$$S_x(\omega) = \frac{\omega^2+4}{\omega^4+10\omega^2+9} \rightarrow \overline{X^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, S_x(\omega) \; ?$$

$$\downarrow$$

$$\omega \rightarrow -js$$

$$= \frac{-s^2+4}{s^4-10s^2+9} = \frac{-(s^2-4)}{(s^2-1)(s^2-9)} = \frac{(s+2)(-s+2)}{(s+1)(s-1)(s+3)(s-3)}$$

$$= \frac{(s+2)(-s+2)}{(s+1)(s+3)(-s+1)(-s+3)}$$

$$\left[\; S_x(\omega) = \frac{c(s)\; c(-s)}{d(s)\; d(-s)} \right.$$

$$\overline{X^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, S_x(\omega)$$

$$= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} ds\, S_x(s) = \text{Table 7-1 pg 276}$$

$$c(s) = C_{n-1}s^{n-1} + C_{n-2}s^{n-2} + \cdots C_0$$

$$d(s) = d_n s^n + d_{n-1}s^{n-1} + \cdots \; d_0$$

$$\begin{cases} n=1 \rightarrow \dfrac{c_0^2}{2d_0 d_1} \\[2ex] n=2 \rightarrow \dfrac{c_1^2 d_0 + c_0^2 d_2}{2d_0 d_1 d_2} \\[2ex] n=3 \rightarrow \dfrac{c_2^2 d_0 d_1 + (c_1^2 - 2c_0c_2)d_0d_3 + c_0^2 d_2 d_3}{2d_0 d_3(d_1d_2 - d_0d_3)} \end{cases}$$

$$\overline{X^2}_{n=2} = \frac{3+2^2}{2\times 3\times 4\times 1} = \frac{7}{24}$$

$$\begin{cases} d(s) = (s+1)(s+3) = s^2+4s+3 \rightarrow \begin{cases} d_0 = 3 \\ d_1 = 4 \\ d_2 = 1 \end{cases} \\[2ex] c(s) = s+2 \quad \begin{cases} c_0 = 2 \\ c_1 = 1 \end{cases} \end{cases}$$

Chapter 8: Response of Linear Systems to Random Inputs

Mean value of system output: (from Linear S.T.I : $Y(t) = X(t) * h(t)$)

$$\overline{Y} = \overline{\int_0^\infty X(t-\lambda) h(\lambda)\, d\lambda} = \overline{X} \int_0^\infty h(\lambda)\, d\lambda .$$

Mean-square value of system input:

$$\overline{Y^2} = \overline{\int_0^\infty X(t-\lambda_1) h(\lambda_1)\, d\lambda_1 \int_0^\infty X(t-\lambda_2) h(\lambda_2)\, d\lambda_2}$$

$$= \int_0^\infty \int_0^\infty d\lambda_1\, d\lambda_2 \; \overline{X(t-\lambda_1) X(t-\lambda_2)} \; h(\lambda_1) h(\lambda_2)$$

$$= \int_0^\infty \int_0^\infty d\lambda_1\, d\lambda_2 \; R_X(\lambda_2 - \lambda_1)\, h(\lambda_1) h(\lambda_2)$$

Autocorrelation function of system output:

$$R_Y(\tau) = \overline{Y(t)\, Y(t+\tau)} = \int_0^\infty \int_0^\infty d\lambda_1\, d\lambda_2 \; \overline{X(t-\lambda_1) X(t+\tau-\lambda_2)} \; h(\lambda_1) h(\lambda_2)$$

$$= \int_0^\infty \int_0^\infty d\lambda_1\, d\lambda_2 \; R_X(\lambda_2 - \lambda_1 - \underset{\substack{\uparrow \\ \text{difference w.r.t. } \overline{Y^2}}}{\tau})\, h(\lambda_1) h(\lambda_2)$$

Cross-correlation function between input & output:

$$R_{XY}(\tau) = \overline{X(t)\, Y(t+\tau)} = \overline{X(t) \int_0^\infty d\lambda\, X(t+\tau-\lambda) h(\lambda)}$$

$$= \int_0^\infty d\lambda \; \overline{X(t) X(t+\tau-\lambda)} \; h(\lambda) = \int_0^\infty d\lambda \; R_X(\tau-\lambda)\, h(\lambda)$$

$$R_{YX}(\tau) = \overline{Y(t) X(t+\tau)} = \int_0^\infty d\lambda \; \overline{X(t-\lambda) X(t+\tau)} \; h(\lambda) = \int_0^\infty d\lambda \; R_X(\underset{\substack{\uparrow \\ \text{difference w.r.t. } R_{XY}}}{\tau+\lambda})\, h(\lambda)$$

Frequency-domain analysis:

$$\text{Fourier} \left\{ Y(t) = X(t) * \underset{\text{"impulse response"}}{h(t)} \right\} \rightarrow Y(\omega) = X(\omega) \underset{\text{"transfer function"}}{H(\omega)} \text{ or } Y(f) = X(f) H(f)$$

$$\text{Laplace} \left\{ Y(t) = X(t) * h(t) \right\} \longrightarrow Y(s) = X(s)\, H(s)$$

Spectral Density of the system output.

$$S_Y(\omega) = F\{R_Y(\tau)\} = F\left\{\int_0^\infty d\lambda_1 \int_0^\infty d\lambda_2 \; R_x(\lambda_2-\lambda_1-\tau)\, h(\lambda_1)\, h(\lambda_2)\right\}$$

$$= \int_{-\infty}^\infty d\tau \; e^{-j\omega\tau}\left\{\int_0^\infty d\lambda_1 \int_0^\infty d\lambda_2 \; R_x(\lambda_2-\lambda_1-\tau)\, h(\lambda_1)\, h(\lambda_2)\right\}$$

$$= \int_0^\infty d\lambda_1 \int_0^\infty d\lambda_2 \underbrace{\int_{-\infty}^\infty d\tau \; e^{-j\omega(\tau-\lambda_2+\lambda_1)} R_x(\lambda_2-\lambda_1-\tau)}_{S_X(\omega)} \; h(\lambda_1)\, e^{+j\omega\lambda_1} \; h(\lambda_2)\, e^{-j\omega\lambda_2}$$

$$= S_x(\omega) \; \underbrace{H^*(\omega)\, H(\omega)} = S_x(\omega)\, |H(\omega)|^2$$

with $s = j\omega \rightarrow S_Y(s) = S_X(s)\, H(s)\, H(-s)$

8.3.3 $\quad X \to \quad R_X(\tau) = 16\, e^{-2|\tau|} + 16$

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$$

$$h(t) = \delta(t) - 2e^{-2t}u(t)$$

a) $\overline{Y} = \overline{X}\int_0^\infty h(\lambda)\,d\lambda = 4\int_0^\infty d\lambda\,\left[\delta(\lambda) - 2e^{-2\lambda}u(\lambda)\right] =$

$R_X(\tau\to\infty) = \overline{X}^2 \to \overline{X} = \sqrt{R_X(\tau\to\infty)} = \sqrt{16} = 4$

$$\hookrightarrow \underbrace{4\int_0^\infty d\lambda\,\delta(\lambda)}_{\substack{\text{(unit } \delta\text{)} \\ 1 \text{ (if the center of } \delta \text{ is} \\ \text{b/w the limits of integration)}}} - 8\int_0^\infty d\lambda\; e^{-2\lambda} = 4 - \frac{8}{2} = 0$$

$$-8\left[\frac{e^{-2\lambda}}{-2}\right]_0^\infty$$

b) $\overline{Y^2} = \int_0^\infty\int_0^\infty d\lambda_1 d\lambda_2\; R_X(\lambda_2-\lambda_1)\,h(\lambda_1)\,h(\lambda_2)$

$$= \int_0^\infty\int_0^\infty d\lambda_1 d\lambda_2\;\left(16\,e^{-2|\lambda_2-\lambda_1|} + 16\right)\left(\delta(\lambda_1) - 2e^{-2\lambda_1}u(\lambda_1)\right)\cdot\left(\delta(\lambda_2) - 2e^{-2\lambda_2}u(\lambda_2)\right)$$

$$= 16\int_0^\infty\int_0^\infty d\lambda_1 d\lambda_2\,\Big\{ e^{-2|\lambda_2-\lambda_1|}\delta(\lambda_1)\delta(\lambda_2) - 2e^{-2|\lambda_2-\lambda_1|}e^{-2\lambda_2}u(\lambda_2)\delta(\lambda_1)$$
$$-2e^{-2|\lambda_2-\lambda_1|}e^{-2\lambda_1}u(\lambda_1)\delta(\lambda_2) + \boxed{4e^{-2|\lambda_2-\lambda_1|}e^{-2\lambda_1}e^{-2\lambda_2}u(\lambda_1)u(\lambda_2)}$$
$$+ \delta(\lambda_1)\delta(\lambda_2) - 2\delta(\lambda_1)e^{-2\lambda_2}u(\lambda_2)$$
$$-2e^{-2\lambda_1}u(\lambda_1)\delta(\lambda_2) + 4e^{-2\lambda_1}e^{-2\lambda_2}u(\lambda_1)u(\lambda_2)\Big\}$$

$$= 16 \Big\{ 1 + \underbrace{2\int_0^\infty d\lambda_2 \, e^{-2|\lambda_2|} e^{-2\lambda_2}}_{-2\int_0^\infty d\lambda_2 \, e^{-4\lambda_2} \, = \, -\frac{2}{4}} \; \underbrace{-2\int_0^\infty d\lambda_1 \, e^{-2|-\lambda_1|} e^{-2\lambda_1}}_{-2\int_0^\infty d\lambda_1 e^{-4\lambda_1} \, = \, -\frac{2}{4}}$$

$$+1 \; \underbrace{-2\int_0^\infty d\lambda_2 \, e^{-2\lambda_2}}_{-\frac{2}{2}} \; \underbrace{-2\int_0^\infty d\lambda_1 \, e^{-2\lambda_1}}_{-\frac{2}{2}} \; +4 \underbrace{\underbrace{\int_0^\infty d\lambda_1 e^{-2\lambda_1}}_{\frac{1}{2}} \underbrace{\int_0^\infty d\lambda_2 e^{-2\lambda_2}}_{\frac{1}{2}}}_{\frac{4}{4}} \; +$$

$$|\lambda_2 - \lambda_1| = \begin{cases} \lambda_2 - \lambda_1 & \text{if } 0 < \lambda_1 < \lambda_2 \\ \lambda_1 - \lambda_2 & \text{if } \lambda_2 < \lambda_1 \end{cases}$$

$$4\int_0^\infty d\lambda_2 \left[\int_0^{\lambda_2} d\lambda_1 \, e^{-2(\lambda_2-\lambda_1)} e^{-2\lambda_1} e^{-2\lambda_2} + \int_{\lambda_2}^\infty d\lambda_1 \, e^{2(\lambda_2-\lambda_1)} e^{-2\lambda_1} e^{-2\lambda_2} \right]$$

$$= 16 \Big\{ 1 - \frac{1}{2} - \frac{1}{2} + 1 - 1 - 1 + 1$$

$$+ 4\int_0^\infty d\lambda_2 \underbrace{\int_0^{\lambda_2} d\lambda_1}_{\lambda_2} e^{-4\lambda_2} + 4\int_0^\infty d\lambda_2 \int_{\lambda_2}^\infty d\lambda_1 \, e^{-4\lambda_1} \Big\}$$

$$= 16 \times 4 \left[\underbrace{\int_0^\infty d\lambda_2 \, \lambda_2 e^{-4\lambda_2}} + \int_0^\infty d\lambda_2 \left[\frac{e^{-4\lambda_1}}{-4} \right]_{\lambda_2}^\infty \right]$$

$$= 64 \left[\underbrace{\int_0^\infty d\lambda_2 \, \lambda_2 e^{-4\lambda_2}}_{\frac{1}{4^2}} + \underbrace{\int_0^\infty d\lambda_2 \, \frac{e^{-4\lambda_2}}{4}}_{\underbrace{\frac{1}{4}\left[\frac{e^{-4\lambda_2}}{-4}\right]_0^\infty}_{\frac{1}{16}}} \right]$$

$$\int_0^\infty dx \, x^n e^{-ax} = \frac{n!}{a^{n+1}} = \frac{\Gamma(n+1)}{a^{n+1}}$$

$$= 64 \left[\frac{1}{16} + \frac{1}{16} \right] = 64 \times \frac{2}{16}$$

$$= 8 \quad ✓$$

c) $\sigma_Y^2 = \overline{Y^2} - \overline{Y}^2 = 8 - 0 = 8$