Continuous random variable:

$$F_X(x) = \begin{cases} 0 & -\infty < x \le 0 \\ 1 - e^{-2x} & 0 \le x < \infty \end{cases} \quad \Bigg| \quad F_X(x) \equiv P_r(X \le x)$$

1) $P_r(X > 0.5) = 1 - F_X(0.5) = 1 - (1 - e^{-2\times 0.5}) = \frac{1}{e} = 36.63\%$

2) $P_r(X \le 0.25) = F_X(0.25) = 1 - e^{-0.5} = 39.36\%$

3) $P_r(0.3 < X \le 0.7) = F(0.7) - F(0.3) = e^{-0.6} - e^{-1.4} = 30.22\%$

2-2.3

$$F_X(x) = \begin{cases} A\{1 - e^{-(x-1)}\} & ; \quad 1 < x < \infty \\ 0 & ; \quad -\infty < x \le 1 \end{cases} \quad \Bigg| \quad F_X(x) \equiv P_r(X \le x)$$

given →

a) Find A such that F_X be a valid probability distribution function:

Axiom #2 (Probability of the certain event is 1)

$$F_X(\infty) = P_r(X \le \infty) = 1 = A\{1 - \underbrace{e^{-\infty}}_{0}\} \Rightarrow A = 1$$

b) $F_X(2) = \dots 1 - e^{-(2-1)} = 1 - \frac{1}{e} = 0.632$

c) $P_r(2 < X < \infty) = F_X(\infty) - F_X(2) = 1 - 1 + \frac{1}{e} = \frac{1}{e} = 0.367$

d) $P_r(1 < X \le 3) = F(3) - F(1) = e^{-0} - e^{-2} = 1 - \frac{1}{e^2} = 0.8648$

<u>Probability Density Function</u>: (derivative of the Probability Distribution Function w.r.t. the random variable)

$$\hookrightarrow f(x) \equiv \frac{dF}{dx}$$

$$= \lim_{\varepsilon \to 0} \frac{F(x+\varepsilon) - F(x)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{P_r(x < X \leq x+\varepsilon)}{\varepsilon}$$

$$\Rightarrow f(x)\,dx = \underbrace{P_r(x < X \leq x+dx)}_{dF} \Bigg\} \rightarrow \boxed{F(x) = \int_{-\infty}^{x} f(\tilde{x})\,d\tilde{x}}$$

This is how we obtain the distribution function from the density function

→ Geometrically: the distribution function is the area under the curve defined by the density function. The density funct. is the slope of the distribution function.

<u>Properties of the density function $f(x)$</u>:

1) Can $f(x)$ be negative? or can the slope of a distribution funct. ever be negative?. $F(x) \equiv P_r(X \leq x)$

$$F(1000) < F(10\,000) < F(1000000)$$

F is always increasing with x (larger interval, larger probability) → its slope or the density function will always be positive.

2) $$\int_{-\infty}^{\infty} f(x)\,dx = F(+\infty) = 1$$

3) $$\int_{-\infty}^{x} f(\tilde{x})\,d\tilde{x} = F(x) - F(-\infty) = F(x)$$

4) $$\int_{x_1}^{x_2} f(\tilde{x})\,d\tilde{x} = F(x_2) - F(x_1) = P_r(x_1 < X \leq x_2)$$

Connection between two random variables X and Y

Linear connection: $Y = AX$ (just a connection b/w X & Y not any inclusion relationship.)

→ Probability for X takes values in an infinitesimal interval

is the same

Probability for Y takes value in another infinitesimal interval

$$\Rightarrow \underbrace{P_r(x < X \le x + dx)}_{} = \underbrace{P_r(y < Y \le y + dy)}_{}$$

3rd axiom

$$\underbrace{F_X(x+dx) - F_X(x)}_{dF_X} \qquad \underbrace{F_Y(y+dy) - F_Y(y)}_{dF_Y}$$

$$\underbrace{\frac{dF_X}{dx}} dx \qquad \underbrace{\frac{dF_Y}{dy}} dy$$

$$f_X(x)\, dx = f_Y(y)\, dy$$

$$\Rightarrow \underbrace{f_Y(y)}_{+} = \underbrace{f_X(x)}_{+} \underbrace{\frac{dx}{dy}}_{?}$$

$$\Rightarrow \boxed{f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|}$$

Context: Quality control: found 10^6 resistors are labeled as 100Ω while they should be labeled as 200Ω

X = current value $\qquad Y$ = actual value

$Y = 2X \quad (\vec{A} = 2)$

$\rightarrow P_2\left(99.99 < X \le 100.01\right) = P_2\left(199.99 < Y \le 200.01\right)$

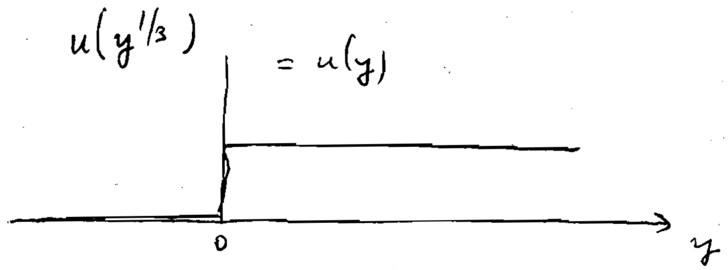
For example: $Y = 2X \rightarrow f_Y(y) = f_X(x)\cdot\frac{1}{2} = f_X\left(\frac{1}{2}y\right)\frac{1}{2}$

Cubic connection: $Y = X^3$ $\left\{\begin{array}{l} X: \text{ actual side of a cube} \\ Y: \text{ actual volume of that cube}\end{array}\right.$

Example: $f_X(x) = e^{-x}u(x)$; $dy = 3x^2\,dx \rightarrow \frac{dx}{dy} = \frac{1}{3x^2} = \frac{1}{3y^{2/3}}$

$$\rightarrow f_Y(y) = f_X\left(y^{1/3}\right)\frac{1}{3y^{2/3}}$$

$$= e^{-y^{1/3}}\underbrace{u(y^{1/3})}\frac{1}{3y^{2/3}} = e^{-y^{1/3}}u(y)\frac{1}{3y^{2/3}}$$



Quadratic connection = $Y = X^2$ $\left\{\begin{array}{l} X: \text{ actual size of a square} \\ Y = \text{ actual area of that square}\end{array}\right.$

$x = \pm\sqrt{y} \rightarrow \frac{dx}{dy} = \pm\frac{1}{2\sqrt{y}}$. the two possible signs need to be included, cannot apply the absolute value blindly:

$$f_Y(y) = f_X(+\sqrt{y})\left|\frac{1}{2\sqrt{y}}\right| + f_X(-\sqrt{y})\left|\frac{1}{-2\sqrt{y}}\right|$$

$$= \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right]$$

Example: $Y = 5X + 3$

$$\rightarrow f_Y(y) = f_X\left(\frac{y-3}{5}\right)\frac{1}{5}$$

HW2: 3.4: $Y = 3X - 4$; $f_X(x) = e^{-2|x|}$; $-\infty < x < \infty$

$$f_Y(y) = ? = f_X\left(\frac{y+4}{3}\right)\frac{1}{3} = \frac{1}{3}e^{-2\left|\frac{y+4}{3}\right|}$$

✓ we know the density function in y → can calculate distribution function in y: $F_Y(y) = \int_{-\infty}^{y} f(\tilde{y})\,d\tilde{y}$

a) $$P_r(Y<0) = F_Y(0) = \int_{-\infty}^{0} f_Y(\tilde{y})\,d\tilde{y}$$

$$= \frac{1}{3}\int_{-\infty}^{0} e^{-2\left|\frac{\tilde{y}+4}{3}\right|}\,d\tilde{y} = \frac{1}{3}\int_{-\infty}^{0} e^{-\frac{2}{3}|\tilde{y}+4|}\,d\tilde{y}$$

There are two cases for any absolute value:

$$|\tilde{y}+4| = \begin{cases} \tilde{y}+4 & \text{if } \tilde{y}+4>0 \text{ or } \tilde{y}>-4 \\ -\tilde{y}-4 & \text{if } \tilde{y}+4<0 \text{ or } \tilde{y}<-4 \end{cases}$$

Since the integral goes from $-\infty$ to 0 we need to split it into two pieces:

$$P_r(Y<0) = \frac{1}{3}\int_{-\infty}^{-4} e^{-\frac{2}{3}(-\tilde{y}-4)}\,d\tilde{y} + \frac{1}{3}\int_{-4}^{0} e^{-\frac{2}{3}(\tilde{y}+4)}\,d\tilde{y}$$

$$= \frac{1}{3}\int_{-\infty}^{-4} e^{\frac{2}{3}(\tilde{y}+4)}\,d\tilde{y} + \frac{1}{3}\int_{-4}^{0} e^{-\frac{2}{3}(\tilde{y}+4)}\,d\tilde{y}$$

$$= \frac{1}{3}\left.\frac{e^{\frac{2}{3}(\tilde{y}+4)}}{\frac{2}{3}}\right]_{\tilde{y}=-\infty}^{\tilde{y}=-4} + \frac{1}{3}\left.\frac{e^{-\frac{2}{3}(\tilde{y}+4)}}{-\frac{2}{3}}\right]_{\tilde{y}=-4}^{\tilde{y}=0}$$

$$= \frac{1}{2}\left\{ 1 - 0 + \left(e^{-\frac{8}{3}} - 1 \right) \right\} = \frac{1}{2}\left\{ 2 - e^{-\frac{8}{3}} \right\} = 0.965$$

b) $Pr(Y > X) =$

Mean values and moments of Random Variables.

Mean value : also a moment of 1st order

$$\overline{X} \equiv \underbrace{\int_{-\infty}^{\infty} dx \; x f_X(x)}_{\substack{\text{more general version of} \\ \text{the wellknown average}}} = \underbrace{E[x]}_{\text{"ensemble average"}}$$

$$\frac{x_1 + x_2 + \cdots x_N}{N} = x_1 \underbrace{\frac{1}{N}}_{\text{is now replaced by } f_X(x)} + x_2 \frac{1}{N} + \cdots x_N \frac{1}{N}$$

$$\overline{X^n} \equiv \int_{-\infty}^{\infty} dx \; x^n f_X(x) \qquad : \quad \text{moment of order } n$$

$$\overline{(X - \overline{X})^n} \equiv \int_{-\infty}^{\infty} dx \; (x - \overline{x})^n f_X(x) \quad : \quad \text{central moment of order } n$$

("centered about the mean")

$$\overline{(X - \overline{X})^2} =$$

1) central moment of order 2 : or Variance
2) $\underbrace{\overline{X^2}}_{\text{moment of order 2}} - \underbrace{\overline{X}^2}_{\text{mean squared}}$

<u>observation:</u> averaging is a linear operation $\Rightarrow$ the average of a sum is the sum of the averages:

$$\overline{(X-\overline{X})^2} = \overline{(X^2 - 2X\overline{X} + \overline{X}^2)} = \overline{X^2} \underbrace{- 2\overline{X}^2 + \overline{X}^2} = \overline{X^2} - \overline{X}^2 \checkmark$$

$\rightarrow$ Average of X is $\overline{X}$ (a number)

$\rightarrow$ Average of $2X\overline{X}$ is $2\overline{X}\,\overline{X} = 2(\overline{X})^2$

$\rightarrow$ Average of $\overline{X}$ is $\overline{X}$:

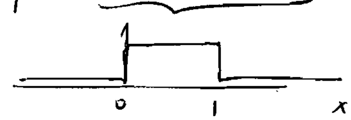
$$\hookrightarrow \int_{-\infty}^{\infty} dx\, \overline{X} f_X(x) = \overline{X} \underbrace{\int_{-\infty}^{\infty} dx\, f_X(x)}_{1 = F(\infty)}$$

$\rightarrow$ Note that $\overline{X^2} = \int_{-\infty}^{\infty} dx\, x^2 f_X(x)$ is different than $\overline{X}^2 = \left[\int_{-\infty}^{\infty} dx\, x f_X(x)\right]^2$

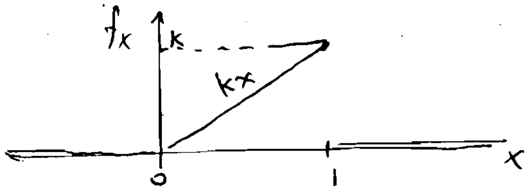
$\rightarrow$ We have used "averaging algebra" to prove

$$\overline{(X-\overline{X})^2} = \overline{X^2} - \overline{X}^2$$

<u>Example:</u> X random variable with $f_X(x) = kx\underbrace{[u(x) - u(x-1)]}$



a) What is k?

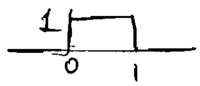


$$F(\infty) = 1 = \int_{-\infty}^{\infty} dx\, f_X(x) = k\int_0^1 dx\, x = k\left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{k}{2} \rightarrow k = 2$$

b) What is $\overline{X}$?

$$\overline{X} = \int_{-\infty}^{\infty} dx \; x \, kx \underbrace{\left[u(x) - u(x-1)\right]} = 2\int_0^1 dx \; x^2 = 2\left[\frac{x^3}{3}\right]_0^1$$



$$\Rightarrow \overline{X} = \frac{2}{3}$$

c) What is $\overline{X^2}$:

$$\overline{X^2} = 2\int_0^1 dx \; x^2 x = 2\int_0^1 dx \; x^3 = 2\left(\frac{x^4}{4}\right)_0^1 = \frac{2}{4} = \frac{1}{2}$$

d) What is $\underbrace{\sigma_x^2}_{\text{Variance}} = \overline{X^2} - \overline{X}^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$

e) $\overline{(X-\bar{X})^4} = 2\int_0^1 dx \; (x-\bar{x})^4 x \overset{?}{=}$

Triangle of Tartaglia or Pascal

1
1 1
1 2 1
1 3 3 1
→ 1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

$$(x-\bar{x})^4 = x^4 - 4x^3\bar{x} + 6x^2\bar{x}^2 - 4x\bar{x}^3 + \bar{x}^4$$

Gaussian Random Variable X

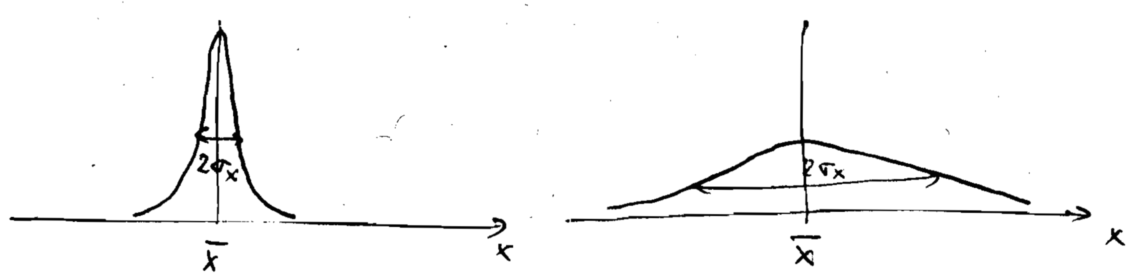
→ Prob. Density function is $f_X(x) = \dfrac{1}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{(x-\overset{\text{mean}}{\bar{X}})^2}{2\underset{\text{variance}}{\sigma_X^2}}}$, $-\infty < x < \infty$

Observations: (assume $\bar{X}$ & σ_X^2 are given)

1) Maximum: when $x = \bar{x}$

$$\underset{\text{height}}{f_{X\,max}} = \frac{1}{\sqrt{2\pi}\,\underset{\text{width of distribution.}}{\sigma_X}}$$

: { larger width → shorter height



2) A very good model for many observed phenomena, like noise

→ 3) If X and Y are both Gaussian random variables: (G.R.V.)
Then $aX + bY$ → also a (G.R.V)

4) Can be determined by : its 1^{st} and 2^{nd} moments
mean variance

5) The only one with complete statistical analysis results

$$F(x) \equiv P_r(\bar{X} < x) = \int_{-\infty}^{x} du\, f_X(u) \qquad (F(+\infty) = 1)$$

Distribution function

$$= \frac{1}{\sqrt{2\pi}\,\sigma_X} \underbrace{\int_{-\infty}^{x} du\, e^{-\frac{(u-\bar{X})^2}{2\sigma_X^2}}}_{\text{evaluated numerically}} \equiv \underset{\text{Phi (found in Math Tables)}}{\Phi}\left(\frac{x-\bar{X}}{\sigma_X}\right)$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du \quad ; \quad Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{u^2}{2}} du$$

$$\Phi(\infty) = 1 = \Phi(x) + Q(x) \Rightarrow \Phi(x) = 1 - Q(x)$$ (They are complement of each other)

Useful formulas for handling Gaussian random variable:

$$\boxed{\int_0^{\infty} dx \, x^n e^{-r^2 x^2} = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2 r^{n+1}}} \quad (\text{any } n)$$

Gamma function $\Gamma(\alpha)$
$$\begin{cases} \Gamma(\frac{1}{2}) = \sqrt{\pi} \\ \Gamma(1) = 1 \\ \Gamma(\alpha+1) = \alpha \Gamma(\alpha) \end{cases}$$

For example: $\Gamma(2) = 1$; $\Gamma(3) = 2$; $\Gamma(4) = 6$; $\Gamma(5) = 24$; ...

$\Gamma(1+1) = 1\Gamma(1) = 1.1$ $\qquad$ $\Gamma(2+1) = 2\Gamma(2) = 2.1$

$$\Phi(\infty) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du \overset{(*)}{=} \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{u^2}{2}} du = \frac{2}{\sqrt{2\pi}} \frac{\Gamma(\frac{1}{2})}{2 \frac{1}{\sqrt{2}}}$$

$n = 0$; $r = \frac{1}{\sqrt{2}}$

$$= \frac{\Gamma(\frac{1}{2})}{\sqrt{\pi}} = \frac{\cancel{\sqrt{\pi}}}{\cancel{\sqrt{\pi}}} = 1$$

(*) $e^{-\frac{u^2}{2}}$ is even in u

2.5.1 → 25% HW 2

2.5.2 → 25% HW 2

Related density functions (to the Gaussian)

1) **Power distribution**: $P = I^2 R$

R → fixed

I is a Gaussian random variable

What is the density function for P?

$$Y = X^2 \rightarrow f_Y(y) = \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right]$$

$$\text{If } f_I(i) = \frac{1}{\sqrt{2\pi}\,\sigma_I} e^{-\frac{(i-\bar{I})^2}{2\sigma_I^2}} \quad \begin{cases} \text{mean}: \bar{I} \\ \text{variance}: \sigma_I^2 \end{cases} \rightarrow \text{what is } f_P(p)?$$

$$f_P\left(\frac{P}{R}\right) = \frac{1}{2\sqrt{\frac{P}{R}}}\left[f_I\left(\sqrt{\frac{P}{R}}\right) + f_I\left(-\sqrt{\frac{P}{R}}\right) \right]$$

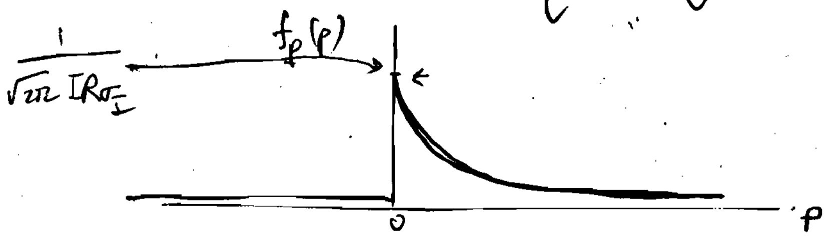
$$f_P(p) = \frac{1}{R} f_P(p/R) = \frac{1}{2\sqrt{PR}}\left[f_I\left(\sqrt{\frac{P}{R}}\right) + f_I\left(-\sqrt{\frac{P}{R}}\right) \right]$$

$(Y = aX)$

$$\boxed{\text{Assume}: \bar{I} = 0} \rightarrow f_P(p) = \frac{1}{\cancel{2} IR}\left[\cancel{2} \frac{1}{\sqrt{2\pi}\,\sigma_I} e^{-\frac{\left(\sqrt{\frac{P}{R}}\right)^2}{2\sigma_I^2}} \right]$$

The density function for the power is

$$f_P(p) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\, IR\sigma_I} e^{-\frac{p}{2R\sigma_I^2}} \; ; & p \geq 0 \\ 0 \; ; & p < 0 \end{cases}$$



→ What is the mean of P ($= I^2 R$) if $\overline{I} = 0$

$$\overline{P} = \int_0^\infty dp\, p f_P(p) = \frac{1}{\sqrt{2\pi}\,\cancel{R}\,\sigma_I} \int_0^\infty dp\, \frac{p\, e^{-\frac{p}{2R\sigma_I^2}}}{\sqrt{\frac{p}{R}}} = \frac{1}{\sqrt{2\pi R}\,\sigma_I} \int_0^\infty dp\, p^{1/2} e^{-\frac{p}{2R\sigma_I^2}}$$

Formula: $$\boxed{\int_0^\infty dx\, x^n e^{-ax} = \frac{n!}{a^{n+1}} = \frac{\Gamma(n+1)}{a^{n+1}}}$$

$$= \frac{1}{\sqrt{2\pi R}\,\sigma_I} \cdot \frac{\Gamma\left(\frac{3}{2}\right)}{\left(\frac{1}{2R\sigma_I^2}\right)^{3/2}} = \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\frac{1}{\sqrt{2R}^3\,\sigma_I^3}} \cdot \frac{1}{\sqrt{2R\pi}\,\sigma_I} = \frac{\frac{1}{2}\cancel{\sqrt{\pi}}}{\frac{1}{\sqrt{2R}^2\,\sigma_I^2}\cancel{\sqrt{\pi}}}$$

$$\Gamma\left(\tfrac{3}{2}\right) = \Gamma\left(\tfrac{1}{2}+1\right) = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right)$$

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

$$\Rightarrow \boxed{\overline{P} = \frac{\cancel{2}R\sigma_I^2}{\cancel{2}} = R\sigma_I^2} \quad \left(\text{if } \overline{I} = 0 \Rightarrow \boxed{\overline{P} = \sigma_I^2 R}\right)$$

$$\rightarrow \sigma_I^2 = \overline{(I - \underset{\uparrow}{\overline{I}})^2} = \overline{I^2} \quad \Rightarrow \quad \overline{P} = \overline{I^2} R$$ (This makes sense!)

→ What is the variance of P ($= I^2 R$) if $\overline{I} = 0$?

$$\sigma_P^2 = \overline{P^2} - \underset{\uparrow \text{just found} \checkmark}{\overline{P}^2}$$ → we need $\overline{P^2}$:

$$\overline{P^2} = \int_0^\infty dp\, p^2 f_P(p) = \frac{1}{\sqrt{2\pi R}\,\sigma_I} \int_0^\infty dp\, p^{3/2} e^{-\frac{p}{2R\sigma_I^2}}$$

$$= \frac{1}{\sqrt{2\pi R}\,\sigma_I} \frac{\Gamma\left(\frac{3}{2}+1\right)}{\frac{1}{(2R\sigma_I^2)^{5/2}}} = \frac{\sqrt{2R}^5\,\sigma_I^{\cancel{5}\,4}}{\sqrt{2R\pi}\,\cancel{\sigma_I}} \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{(2R)^2\sigma_I^4}{\cancel{\sqrt{\pi}}} \frac{3}{2}\cdot\frac{1}{2}\cancel{\sqrt{\pi}}$$

$$\boxed{\overline{P^2} = 3R^2\sigma_I^4}$$

$$\boxed{\sigma_P^2 = \overline{P^2} - \overline{P}^2 = 3R^2\sigma_I^4 - \sigma_I^4 R^2 = 2R^2\sigma_I^4}$$

---

2) Rayleigh distribution.

Hypothenuse or length of a 2D vector : $R = \sqrt{X^2 + Y^2}$

X & Y are Gaussian random variables with zero means and same variance σ^2, then R follows the Rayleigh distribution.

$Z = \sqrt{W}$; if W is G.R.V. $\rightarrow f_Z(z) = f_W(w) \left| \frac{dw}{dz} \right|$

$\frac{dz}{dw} = -\frac{1}{2w^{1/2}} = -\frac{1}{2z} \rightarrow \left| \frac{dw}{dz} \right| = 2z \rightarrow f_Z(z) = f_W(z^2)\, 2z$

$$f_R(r) = f_{X^2+Y^2}(r^2)\, 2r = \begin{cases} \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} & r \geq 0 \\ 0 & r < 0 \end{cases}$$

$(\overline{X} = \overline{Y} = 0$ and $\sigma_X^2 = \sigma_Y^2 \equiv \sigma^2)$

Derive this from X & Y Gaussian density functions
25% HW3