Exercise 2-5-1 :

X = G.R.V. $\overline{X} = 1$ & $\sigma_X^2 = 4$

$$\Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}\,2} e^{-\frac{(x-1)^2}{8}}$$

a) $P_r(X < 0) = F(0) = \underbrace{\frac{1}{\sqrt{2\pi}\,2}\int_{-\infty}^{0} e^{-\frac{(x-1)^2}{8}} dx}$ = (*)

(Prob. that X has a negative value)

$$= \Phi\left(\frac{\overset{0}{x} - 1}{2}\right) = \Phi(-0.5) = 1 - \Phi(0.5) = 1 - 0.6915$$

Table pg 432

$$= 0.3085$$

$$\boxed{\Phi(-x) = 1 - \Phi(x)}$$

Proof

$$\Phi(-x) = \frac{1}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{-x} e^{-\frac{u^2}{2}} du} = \frac{1}{\sqrt{2\pi}}\left\{\underbrace{\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du}_{\sqrt{2\pi}} - \int_{-x}^{\infty} e^{-\frac{u^2}{2}} du\right\}$$

$$= 1 - \underbrace{\boxed{\frac{1}{\sqrt{2\pi}}\int_{-x}^{\infty} e^{-\frac{u^2}{2}} du}}_{\text{change of dummy } u:\ u \to -u} = \boxed{1 - \Phi(x)}$$

$$\frac{1}{\sqrt{2\pi}}\int_{x}^{-\infty} e^{-\frac{u^2}{2}}(-du) = \boxed{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{u^2}{2}} du} = \Phi(x)$$

(*) $F(0) = \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{0} e^{-\frac{(x-1)^2}{8}} dx = \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{-1} e^{-\frac{y^2}{8}} dy$

change of var.

$x - 1 = y \rightarrow dx = dy$

$x = -\infty \rightarrow y = -\infty$

$x = 0 \rightarrow y = -1$

Limits not conforming with the formula we had.

$$F(0) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\left(\frac{x^2 - 2x + 1}{8}\right)} dx$$

→ Shorter to use $\Phi(x)$ Table in pg 432.

b) $Pr(1 < X \leq 2) = F(2) - F(1)$

$$= \Phi\left(\frac{2-1}{2}\right) - \Phi(0) = \Phi(0.5) - \Phi(0)$$

$$= 0.6915 - 0.5 = 0.1915$$

$\boxed{\text{G.R.V: } F(x) = \Phi\left(\frac{x - \bar{X}}{\sigma_X}\right)}$

$\begin{cases} \bar{X} = 1 \\ \sigma_X^2 = 4 \end{cases}$

$F(x) = Pr(X \leq x)$

c) $\underbrace{Pr(X > 4)} = 1 - Pr(X \leq 4) = 1 - F(4) = 1 - \Phi\left(\frac{4-1}{2}\right) = 1 - \Phi(1.5)$

$$= 1 - 0.9332 = 0.0668$$

Exercise 2.5.2:

a) $\overline{(X - \bar{X})^4} = \overline{X^4} - 4\,\overline{X^3}\,\bar{X} + 6\,\overline{X^2}\,\bar{X}^2 \underbrace{- 4\bar{X}^4 + \bar{X}^4}_{-3\bar{X}^4}$

$\sigma_X^2 = \overline{(X - \bar{X})^2} = \overline{X^2} - \bar{X}^2 \rightarrow \sigma_X^4 = \overline{X^2}^2 - \bar{X}^4 - 2\overline{X^2}\,\bar{X}^2$

$$\Rightarrow \sigma_X^4 \neq \overline{(X - \bar{X})^4}$$

1st method: $\overline{X^4}$; $\overline{X^3}$; $\overline{X^2}$ → plug in

2nd method: $\int_{-\infty}^{\infty} dx \left(x^4 - 4x^3\bar{x} + 6x^2\bar{x}^2 - \cancel{4x\bar{x}^3} + \bar{x}^4\right) f_X(x)$

or change of variable: $\int_{-\infty}^{\infty} dx\, (x - \bar{x})^4 \frac{1}{\sqrt{2\pi}\, 2} e^{-\frac{(x-\bar{x})^2}{8}}$

$\bar{X} = 1$
$\sigma_X^2 = 4$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, (x-1)^4 e^{-\frac{(x-1)^2}{8}}$$

$$\overline{(X-\bar{x})^4} = \frac{1}{2\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} dy\, \underset{\text{even}}{y^4}\, \underset{\text{even}}{e^{-\frac{y^2}{8}}}}_{2\int_0^\infty dy\, y^4\, e^{-\frac{y^2}{8}}} = \frac{1}{\sqrt{2\pi}} \frac{\Gamma\left(\frac{5}{2}\right)}{2\,\frac{1}{8^{5/2}}} .$$

$$y = x - 1 \qquad \begin{cases} n = 4 \\ r^2 = \frac{1}{8} \end{cases} \qquad r = \frac{1}{\sqrt{8}}$$

$$= \frac{8^{5/2}}{2\sqrt{2\pi}}\, \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{8^{5/2}}{2\sqrt{2\pi}}\, \frac{3}{2}\,\frac{1}{2}\sqrt{\pi} = \frac{8^{5/2}\, 3}{8\sqrt{2}} = \frac{8^{3/2}\cdot 3}{2^{1/2}} = \frac{2^{3/2}\, 4^{3/2}\, 3}{2^{1/2}} = 48$$

$$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(1+\frac{3}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\Gamma\left(1+\frac{1}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\,\underbrace{\Gamma\left(\frac{1}{2}\right)} = \frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$$

If I kept σ_x in:

$$\boxed{\overline{(X-\bar{x})^4}} = \frac{1}{\sqrt{2\pi}\,\sigma_x}\int_{-\infty}^{\infty} dx\,(x-\bar{x})^4\, e^{-\frac{(x-\bar{x})^2}{2\sigma_x^2}} = \frac{1}{\sqrt{2\pi}\,\sigma_x}\underbrace{\int_{-\infty}^{\infty} dy\, y^4\, e^{-\frac{y^2}{2\sigma_x^2}}}_{n=4;\ r = \frac{1}{\sqrt{2}\,\sigma_x}}$$

$$y \equiv x - \bar{x}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x}\,\frac{\Gamma\left(\frac{5}{2}\right)}{2\left(\frac{1}{\sqrt{2}\,\sigma_x}\right)^5} = \boxed{3\sigma_x^4}$$

Generally (repeating previous for any even n):

$$\overline{(X-\bar{X})^n} = \begin{cases} 0 & \text{if } n \text{ is odd. (Why?)} \\ 1.3.5 \ldots (n-1)\,\sigma_x^n & \text{if } n \text{ is even.} \end{cases}$$

(e.g. $n = 4 \rightarrow \overline{(X-\bar{X})^4} = 1.3.\sigma_x^4$)

Ch1 : 7.1 Digital communication : 0 or 1

$Pr(T_0) = 0.4$; $Pr(T_1) = 0.6$

$Pr(R_1 | T_0) = 0.08$; $\underline{Pr(R_0 | T_1)} = 0.05$ (errors)

a) Pr. that a received 0 was transmitted as 0 : ("received as 0" is the condition)

$\underbrace{Pr(T_0 | R_0)}$: prob. that given a zero was received, what is the prob. it was transmitted as zero

1) $$\boxed{Pr(T_0 | R_0)} = \frac{Pr(T_0, R_0)}{Pr(R_0)} = \frac{Pr(R_0, T_0)}{Pr(R_0)}$$

$$= \frac{Pr(R_0 | T_0)\, Pr(T_0)}{Pr(R_0)} = \frac{0.92 \times 0.4}{0.398} = \boxed{0.9246}$$

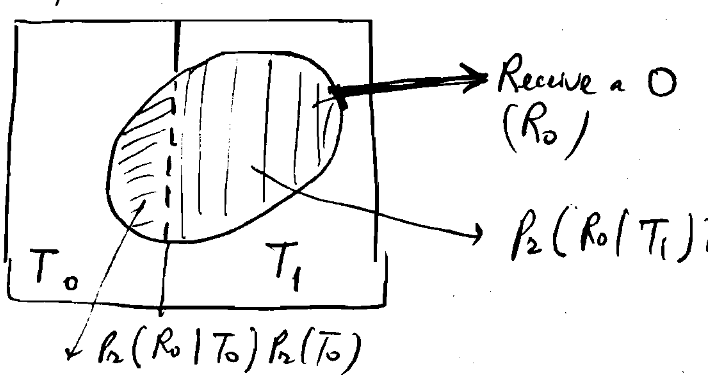
$Pr(T_0)$: 0.4

$$Pr(R_0 | T_0) = 1 - Pr(R_1 | T_0) = 1 - 0.08 = 0.92$$

$$Pr(R_0) = Pr(R_0 | T_0)\, Pr(T_0) + Pr(R_0 | T_1)\, Pr(T_1)$$

Total prob.

$$= 0.92 \times 0.4 + 0.05 \times 0.6 = 0.398$$



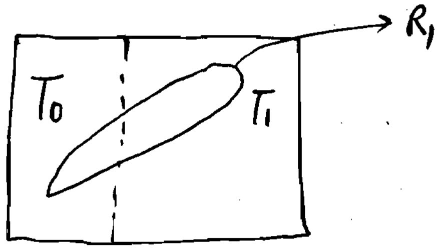
b) Pr that a receive 1 was transmitted as 1 ("received as 1" is the condition) ✓

$$\boxed{Pr(T_1 \mid R_1)} = \frac{Pr(T_1, R_1)}{Pr(R_1)} = \frac{Pr(R_1, T_1)}{Pr(R_1)} = \frac{Pr(R_1|T_1)\,Pr(T_1)}{Pr(R_1)}$$

$$= \frac{0.95 \times 0.6}{0.602} = \boxed{0.9468}$$

$$Pr(R_1|T_1) = 1 - Pr(R_0|T_1) = 1 - 0.05 = 0.95$$

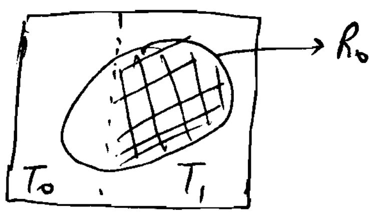
$$Pr(R_1) = Pr(R_1|T_0)\,Pr(T_0) + Pr(R_1|T_1)\,Pr(T_1)$$

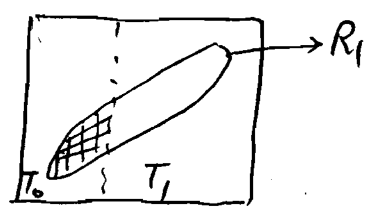


$$= 0.08 \times 0.4 + 0.95 \times 0.6$$

$$= 0.602 \qquad (\text{check: } = 1 - Pr(R_0))$$

c) Prob. that any symbol is received in error





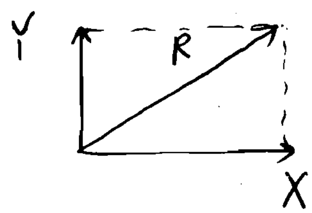
$$Pr(R_0|T_1)\,Pr(T_1) + Pr(R_1|T_0)\,Pr(T_0)$$

$$= 0.05 \times 0.6 + 0.08 \times 0.4 = 0.062$$

Probability density functions:

- Gaussian $(\bar{X}, \sigma_x^2)$ ✓
- Power ✓
- Rayleigh ✓
- Maxwell ✓
- Chi-Square
- t-student
- Uniform.

Rayleigh:



$$R = \sqrt{X^2 + Y^2}$$

X, Y: Gaussian w/ zero mean and same variance σ^2

$\rightarrow R$: follows Rayleigh distribution

$$f_R(r) = \begin{cases} \dfrac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} & r \geq 0 \\ 0 & r < 0 \end{cases}$$

(a magnitude cannot be negative)

Not symmetrical!

$$\bar{R} = \int_0^\infty dr\, r f_R(r) = \int_0^\infty dr\, r \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} = \frac{1}{\sigma^2}\int_0^\infty dr\, r^2 e^{-\frac{r^2}{2\sigma^2}} = \frac{1}{\sigma^2} \frac{\Gamma\left(\frac{3}{2}\right)}{2\left(\frac{1}{2\sigma^2}\right)^{3/2}}$$

$$\Rightarrow \bar{R} = \sigma\sqrt{\frac{\pi}{2}}$$

$$\overline{R^2} = \int_0^\infty dr\, r^2 f_R(r) = \frac{1}{\sigma^2}\int_0^\infty dr\, r^3 e^{-\frac{r^2}{2\sigma^2}} = \frac{1}{\sigma^2} \frac{\Gamma(2)}{2\left(\frac{1}{2\sigma^2}\right)^2}$$

$$\Rightarrow \overline{R^2} = 2\sigma^2 \quad \Rightarrow \boxed{\sigma_R^2 = \overline{R^2} - \bar{R}^2 = 2\sigma^2 - \frac{\pi}{2}\sigma^2 = 0.429\,\sigma^2}$$

Maxwell: 3D version of the Rayleigh

If v is the speed of gas molecules with cartesian components v_x, v_y, v_z that are Gaussian random variables with zero mean and variance $\sigma^2 = \frac{kT}{m}$ (k = Boltzmann constant $= 1.38 \times 10^{23} \frac{W\cdot s}{°K}$; T is the temp in °K, m: mass of gas molecule in kg)

→ The speed is not a Gaussian but Maxwellian:

$$f_v(v) = \begin{cases} \sqrt{\frac{2}{\pi}} \frac{v^2}{\sigma^3} e^{-\frac{v^2}{2\sigma^2}} & v \geq 0 \\ 0 & v < 0 \end{cases}$$

(speed cannot be negative)

1) $$\bar{v} = \int_0^\infty dv\, v\, f_v(v) = \sqrt{\frac{2}{\pi}} \int_0^\infty dv \frac{v^3}{\sigma^3} e^{-\frac{v^2}{2\sigma^2}} = \sqrt{\frac{2}{\pi}} \frac{1}{\sigma^3} \frac{\Gamma(2)}{2\left(\frac{1}{\sqrt{2\sigma^2}}\right)^4}$$

$$\Rightarrow \bar{v} = \sqrt{\frac{8}{\pi}}\,\sigma$$

2) $$\overline{v^2} = \int_0^\infty dv\, v^2 f_v(v) = \sqrt{\frac{2}{\pi}} \frac{1}{\sigma^3} \int_0^\infty dv\, v^4 e^{-\frac{v^2}{2\sigma^2}} = \sqrt{\frac{2}{\pi}} \frac{1}{\sigma^3} \frac{\Gamma(\frac{5}{2})}{2\left(\frac{1}{\sqrt{2\sigma^2}}\right)^5}$$

$$\Rightarrow \overline{v^2} = 3\sigma^2$$

variance for each cartesian component.

$\updownarrow$

equipartition theorem: each cartesian component carries $\frac{1}{3}$ of the average kinetic energy.

3) $$\boxed{\sigma_v^2 = \overline{v^2} - \bar{v}^2 = 3\sigma^2 - \frac{8}{\pi}\sigma^2 = 0.453\,\sigma^2}$$