B - A; A \cap B;
\[
A \cup B = (A - B) \cup (B - A) \cup (A \cap B)
\]

These 3 sets are mutually exclusive or disjoint.

Property:
\[
\overline{A} = S - A
\]
\[
A - B = A \cap \overline{B}
\]
\[
= A - (A \cap B)
\]

Example 1-5.1: \(A, B\) subsets of \(S\)

a) \((A \cap B) \cup (A - B) = A \checkmark\)

\[
\begin{array}{c}
\begin{array}{c}
\text{Shaded: } A
\end{array}
\end{array}
\cup
\begin{array}{c}
\begin{array}{c}
\text{Shaded } A - B
\end{array}
\end{array}
\]

b) \(\overline{A} \cap (A - B) = \emptyset\)

\[
\begin{array}{c}
\begin{array}{c}
\text{Shaded: } A
\end{array}
\end{array}
\land
\begin{array}{c}
\begin{array}{c}
\text{Shaded } A - B
\end{array}
\end{array}
\]

c) \((A \cap B) \cap (B \cup A) = A \cap B\)

\[
\begin{array}{c}
\begin{array}{c}
\text{Shaded: } A
\end{array}
\end{array}
\land
\begin{array}{c}
\begin{array}{c}
\text{Shaded } A \cap B
\end{array}
\end{array}
\]
Corollary (consequence)

1) \( A \cap \overline{A} = \emptyset \rightarrow p_s(A \cup \overline{A}) = p_s(A) + p_s(\overline{A}) \)

\[ \therefore \]

\[ p_s(\overline{A}) = 1 - p_s(A) \]

2) If \( A \cap B \neq \emptyset \rightarrow p_s(A \cup B) = p_s(A) + p_s(B) - p_s(A \cap B) \)

**Proof**: \( A \cup B = A \cup (\overline{A} \cap B) \) (A and \( \overline{A} \cap B \) are disjoint)

In example 1.5.2: \( \overline{A} \cap B = B - A \)

\[ \therefore \]

\[ p_s(A \cup B) = p_s(A) + p_s(\overline{A} \cap B) \]

\[ \therefore \]

\[ B = (A \cap B) \cup (\overline{A} \cap B) \]

\[ \therefore \]

\[ B = (A \cap B) \cup (\overline{A} \cap B) \]

\[ \therefore \]

\[ B = (A \cap B) \cup (\overline{A} \cap B) \]

Observation: \( A \cap B \) and \( \overline{A} \cap B \) are disjoint \( \Rightarrow 2^{nd} \) axiom

\[ p_s(B) = p_s(A \cap B) + p_s(\overline{A} \cap B) \] (ii)
So far we have derived:

$$P(A \cup B) = P(A) + P(A' \cap B) \quad (i)$$

$$P(B) = P(A \cap B) + P(A' \cap B) \quad (ii)$$

Then:

$$P(A' \cap B) = P(B) - P(A \cap B)$$

Now, use this in (i)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

When $$A \cap B \neq \emptyset$$

---

**Interpretation of the Venn diagram for this corollary:**

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- The overlap area between $A$ and $B$ is double-counted, so we need to subtract once.

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**Random experiment:** Throwing a dice $S = \{1, 2, 3, 4, 5, 6\}$

Event of getting a "1" $\frac{1}{6}$

Dice is not tricked $\rightarrow$ all of the six single events are equally probable $\rightarrow A =$ event of getting 1 $\rightarrow P(A) = \frac{1}{6}$

Let's define event $\hat{A}$ of getting a "1" or a "3" $\rightarrow \hat{A} = \{1, 3\}$
Let's define event $B$ of getting a "3" or a "5" $\rightarrow B = \{3, 5\}$

Using the axiomatic approach to probability, calculate $P_2(\hat{A} \cup \hat{B})$.

$$P_2(\hat{A} \cup \hat{B}) = P_2(\hat{A}) + P_2(\hat{B}) - P_2(\hat{A} \cap \hat{B})$$

$$P_2(\{1, 3, 5\}) = P_2(\{1, 3\}) + P_2(\{3, 5\}) - P_2(\{3\})$$

$$= \frac{2}{6} + \frac{2}{6} - \frac{1}{6} = \frac{3}{6} = 0.5$$

HW1: (Ch1) 4.5; 6.1; 7.1, 8.4