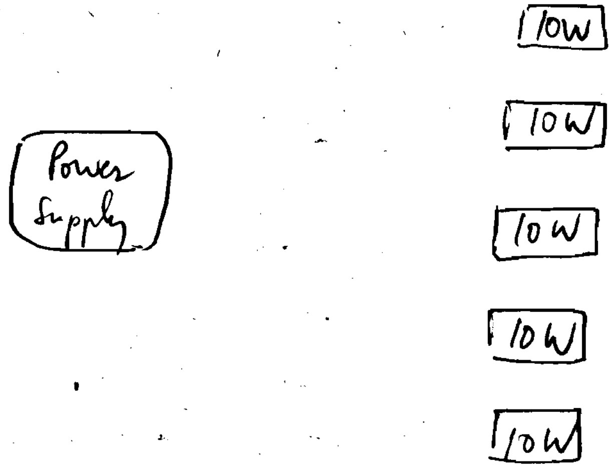
2.4.4



Each load in operation 25%

a) Average power required by loads ?

Total Power $= X$; $\bar{X} = E\{X\} = [10 \times \underbrace{0.25}_{\text{probability}} \times 5] = 12.5\,W$

$\hookrightarrow E\{X_1 + X_2 + X_3 + X_4 + X_5\}$

b) $\sigma_x^2 = \overline{X^2} - \bar{X}^2$

(Variance of power required)

$$E\{X^2\} = E\{(X_1 + X_2 + X_3 + X_4 + X_5)^2\}$$

$$= E\{\underbrace{X_1^2 + \dots X_4^2 + X_5^2}_{5 \text{ terms}} + \underbrace{2X_1X_2 + \dots}_{10 \text{ terms}}\}$$

$$= \underbrace{E\{X_1^2 + \dots X_4^2 + X_5^2\}}_{} + 2E\{\underbrace{X_1X_2 + \dots}_{10 \text{ terms}}\}$$

$$= 10^2 \times 0.25 \times 5 \quad + \quad 2 \times 10 \times 0.25 \times 10 \times 0.25 \times 10$$

$$= 125 \quad + \quad 125$$

$$\overline{X^2} = 250$$

$$\rightarrow \sigma_x^2 = \overline{X^2} - \bar{X}^2 = 250 - 12.5^2 = 250 - 156.25 = 93.75\,W^2$$

c) Prob. power supplied is overloaded ($X > 40W$)

$P_r(X > 40W) = P_r$ (Having five loads connected at once) $= \left(\frac{1}{4}\right)^5 = 0.098\%$

Chi-Square or X-square or χ^2 distribution:

$$\chi^2 = Y_1^2 + Y_2^2 + \dots Y_n^2$$ where Y_i are Gaussian Random Variables with zero mean and unit variance ($\sigma^2 = 1$)

$$f(x^2) = \begin{cases} \dfrac{(x^2)^{\frac{n}{2}-1}}{2^{\frac{n}{2}}\,\Gamma(\frac{n}{2})}\, e^{-\frac{x^2}{2}} & x^2 \geq 0 \\ 0 & x^2 < 0 \end{cases}$$

1) $$\overline{\chi^2} = \int_0^\infty dx^2 \; x^2 \, \frac{(x^2)^{\frac{n}{2}-1}}{2^{\frac{n}{2}}\,\Gamma(\frac{n}{2})}\, e^{-\frac{x^2}{2}} = \frac{1}{2^{\frac{n}{2}}\,\Gamma(\frac{n}{2})} \underbrace{\int_0^\infty d(x^2)\, x^n\, e^{-\frac{x^2}{2}}}_{y = x^2}$$

$$\boxed{\int_0^\infty dx\, x^n e^{-ax} = \frac{\Gamma(n+1)}{a^{n+1}}} \qquad \boxed{\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)}$$

$$= \frac{1}{2^{\frac{n}{2}}\,\Gamma(\frac{n}{2})} \underbrace{\int_0^\infty dy\; y^{\frac{n}{2}}\, e^{-\frac{y}{2}}} = \frac{1}{2^{\frac{n}{2}}\,\Gamma(\frac{n}{2})} \, \frac{\Gamma(\frac{n}{2}+1)}{(\frac{1}{2})^{\frac{n}{2}+1}}$$

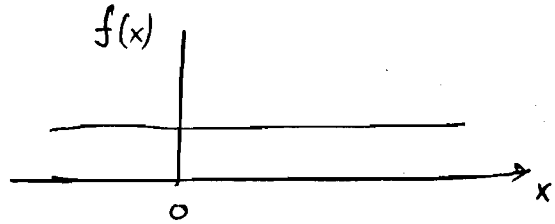
$$= \frac{1}{\cancel{2^{\frac{n}{2}}}\,\cancel{\Gamma(\frac{n}{2})}} \, \frac{\frac{n}{\cancel{2}}\cancel{\Gamma(\frac{n}{2})}}{\cancel{(\frac{1}{2})^{\frac{n}{2}}} \cdot \frac{1}{\cancel{2}}} = n$$

2) $\overline{\chi^4} = 2n + n^2$ (Prove → 5 exam points)

3) $\sigma_{\chi^2}^2 = \overline{\chi^4} - \overline{\chi^2}^{\,2} = 2n + n^2 - n^2 = 2n$

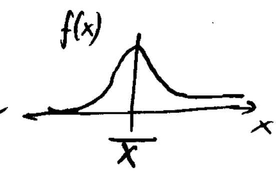
Uniform distribution:

↳ No preferred value



Previously:

- Gaussian distribution :→ preferred value was the mean



- Rayleigh distribution → preferred value was:

$\boxed{R = \sqrt{X^2 + Y^2}}$

↳ $f_R(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \rightarrow \frac{df_R}{dr} = 0 = \frac{1}{\sigma^2} \cancel{e^{-\frac{r^2}{2\sigma^2}}} - \frac{r^2}{\sigma^4} \cancel{e^{-\frac{r^2}{2\sigma^2}}}$

$$\frac{d}{dr}\left(-\frac{r^2}{2\sigma^2}\right) = -\frac{2r}{2\sigma^2} = -\frac{r}{\sigma^2}$$

$$0 = \frac{1}{\sigma^2} - \frac{r^2}{\sigma^4} \quad \text{or} \quad 0 = 1 - \frac{r^2}{\sigma^2} \Rightarrow \boxed{r = \sigma}$$

X & Y are Gaussian w/ zero mean and st. dev. σ

- Maxwell distribution:

$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$, v_x, v_y, v_z are Gaussian with zero mean and st. dev. σ

$$f_v = \sqrt{\frac{2}{\pi}} \frac{v^2}{\sigma^3} e^{-\frac{v^2}{2\sigma^2}} \rightarrow \frac{df_v}{dv} = 0 = \cancel{\sqrt{\frac{2}{\pi}}}\left(\frac{2v}{\sigma^3} \cancel{e^{-\frac{v^2}{2\sigma^2}}} - \frac{v^3}{\sigma^5} \cancel{e^{-\frac{v^2}{2\sigma^2}}}\right)$$

$$\rightarrow 0 = \frac{2v}{\sigma^3} - \frac{v^3}{\sigma^5} \quad \text{or} \quad 0 = \frac{2}{\sigma^2} - \frac{v^2}{\sigma^4}$$

$$\rightarrow 2 = \frac{v^2}{\sigma^2} \rightarrow \boxed{v = \sigma\sqrt{2}}$$

- <u>χ^2 distribution:</u> $\chi^2 = Y_1^2 + Y_2^2 + \dots Y_n^2$

  Y_i are Gaussian w/ unit variance $\sigma^2 = 1$

Preferred value: $$f(\chi^2) = \frac{(\chi^2)^{\frac{n}{2}-1}}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} e^{-\frac{\chi^2}{2}}$$

$$\frac{df}{d\chi^2} = 0 = \cancel{\frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}}\left[\left(\frac{n}{2}-1\right)(\chi^2)^{\frac{n}{2}-2} \cancel{e^{-\frac{\chi^2}{2}}} - \frac{1}{2}(\chi^2)^{\frac{n}{2}-1} \cancel{e^{-\frac{\chi^2}{2}}}\right]$$

or $0 = \left(\frac{n}{2}-1\right)(\chi^2)^{-1} - \frac{1}{2} \rightarrow \frac{1}{2(\chi^2)^{-1}} = \frac{n}{2} - 1$

$$\rightarrow \boxed{\chi^2 = 2\left(\frac{n}{2} - 1\right) = n - 2}$$

$\left(\overline{\chi^2} = n\right)$ $\leftarrow$ peak is not the mean
mean for χ^2
or peak is not in the middle $\rightarrow$ asymmetrical!

<u>Uniform distribution</u> has no preferred value!

$$\hookrightarrow f(x) = \begin{cases} \frac{1}{x_2 - x_1} & x_1 < x \leq x_2 \\ 0 & \text{otherwise} \end{cases}$$

1) $\bar{x} = \frac{x_1 + x_2}{2}$

2) $\overline{x^2} = \frac{x_2^3 - x_1^3}{3(x_2 - x_1)}$

3) $\sigma_x^2 = \overline{x^2} - \bar{x}^2 = \frac{(x_2 - x_1)^2}{12}$ $\rightarrow$ Proof $\rightarrow$ 5 exam x-credit points

Conditional Probability Distribution and Density Function:

$$F(x|M) = P_r(X \le x \mid M) = \frac{P_r\{X \le x, M\}}{P_r(M)}$$

$\{X \le x, M\}$: is joint ~~probability~~ event of all outcomes ξ such that $X(\xi) \le x$ and $\xi \subset M$

xi ('x' in Greek alphabet)

$F(x|M)$ is a valid probability distribution, if we check the 3 (three) axioms:

1) $0 \le F(x|M) \le 1$ ✓ (since $P_r(M) \ge P_r\{X \le x, M\}$)

2) $F(-\infty|M) = P_r(X \le -\infty \mid M) = 0$ ✓

$F(\infty|M) = P_r(X \le \infty \mid M) = 1$ ✓

3) $P_r(x_1 < X \le x_2 \mid M) = F(x_2|M) - F(x_1|M) \ge 0$ ✓

$\boxed{\text{Ch. 3: Several Random Variables.}}$

We will work with 2 random variable, extension to more can be done easily.

2D Distribution & Density Functions: $F(x,y) \equiv P_r(X \le x, Y \le y)$

Three-axioms check:

1) $0 \le F(x,y) \le 1$ ✓ (still probability)

2) $F(-\infty, 0) = 0$; $F(-\infty, y) = P_r(X \le -\infty, Y \le y) = 0$

$F(x, -\infty) = F(-\infty, -\infty) = 0$

$F(x, +\infty) = P_r(X \le x, Y < +\infty) = F_X(x)$ or "Marginal Dist. in X"

$F(+\infty, y) = P_r(X < +\infty, Y \le y) = F_Y(y)$ or "Marginal Dist. in Y"

$F(+\infty, +\infty) = P_r(X < +\infty, Y < +\infty) = \underline{1}$

3) $\Pr\left(x_1 \le X \le x_2,\ y_1 \le Y \le y_2\right) = F(x_2, y_2) - F(x_1, y_1)$

<u>2D density function</u>: $f(x,y) = \dfrac{\partial^2 F}{\partial x \partial y}$

partial derivatives since now we have more than one variable.

$$f(x,y)\,\cancel{dx}\,dy = \Pr\left(x < X \le x+dx,\ y < Y \le y+dy\right)$$

<u>Properties</u>: 1) Always positive: $f(x,y) \geq 0$, since $F(x,y)$ is non-decreasing in both x & y. Same as with 1D.

2) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\, f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cancel{dx\,dy} \frac{\partial^2 F}{\cancel{\partial x}\partial y} = F(\infty,\infty) - F(-\infty,-\infty)$$
$$= 1$$

3) $$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} du\,dv\, f(u,v)$$

4) Marginal density functions: $\displaystyle\int_{-\infty}^{\infty} dy\, f(x,y) = \underbrace{f_X(x)}_{\text{m.d.f. in } x}$

$$\int_{-\infty}^{\infty} dx\, f(x,y) = f_y(y) : \quad \text{m.d.f. in } y$$

5) $$\Pr\left(x_1 < X \le x_2,\ y_1 < Y \le y_2\right) = \int_{x_1}^{x_2}\int_{y_1}^{y_2} dx\,dy\ f(x,y)$$
$$= F(x_2, y_2) - F(x_1, y_1)$$

<u>2D Uniform Distributions</u> : $X \subset [x_1, x_2]$; $Y \subset [y_1, y_2]$

$f(x,y)$: density function : any point within this rectangular region has equal probability :

$$f(x,y) = \begin{cases} \dfrac{1}{(x_2-x_1)(y_2-y_1)} & \text{if } x_1 < X \le x_2 \; ; \; y_1 < Y \le y_2 \\ 0 & \text{otherwise} \end{cases}$$

Total area of the rectangle

What is the marginal density function in x ?

$$f_X(x) = \int_{-\infty}^{\infty} dy\, f(x,y) = \int_{y_1}^{y_2} dy \frac{1}{(x_2-x_1)(y_2-y_1)} = \frac{1}{(x_2-x_1)\cancel{(y_2-y_1)}} \cancel{(y_2-y_1)}$$

$$f_X(x) = \frac{1}{x_2-x_1}$$

$$f_Y(y) = \frac{1}{y_2-y_1}$$

$$f(x,y) = f_X(x) \cdot f_Y(y)$$

X & Y are statistically independent

Correlation between X & Y :

$$\overline{XY} = E[XY] = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\; xy\, f(x,y)$$

1D : $\overline{X} = \int_{-\infty}^{\infty} dx\; x\, f(x)$

What is the Correlation between 2 uniform random variable:

$$\overline{XY} = E[XY] = \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} dy \; xy \, \underbrace{\frac{1}{(x_2-x_1)(y_2-y_1)}}$$

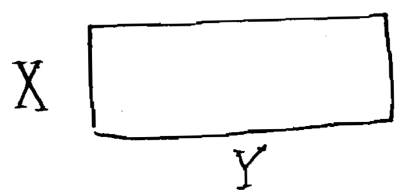
$$= \frac{1}{(x_2-x_1)(y_2-y_1)} \; \frac{(x_2^2-x_1^2)}{2} \cdot \frac{(y_2^2-y_1^2)}{2}$$

$$= \underbrace{\frac{x_1+x_2}{2}} \cdot \underbrace{\frac{y_1+y_2}{2}}$$

$$= \overline{X} \cdot \overline{Y}$$

(X & Y are stat. independent !)

Semiconductor substrate: (3.1.1.)



X, Y : Gaussian random variable

$\overline{X} = 1cm$; $\overline{Y} = 2cm$

$\sigma_x = 0.1cm = \sigma_y$

a) $P_r(X \geq 1.05cm, Y \geq 2.05cm) = 1 - F(1.05, 2.05)$

$$= Q\left(\frac{1.05-1}{0.1}, \frac{2.05-2}{0.1}\right) = Q\left(\frac{0.05}{0.1}\right)^2$$

$$= Q\left(\frac{0.05}{0.1}, \frac{0.05}{0.1}\right) = Q(0.5).Q(0.5)$$

$$= [Q(0.5)]^2 = 0.3085^2 = 0.09517$$

$$\begin{cases} F(x,y) = P_r(X \leq x, Y \leq y) \\ \text{1D: } F(x) = P_r(X \leq x) \\ \quad F(x) = 1 - Q\left(\frac{x-\overline{X}}{\sigma_x}\right) \end{cases}$$

$\downarrow$

X Gaussian

$Pr(x \le 0.95, Y \ge 2.05)$ $\bar{X} = 1$ $\bar{Y} = 2$

$\sigma = 0.1$

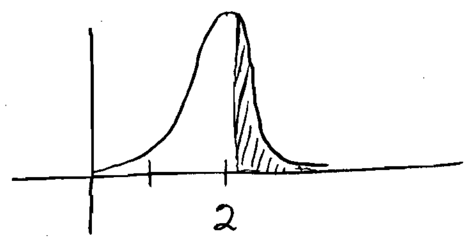
$F(x) = 1 - Q\left(\frac{x - \bar{x}}{\sigma}\right)$; $Q(-x) = 1 - Q(x)$

$= 1 - Q\left(\frac{0.95 - 1}{0.1}\right)$

$= 1 - Q(-0.5)$

$= 1 - (1 - Q(0.5))$

$= Q(0.5)$



$F(y) = \cancel{1-} Q\left(\frac{2.05 - 2}{0.1}\right)$

$= \cancel{1-} Q(0.5)$

$Q(0.5)(\cancel{1-} Q(0.5))$

$\cancel{= 0.3085(1 - .3085) = 0.2133}$

$\cancel{= 0.2133}$

$(0.3085)^2 = 0.0952$

b) $$P_r(X \le 0.95,\ Y \ge 2.05) = \frac{1}{\sqrt{2\pi}\,0.1}\int_{-\infty}^{0.95} dx\, e^{-\frac{(x-1)^2}{2\times 0.01}} \times \frac{1}{\sqrt{2\pi}\,0.1}\int_{2.05}^{\infty} dy\, e^{-\frac{(y-2)^2}{2\times 0.01}}$$

Change of variable. $x'^2 \equiv \frac{(x-1)^2}{0.01}$ or $x' = \frac{x-1}{0.1} \rightarrow dx = dx'\,0.1$ $\begin{cases} x = -\infty \rightarrow x' = -\infty \\ x = 0.95 \rightarrow x' = -0.5 \end{cases}$

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}\,dt \quad \text{(Appendix D)}$$

$$Q(x) = \frac{1}{\sqrt{2\pi}}\int_{x}^{\infty} e^{-\frac{u^2}{2}}\,du \quad \text{(Appendix E)}$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}}\frac{\cancel{0.1}}{\cancel{0.1}}\int_{-\infty}^{-0.5} dx'\, e^{-\frac{x'^2}{2}}}_{\phi(-0.5) = 1-\phi(0.5) = \underbrace{1 - 0.6915}_{0.3085}} \times \underbrace{\frac{1}{\sqrt{2\pi}}\frac{0.1}{\cancel{0.1}}\int_{0.5}^{\infty} dy'\, e^{-\frac{y'^2}{2}}}_{\underbrace{Q(0.5)}_{0.3085}} = 0.3085^2 = 0.09517$$

c) $\overline{XY} = \overline{X}.\overline{Y} = 2\ cm^2$

3.1.2

$$f(x,y) = \begin{cases} A e^{-(2x+3y)} & x \ge 0;\ y \ge 0 \\ 0 & x < 0,\ y < 0 \leftarrow \end{cases}$$

a) Find A: $F(\infty,\infty) = 1 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, f(x,y) = A \underbrace{\int_0^{\infty} e^{-2x}\,dx}_{\underbrace{\left[\frac{e^{-2x}}{-2}\right]_0^{\infty}}_{\frac{1}{2}}} \underbrace{\int_0^{\infty} e^{-3y}\,dy}_{\underbrace{\left[\frac{e^{-3y}}{-3}\right]_0^{\infty}}_{\frac{1}{3}}}$

$A = 6$

b)

(55)

b) $$Pr\left(X<\tfrac{1}{2},\ Y<\tfrac{1}{4}\right) = \underbrace{F_X(x)}_{\substack{\text{Marginal}\\ \text{Distribution}\\ \text{in}\\ x}} \cdot \underbrace{F_Y(y)}_{\substack{\text{Marginal}\\ \text{Distribution}\\ \text{in}\\ y}} = \int_{-\infty}^{x} \underbrace{f_X(\lambda)}_{\substack{\text{marginal}\\ \text{density}\\ \text{function}\\ \text{in } x}} d\lambda \times \int_{-\infty}^{y} \underbrace{f_Y(\varphi)}_{\substack{\text{marginal}\\ \text{density}\\ \text{function}\\ \text{in } y}} d\varphi$$

How to get $f_X(x)$ & $f_Y(y)$ from the 2D density function $f(x,y)$?

$$f_X(x) = \int_{-\infty}^{\infty} dy\, f(x,y) = 6 \underbrace{\int_{-\infty}^{\infty} dy\, e^{-3y}}_{\frac{1}{3}}\, e^{-2x} = 2e^{-2x}$$

$$f_Y(y) = \int_{-\infty}^{\infty} dx\, f(x,y) = 6 \underbrace{\int_{0}^{\infty} dx\, e^{-2x}}_{\frac{1}{2}}\, e^{-3y} = 3e^{-3y}$$

$$Pr\left(X<\tfrac{1}{2},\ Y<\tfrac{1}{4}\right) = \underbrace{\int_{0}^{\frac{1}{2}} 2e^{-2x}\,dx}\ * \underbrace{\int_{0}^{\frac{1}{4}} 3e^{-3y}\,dy}$$

$$= 2\left[\frac{e^{-2x}}{-2}\right]_0^{\frac{1}{2}} * 3\left[\frac{e^{-3y}}{-3}\right]_0^{\frac{1}{4}}$$

$$= \left(1-e^{-1}\right) * \left(1-e^{-\frac{3}{4}}\right)$$

$$= 0.3335$$

c) $$\overline{XY} = 6\int_0^{\infty} dx\, x e^{-2x} \int_0^{\infty} dy\, y e^{-3y} = 6\,\frac{\Gamma(2)}{2^2}\,\frac{\Gamma(2)}{3^2} = \frac{1}{6} = 0.1667$$

not studied distribution like
Gauss, Poisson, Maxwell, Rayleigh, χ^2 etc.

Conditional Probability involving 2 random variables :

So far : $$\Pr(A|B) = \frac{\Pr(A,B) \rightarrow \text{joint}}{\Pr(B) \longrightarrow \text{marginal}}$$

With 2 random variables : we can place the condition on the second random variable.

a) $B = \{Y \leq y\}$ B is the event that Y takes a value less or equal than y

$$\Pr(B) = \underbrace{F_Y(y)}_{\text{Marginal Pr. Distribution in } Y}$$

$$\underbrace{F_X(x \mid Y \leq y)}_{\text{Marg. P. Distribution in } X} = \frac{\Pr(X \leq x, Y \leq y)}{\Pr(Y \leq y)} = \frac{\overbrace{F(x,y)}^{\text{2D Pr. Distribution}}}{F_Y(y)}$$

b) Define now event B differently : $B = \{y_1 < Y \leq y_2\}$

$$F_X(x \mid y_1 < Y \leq y_2) = \frac{\Pr(X \leq x, y_1 < Y \leq y_2)}{\Pr(y_1 < Y \leq y_2)} = \frac{F(x,y_2) - F(x,y_1)}{F_Y(y_2) - F_Y(y_1)}$$

c) Write this other event $B = \{Y = y\}$ as $\lim_{\Delta y \to 0} \underbrace{\{y < Y \leq y + \Delta y\}}_{\text{similar to b)}}$

where $y_1 \rightarrow y$
$y_2 \rightarrow y + \Delta y$

$$F(x \mid Y = y) = \lim_{\Delta y \to 0} \frac{F(x, y+\Delta y) - F(x,y)}{F_Y(y+\Delta y) - F_Y(y)}$$

$$= \lim_{\Delta y \to 0} \frac{\dfrac{F(x, y+\Delta y) - F(x,y)}{\Delta y}}{\dfrac{F_Y(y+\Delta y) - F_Y(y)}{\Delta y}} = \frac{\dfrac{\partial F(x,y)}{\partial y}}{\dfrac{dF_Y}{dy}}$$

Recall: Calculus: $\underbrace{\dfrac{\partial f}{\partial y}}_{\text{partial derivative}} = \lim_{\Delta y \to 0} \dfrac{f(y+\Delta y) - f(y)}{\Delta y}$

Recall: $F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} du\,dv\, f(u,v) \rightarrow \dfrac{\partial F}{\partial y} = \int_{-\infty}^{x} du\, f(u,y)$

$\underbrace{F_Y(y)}_{\text{marg. dist.}} = \int_{-\infty}^{y} \underbrace{f_Y}_{\text{marg. density}}(v)\,dv \rightarrow \dfrac{dF_Y}{dy} = f_Y(y)$

$$\Rightarrow \boxed{F(x \mid Y=y) = \frac{\int_{-\infty}^{x} du\, f(u,y)}{f_Y(y)}}$$

d) $\dfrac{\partial}{\partial x}\left[F(x \mid Y=y) = \dfrac{\int_{-\infty}^{x} du\, f(u,y)}{f_Y(y)} \right]$

$$\boxed{f(x \mid Y=y) = \frac{f(x,y)}{f_Y(y)}}$$

This should be expected. We made a proof.

Switching $x \leftrightarrow y$ is also correct:

$$\boxed{f(y \mid X=x) = \frac{f(x,y)}{f_X(x)}}$$

$\hookrightarrow f(y \mid X=x) = \dfrac{f(x \mid Y=y) \cdot f_Y(y)}{f_X(x)}$

If we use: $f(y|X=x) = f(y|x)$ (a notation)

$$\rightarrow \boxed{f(y|x) = \frac{f(x|y)\cdot f_Y(y)}{f_X(x)}}$$ Bayes Theorem.

An application: Random noise elimination (signal processing)

Assume we are measuring X, noise will come along as a random variable N; so we are recording:

$$Y = X + N$$

The problem at hand is to retrieve X from the noisy signal Y → by working on $f_X(x|y)$

From the Bayes Theorem: $f_X(x|y) = \dfrac{f_Y(y|X=x)\, f_X(x)}{f_Y(y)}$

1) $f_Y(y|X=x) = f_N(n) = f_N(y-x)$

2) $f_Y(y) = \int_{-\infty}^{\infty} dx\, f(x,y) = \int_{-\infty}^{\infty} dx\, f_Y(y|X=x)\, f_X(x)$

$$f_Y(y|X=x) = \frac{f(x,y)}{f_X(x)}$$

$$= \int_{-\infty}^{\infty} dx\, f_N(y-x) f_X(x)$$

$$\rightarrow f_X(x|y) = \frac{f_N(y-x)\cdot f_X(x)}{\int_{-\infty}^{\infty} dx\, f_N(y-x)\, f_X(x)} = \frac{\partial}{\partial x} \ln\left[\int_{-\infty}^{\infty} dx\, f_N(y-x) f_X(x)\right]$$

To retrieve X : $\dfrac{\partial f_X(x|y)}{\partial x} = 0$

This is to find what value x is most probable given $Y=y$.

<u>Example:</u> Assume we would like to measure X where

$$f_X(x) = \begin{cases} b e^{-bx} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Assume broad-band noise (Gaussian) : $f_N(n) = \dfrac{1}{\sqrt{2\pi}\,\sigma_N} e^{-\frac{n^2}{2\sigma_N^2}}$

↓

(zero mean, st. dev is σ_N).

With these information we can retrieve X from $\underbrace{Y}_{\text{contaminated}} = \underbrace{X}_{\text{non-contaminated}} + N$

→ Apply the method described earlier:

$$\frac{\partial f_X(x|y)}{\partial x} = 0 = \frac{\partial}{\partial x}\left\{ \frac{f_N(y-x)\cdot f_X(x)}{\underbrace{\int_{-\infty}^{\infty} dx\, f_N(y-x)\, f_X(x)}_{\text{Not a function of } x \text{ after the integral}}} \right\}$$

$$\to\ 0 = \frac{\partial}{\partial x}\left[f_N(y-x)\cdot f_X(x) \right] = \frac{b}{\sqrt{2\pi}\,\sigma_N}\,\frac{\partial}{\partial x}\left[e^{-\frac{(y-x)^2}{2\sigma_N^2}} \cdot e^{-bx} \right]$$

$$\to\ 0 = \frac{\cancel{2}(y-x)}{\cancel{2}\sigma_N^2}\, e^{-\frac{(y-x)^2}{2\sigma_N^2}} \cdot e^{-bx} - b\, e^{-\frac{(y-x)^2}{2\sigma_N^2}} \cdot e^{-bx}$$

$$\to\ 0 = \frac{y-x}{\sigma_N^2} - b \quad \to \quad \boxed{x = y - b\sigma_N^2}$$

just plug in the measured Y to find the wanted X.

Statistical Independence b/w two Random Variables:

Def: X & Y are stat. independent if $f(x,y) = f_X(x) \cdot f_Y(y)$ (joint density function can be factored into the marginal density functions)

Consequences:

1) $E[XY] = E[X] \cdot E[Y]$ (can do integral in x and in y separately since $f(x,y) = f_X(x) f_Y(y)$)

2) $\underbrace{f_X(x|y)} = \dfrac{f(x,y)}{f_Y(y)} = \dfrac{f_X(x) \cdot \cancel{f_Y(y)}}{\cancel{f_Y(y)}} = \underbrace{f_X(x)}$

makes sense since X & Y are independent

3) $f_Y(y|x) = \dfrac{f(x,y)}{f_X(x)} = f_Y(y)$

Correlation b/w two random variables (X & Y)

$E[(X-\bar{X})(Y-\bar{Y})]$: covariance b/w X & Y

$E\left[\underbrace{\dfrac{X-\bar{X}}{\sigma_x}}_{\equiv \xi \text{ (zeta)}} \cdot \underbrace{\dfrac{Y-\bar{Y}}{\sigma_y}}_{\equiv \eta \text{ (eta)}}\right]$: normalized covariance b/w X & Y: ρ ("rho")

$\longrightarrow \boxed{\rho = \overline{\xi \cdot \eta}}$

$\bar{\xi} = \overline{\left(\dfrac{X-\bar{X}}{\sigma_X}\right)} = \dfrac{1}{\sigma_X} \cdot (\bar{X} - \bar{X}) = 0$; $\bar{\eta} = 0$

$\sigma_\xi^2 = \overline{(\xi - \bar{\xi})^2} = \overline{\xi^2} = \dfrac{1}{\sigma_X^2}\overline{(X-\bar{X})^2} = \dfrac{1}{\cancel{\sigma_X^2}} \cdot \cancel{\sigma_X^2} = 1$

$\sigma_\eta^2 = 1$

ξ, η are random variable with zero means and unit variances.

A property for $\rho = \overline{\xi \cdot \eta}$:

$$0 \le \underbrace{\overline{(\xi \pm \eta)^2}} = \overline{(\xi^2 + \eta^2 \pm 2\xi\eta)} = \overline{\xi^2} + \overline{\eta^2} \pm 2\overline{\xi\eta}$$

$$= 1 + 1 \pm 2\rho$$

$$0 \le 2 \pm 2\rho = 2(1 \pm \rho) \quad \Big\{ \quad 0 \le 1 \pm \rho \quad \begin{cases} 0 \le 1+\rho \to \rho \ge -1 \\ 0 \le 1-\rho \to \rho \le 1 \end{cases}$$

$$\downarrow \quad \boxed{-1 \le \rho \le 1}$$

The normalized covariance ρ can only takes values b/w -1 and 1

→ What is ρ if X & Y are stat. independent?

$$\rho = \overline{\xi \cdot \eta} \underset{\downarrow}{=} \overline{\xi} \cdot \overline{\eta} = 0$$

$f(x,y) = f_X(x) \cdot f_Y(y)$
→ can do the integral (for the ensemble average) separably for X and for Y

HW4 is due next Tuesday 4/17. (1.2; 2.1; 3.3; 4.4)

Ch.4 Sampling Theory (on Random Variables)

Goals: difference between $\begin{cases} \text{mean} & \& \text{ sample mean} \\ \text{variance} & \& \text{ sample variance} \end{cases}$

- Population with N elements (very large number) $\rightarrow \begin{cases} \text{Mean}: \overline{X} = \int_{-\infty}^{\infty} dx\, x f(x) \\ \text{Variance}: \sigma_X^2 = \overline{X^2} - \overline{X}^2 \end{cases}$

$\downarrow$

(of resistors, transistors, etc).

Quality control: take samples of this large population, and test the samples.

- Samples: with n elements (usually manageable)
- To reflect the population behavior, samples are picked randomly

  ↳ Sample mean: $\hat{\overline{X}} = \frac{1}{n} \sum_{i=1}^{n} x_i$

  ↳ Sample variance $s^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \hat{\overline{X}})^2$

- Difference: $\overline{X}, \sigma_X^2$ ↑ ↑ one one

  $\hat{\overline{X}}, s^2$ ↑ ↑ $\binom{N}{n}$ many

  sample-specific

  (by picking randomly you can reduce the variation from sample to sample)

  They are random variables themselves

  ↳ Can define: $\begin{cases} \text{Mean of the sample means} \\ \text{Variance of the sample means} \\ \text{Mean of the sample variance} \end{cases}$

→ Mean of the sample mean: $\overline{\hat{X}} = \overline{\frac{1}{n}\sum_{i=1}^{n} x_i} = \frac{1}{n}\sum_{i=1}^{n} \underbrace{\overline{x_i}} = \overline{X}$

"Mean for each element in a sample"

Population of 10^7 resistors: each resistor carries a value for X (the resistance). Each resistor could be an element in a sample.
↓
a random variable → the mean for an element in a sample is: $\overline{X}$

→ Variance of the sample mean: $var(\hat{X}) = \overline{\hat{X}^2} - \overline{X}^2$

$$= \frac{\sigma_x^2}{n}\left(\frac{N-n}{N-1}\right)$$

If the population is very large: $N \to \infty$: $var(\hat{X}) \xrightarrow{N\to\infty} \frac{\sigma_x^2}{n}$

This makes sense: if we take samples of larger n the variance between the sample means is less than that for samples with less elements.

→ Mean of sample variance: $\overline{S^2} = \frac{n-1}{n}\sigma_x^2$:

we can define a different sample variance $\tilde{S}^2$:

$$\tilde{S}^2 = \frac{n}{n-1} S^2 \Rightarrow \overline{\tilde{S}^2} = \frac{n}{n-1}\overline{S^2}$$

$$= \frac{\cancel{n}}{\cancel{n-1}}\frac{\cancel{n-1}}{\cancel{n}}\sigma_x^2$$

such that its mean, $\overline{\tilde{S}^2}$, is the actual variance σ_x.

4.4 Sampling Distribution and Confidence Interval:

Example 4.4.2:

We have a very large population of resistors $\bar{R} = 100\Omega$; $S = 4\Omega$. What are the interval limits on the sample mean for a confidence level of 95% ?

Sample mean $\bar{\hat{X}} = \frac{1}{n}\sum_{i=1}^{n} \underbrace{X_i}_{\text{value of one resistor, a random variable}}$

$n > 30 \rightarrow X_i$ Gaussian distribution

$\rightarrow Z \equiv \dfrac{\bar{\hat{X}} - \bar{X}}{\frac{\sigma_x}{\sqrt{n}}}$

$\frac{\sigma_x}{\sqrt{n}}$: St. dev. of the sample mean: it is a number

Since $\bar{\hat{X}}$ is a sum of gaussian random variables.

$\rightarrow Z$ is a gaussian random variable:

$$\bar{Z} = \frac{\bar{\hat{X}} - \bar{X}}{\frac{\sigma_x}{\sqrt{n}}} = 0$$

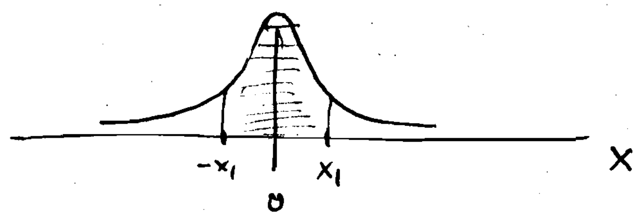
$$\mathrm{var}(Z) = 1$$

$n < 30$, X_i follows Student's T distribution

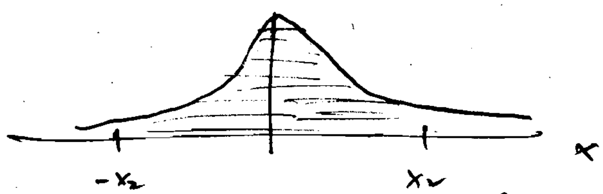
$$T \equiv \frac{\bar{\hat{X}} - \bar{X}}{\frac{\tilde{S}}{\sqrt{n}}}$$

degree of freedom $\nu = n - 1$ ("nu")

Interval limits & confidence level are connected via a probability distribution:



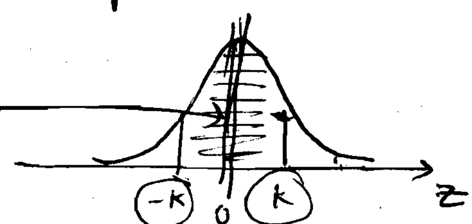
$(-x_1, x_1) \longrightarrow$ prob = area under the curve (shaded area)



For the same distribution: interval limits $(-x_2, x_2)$ are associated with larger probability.

We can also ~~associate~~ associate each probability or confidence level with certain interval limits:

95% of confidence level $\longrightarrow$



(z is Gaussian w/ zero mean and unit variance)

where $(-k, k)$ are limits for z

Table 4-1:

Conf level	Limits
95%	$k = 1.96$
90% How?	$k = 1.64$

Two-sided test:

$$\underbrace{P_r(-k \le z \le k)}_{} = 0.9$$

$$\boxed{F(z) = 1 - Q\left(\frac{z - \bar{z}}{\sigma_z}\right)} \quad F(k) - F(-k) = \cancel{1} - Q(k) \cancel{-1} + \underbrace{Q(-k)}_{1 - Q(k)}$$

$0.9 = 1 - 2Q(k)$

$$\rightarrow Q(k) = \frac{-0.1}{-2} = 0.05 \longrightarrow Q\text{ table} \rightarrow k \cong 1.645$$

pg 434

Confidence Level	Limits
99.99 %	$k = 3.89$
99.9 %	$k = 3.29$
99 %	$k = 2.58$
95 %	$k = 1.96$
90 %	$k = 1.645$

(Two-sided tests)

For a confidence level of 99% : $P_r(-k \le z \le k) = 0.99$

$$F(k) - F(-k) = 1 - 2Q(k)$$

$$Q(k) = \frac{0.01}{2} = 0.005 \rightarrow \text{Q-Table}$$

$$k = 2.575 \approx 2.58$$

From each confidence level we found limits for k, now what are the interval limits for the sample mean $\hat{X}$?

$$-k \le z \le k \quad \text{or} \quad \left\{ -k \le \frac{\hat{X} - \overline{X}}{\frac{\sigma}{\sqrt{n}}} \le k \right\} \frac{\sigma}{\sqrt{n}} \text{ then add } \overline{X}$$

$$\underbrace{\overline{X} - k\frac{\sigma}{\sqrt{n}}}_{\text{lower limit}} \le \hat{X} \le \underbrace{\overline{X} + k\frac{\sigma}{\sqrt{n}}}_{\text{upper limit}}$$

(depend on the true mean $\overline{X}$, k (from the confidence level); the standard dev. σ; and the number of elements in the sample)

Consequences: 1) ↑ confidence level, ↑ k → broader interval } consequence of randomness (gain certainty, loose information and viceversa)

2) ↑ in sample size (by 100x) narrow the interval (by 10^{-1})

In our example: $\overline{X} = 100\,\Omega$

Sample variance: $\frac{\sigma^2}{n}\left(\frac{N-n}{N-1}\right) \xrightarrow{N\to\infty} \frac{\sigma^2}{n}$

Sample standard dev: $s = \longrightarrow \frac{\sigma}{\sqrt{n}}$

95% $\to$ k = 1.96

$$\to \quad 100 - 1.96\times 4 \le \hat{\overline{X}} \le 100 + 1.96\times 4$$

$$92.16 \le \hat{\overline{X}} \le 107.84$$

These are the interval limits for the sample mean at 95% confidence level.

At 99% confidence level the limits are:

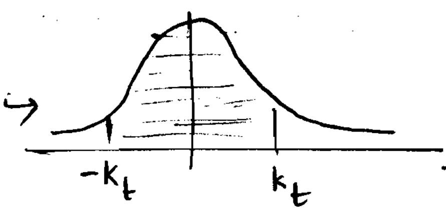
$$100 - 2.58\times 4 \le \hat{\overline{X}} \le 100 + 2.58 \times 4$$

$$89.68 \le \hat{\overline{X}} \le 110.32$$

All this was for sample size $n > 30$. For smaller samples at $n < 30$: use Student's T distribution on

$$T \equiv \frac{\hat{\overline{X}} - \overline{X}}{\frac{\tilde{s}}{\sqrt{n}}} \; ;$$

$\to$ degree of freedom $\nu = n-1$ ("nu")



$$P_r(-k_t \le T \le k_t) = 0.95 \quad (95\%)$$

$$F_t(k_t) - F_t(-k_t) = 0.95$$

F_t is given in Appendix F (pg 436): can verify: $F_t(-t) = 1 - F_t(t)$

$$2F_t(k_t) - 1 = 0.95 \longrightarrow F_t(k_t) = \frac{1.95}{2} = 0.975$$

$\to$ Table (pg 436) $\to k_t = 2.306$

$\nu = 8$
$(n = 9)$

Confidence Level	$\nu = 8$ k_t (limits)	$\nu = 20$ k_t (limits)
99.99%		
99.9%		
99%		
95%	2.306	2.086
90%		

Interval limits for sample mean.

95%; $\nu = 8$: $\quad -k_t \le T \le k_t \quad$ or $\quad -k_t \le \dfrac{\hat{\bar{X}} - \bar{X}}{\frac{\tilde{S}}{\sqrt{n}}} \le k_t$

$$\rightarrow \bar{X} - k\frac{\tilde{S}}{\sqrt{n}} \le \hat{\bar{X}} \le \bar{X} + k\frac{\tilde{S}}{\sqrt{n}}$$

Our example: $\bar{X} = 100\,\Omega$; $\quad S = 4\,\Omega$

$\tilde{S}^2 = \dfrac{n}{n-1} S^2 \quad$ or $\quad \tilde{S} = \sqrt{\dfrac{n}{n-1}}\, S$

$$\bar{X} - k\frac{S}{\sqrt{n-1}} \le \hat{\bar{X}} \le \bar{X} + k\frac{S}{\sqrt{n-1}}$$

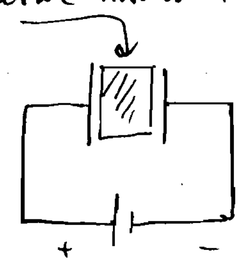
$$100 - 2.306\frac{4}{\sqrt{8}} \le \hat{\bar{X}} \le 100 + 2.306 \times \frac{4}{\sqrt{8}}$$

$$96.74 \le \hat{\bar{X}} \le 103.26$$

Interval Limits, Confidence Levels, Hypothesis Testing:

One-sided Testing:

Manufacturer claims their capacitors have $V_{breakdown} = 300V$ (dielectric inserted b/w the plates).



At $X = \cancel{300V}$ $V_{Breakdown}$, the capacitor stops working.

Hypothesis:

- Manufacturer is claiming $X \geq 300\,V$ (higher side, or one-sided)
- $n = 100$; $\hat{X} = 290V$; $\tilde{S} = 40V$
- Confidence level of 99%

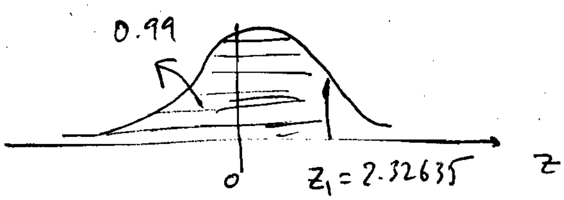
→ Reject?
→ Confirm?

Since $n = 100 > 30$ → we work with z (Gaussian). Let's find the limits for z at 99%:

$$P_r(z \leq z_1) = 0.99 = F(z_1) = 1 - Q(z_1)$$

$$Q(z_1) = 1 - 0.99 = 0.01 \rightarrow \text{Appendix E}: \; z_1 = 2.32$$

(Bottom): $z_1 = 2.32635$



$$z_1' = \frac{\hat{X} - \bar{X}}{\frac{\sigma}{\sqrt{n}}} = \frac{\hat{X} - \bar{X}}{\frac{40}{\sqrt{100}}} = \frac{290 - 300}{\frac{40}{\sqrt{100}}}$$

3.2.1

~~pg~~ 58 (Notes)

$$f_X(x|y) = \frac{f_N(y-x)\cdot f_X(x)}{\int_{-\infty}^{\infty} dx\, f_N(y-x)\, f_X(x)} \quad \text{or} \quad f(x|y) = \frac{f_N(y-x)\cdot f(x)}{\int_{-\infty}^{\infty} dx\, f_N(y-x) f(x)}$$

Noise: uniform distribution $\begin{cases} 0 = \frac{x_1+x_2}{2} \longrightarrow \\ 12 = \frac{(x_2-x_1)^2}{12} \rightarrow x_2 - x_1 = 12 \end{cases}$ $\Big\}\ x_1 = -6,\ x_2 = 6$

pg48 (Notes)

$$f(x) = \begin{cases} \frac{1}{x_2-x_1} & x_1 \le x \le x_2 \\ 0 & \text{elsewhere} \end{cases} \quad \Bigg| \quad \bar{X} = \frac{x_1+x_2}{2}, \quad \sigma_X^2 = \frac{(x_2-x_1)^2}{12}$$

$$f_N(n) = \begin{cases} \frac{1}{12}; & -6 < x \le 6 \\ 0 & \text{elsewhere} \end{cases}$$

X: Rayleigh with $\bar{X} = 10$.

pg43 (Notes)

$$f_R(r) = \begin{cases} \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} & r \ge 0 \\ 0 & r < 0 \end{cases} \quad \Bigg| \quad \bar{R} = \sigma\sqrt{\frac{\pi}{2}} \rightarrow \sigma = \frac{10}{\sqrt{\frac{\pi}{2}}}, \quad \sigma_R^2 = 0.429\,\sigma^2$$

$$f(x) = \left\{ x\frac{\pi}{200} e^{-\frac{x^2}{\frac{400}{\pi}}} \right.$$

$\frac{1}{12}(-6 < y-x \le 6)$

c) $$f(x|y) = \frac{f_N(y-x)\cdot \frac{\pi x}{200} e^{-\frac{x^2}{\frac{400}{\pi}}}}{\int_{-\infty}^{\infty} dx\, f_N(y-x)\, \frac{\pi x}{200} e^{-\frac{x^2}{\frac{400}{\pi}}}} = \frac{\cancel{\frac{\pi}{2400}}\, x e^{-\frac{x^2}{\frac{400}{\pi}}}}{\int_{y-6}^{y+6} dx\, \cancel{\frac{\pi}{2400}}\, x e^{-\frac{x^2}{\frac{400}{\pi}}}}$$

$-6 < y - x \rightarrow x < y+6$

$y - x \le 6 \rightarrow y-6 < x$

integrals.wolfram.com: $\int dx\, x e^{-ax^2} = -\frac{e^{-ax^2}}{2a}$

$a = \frac{\pi}{400}$

$$\boxed{f(x|y) = \frac{x e^{-\frac{x^2}{\frac{400}{\pi}}}}{\frac{200}{\pi}\left[e^{-\frac{\pi}{400}(y-6)^2} - e^{-\frac{\pi}{400}(y+6)^2}\right]}}$$

Not good for $y = 0; 6$

(Good for $y \geq 12$)

$y=0:\quad \int_{-6}^{6} dx\; x\, e^{-\frac{x^2}{\frac{400}{\pi}}} = \int_{0}^{6} dx\; x\, e^{-\frac{x^2}{\frac{400}{\pi}}}$

Rayleigh distribution $f(x)$ is zero for $x \leq 0$

No truncation needed

$y=6:\quad \int_{0}^{12} dx\; x\, e^{-\frac{x^2}{\frac{400}{\pi}}} =$

$y=12=\quad \int_{6}^{18} dx\; x\, e^{-\frac{x^2}{\frac{400}{\pi}}}$

$$= \frac{200}{\pi}\left[1 - e^{-\frac{36\times\pi}{400}}\right]$$

$0 \leq y \leq 6 \;\rightarrow\; \int_{y-6}^{y+6} dx\; x\, e^{-\frac{x^2}{\frac{400}{\pi}}} = \frac{200}{\pi}\left[1 - e^{-\frac{(y+6)^2\pi}{400}}\right]$

$6 < y \;\rightarrow\; \int_{y-6}^{y+6} dx\; x\, e^{-\frac{x^2}{\frac{400}{\pi}}} = \frac{200}{\pi}\left[e^{-\frac{(y-6)^2\pi}{400}} - e^{-\frac{(y+6)^2\pi}{400}}\right]$