Next meeting: will finish this chapter 1, will start looking at HW1: if you bring in a solved problem to present.

→ get 100% credit for that set

→ get 50% credit if solution is not correct.

Conditional Probability in the Axiomatic Approach:

$$\boxed{Pr(A|B)} = \frac{Pr(A, B)}{Pr(B)} = \boxed{\frac{Pr(A \cap B)}{Pr(B)}}$$

given that B happened

conditional

marginal

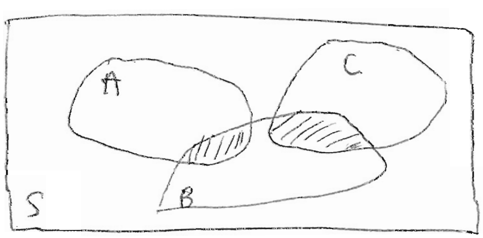
joint

Check on the 3 axioms to see if this is a valid probability.

$$\begin{cases} Pr(A|B) \geq 0 & \text{True since it is a ratio of probabilities} \\ Pr(S|B) = 1 & \\ A \cap C = \phi : \boxed{Pr(A \cup C | B)} = \dfrac{Pr((A \cup C) \cap B)}{Pr(B)} = \dfrac{Pr[(A \cap B) \cup (C \cap B)]}{Pr(B)} & \end{cases}$$

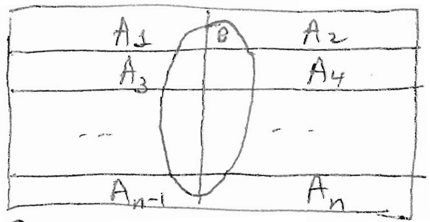
$$= \frac{Pr(A \cap B)}{Pr(B)} + \frac{Pr(C \cap B)}{Pr(B)}$$

$$= \boxed{Pr(A|B) + Pr(C|B)}$$



Checked all 3 axioms → This conditional probability is indeed a probability based on the axiomatic approach.

Total probability:



S

$$S = \bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n$$

The way we divide the space is such that

$$A_1 \cap A_2 = \phi = A_1 \cap A_3 = A_1 \cap A_n = \dots$$

(We make a partition of S into n subsets)

Set B overlaps with all A_i's

$$B = B \cap S = \underset{\substack{\downarrow \\ \text{product}}}{B \cap} (A_1 \underset{\substack{\downarrow \\ \text{sum}}}{\cup} A_2 \underset{\substack{\downarrow \\ \text{sum}}}{\cup} \dots \cup A_n) = \underbrace{(B \cap A_1) \cup (B \cap A_2) \cup \dots \cup (B \cap A_n)}_{\text{are disjoint}}$$

$$\Rightarrow P_r(B) = P_r(B \cap A_1) + P_r(B \cap A_2) + \dots + P_r(B \cap A_n)$$ (using the axiomatic approach)

↓ cond. prob.

$$\boxed{P_r(B) = P_r(B|A_1) P_r(A_1) + P_r(B|A_2) P_r(A_2) + \dots + P_r(B|A_n) P_r(A_n)}$$

↓ Total probability formula (in analogy with the total & partial derivative in calculus)

Calculus: Total derivative d Partial derivative ∂

$f(x,y)$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

Statistical Independence:

- Definition: A & B are stat. independent iff $\boxed{P_r(A \cap B) = P_r(A) \cdot P_r(B)}$

For two events A & B:

- $A \cap B = \phi$ (Mutually exclusive) → A & B can be independent only if A or B is the ϕ itself. or $\boxed{\text{Two non-empty events that are disjoint are NOT statistical independent}}$
- $A \cap B \neq \phi$ → can be stat. independent or not

$\boxed{\text{Rolling a die :}}$ $A = \{2,3\}$; $B = \{3,4\}$

$A \cap B \neq \emptyset$ since $A \cap B = \{3\}$

A & B statistical independent if $Pr(A \cap B) = Pr(A) \cdot Pr(B)$

$$Pr\{3\} = Pr\{2,3\} \cdot Pr(3,4)$$

$$\frac{1}{6} \qquad \underbrace{\frac{2}{6} \cdot \frac{2}{6}}_{\frac{4}{36}}$$

→ A & B are NOT statistical independent.

$A = \{1,2,3\}$; $B = \{3,4\}$; $A \cap B \neq \emptyset$

$$Pr\{3\} = \underbrace{Pr\{1,2,3\}} \cdot Pr(3,4)$$

$$\frac{1}{6} = \frac{3}{6} \cdot \frac{2}{6} = \frac{6}{36} \quad ✓$$

→ A & B are statistical independent.

$\boxed{\text{Rolling two die}}$ and add the dots

1) $A = \{(\text{getting } 7)\}$; $B = \{\text{getting } 11\}$

$A \cap B = \emptyset$ and A & B are non-empty ⟶ They are not statistical independent

2) $C = \{\text{getting an odd sum}\}$; $D = \{\text{getting } 11\}$

$C \cap D \neq \emptyset = D$

$Pr(D) = \frac{2}{36} = \frac{1}{18}$

$Pr(C) = Pr\{3, 5, 7, 9, 11\} = \frac{18}{36} = \frac{1}{2}$

(2) (4) (6) (4) (2)

$\cancel{X} = aX$ ↓ $Pr(C) \neq 1$ since C is not S

$Pr(C \cap D) = Pr(C) \cdot Pr(D)$?

No C & D are NOT statistical indep

3000 motors

4.5

a) Probability hp of 0.5?

Hp	120 Vac	240 Vac	24V 3Ø	
0.1	900	400	0	
0.5	200	500	100	= 800
1.0	100	200	600	
		1100		

$Pr(a) = \frac{800}{3000} = \frac{4}{15} = 0.267$

b) 240 V single phase?

$Pr(b) = \frac{1100}{3000} = \frac{11}{30} = 0.367$

c) motor 1.0 hp, 240 v three phase?

$Pr(c) = \frac{600}{3000} = \cancel{\frac{4}{15}} = 0.2$

d) motor 0.1 hp, 120 v operation

$Pr(d) = \frac{900}{3000} = \frac{3}{10} = 0.30$

6.1 $S = \{1, 3, 5, 7, 9, 11\}$

$A = \{1, 3, 5\}$, $B = \{7, 9, 11\}$, $C = \{1, 3, 9, 11\}$

$\frac{1}{2}$ $\qquad$ $\frac{2}{3}$

a. $Pr(A) = 50\%$

b. $Pr(B) = 50\%$

c. $Pr(C) = \frac{4}{6} \approx 0.6667 \Rightarrow 66.67\%$

d. $Pr(A \cup B) = 100\%$

e. $Pr(A \cup C) = Pr(A) + Pr(C) - Pr(A \cap C)$

$= \cancel{\frac{3}{6}}\frac{1}{2} + \frac{2}{3} - \cancel{\frac{2}{6}}\frac{1}{3} \approx 0.833$

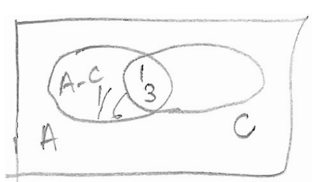
$\Rightarrow 83.3\%$

f. $Pr[(A-C) \cup B] = Pr(A-C) + Pr(B) - Pr[(A-C) \cap B]$

$= \frac{1}{6} + \frac{3}{6} - \phi$

$= \frac{4}{6} \approx 0.6667 \Rightarrow 66.67\%$

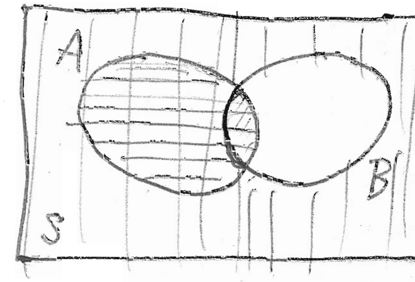
$Pr(A-C) = Pr(A) - Pr(C)$ ✗



1-8.4 A is independent of B, i.e.,

$$Pr(A\cap B) = Pr(A)\cdot Pr(B)$$

a) Prove A is independent of $\overline{B}$



$$\overline{B} = S - B$$

$$\underbrace{Pr(A\cap\overline{B})}_{\mathrm{I}} \overset{?}{=} \underbrace{Pr(A\cap(S-B))}_{\mathrm{II}}$$

II: $A\cap(S-B) = \cancel{(A\cap S) - (A\cap B)}$

$$= A - A\cap B$$

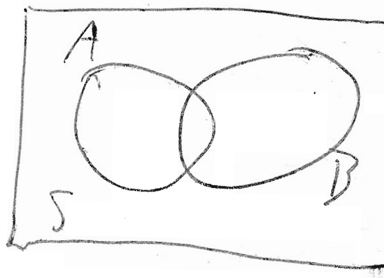
$$Pr(A - A\cap B) = Pr(A) - Pr(A\cap B) =$$

$$= Pr(A) - Pr(A)\cdot Pr(B) = \cancel{Pr(A)\,[Pr(S)\cdot Pr(\ \ ]}$$

$$= Pr(A)\underbrace{[Pr(S) - Pr(B)]}_{Pr(\overline{B})} = \underbrace{Pr(A)\,Pr(\overline{B})}_{Pr(A\cap\overline{B})}$$

b) $\bar{A}$ is independent of $\bar{B}$:

iff $P_r(\bar{A} \cap \bar{B}) = P_r(\bar{A}) \cdot P_r(\bar{B})$.



$$\bar{A} \cap \bar{B} = (S - A) \cap \underbrace{(S - B)}_{\bar{B}} =$$

$$= \cancel{S \cap S - A \cap S + S \cap \bar{B} = A \cap (-B)}$$

$$= \cancel{S - A - B + A \cap B}$$

$$= \cancel{\bar{A} - B + A \cap B} =$$

$$= (S - A) \cap \bar{B} = S \cap \bar{B} - A \cap \bar{B} = \underbrace{\bar{B} - A \cap \bar{B}}_{\mathrm{I}}$$

$$P_r(\mathrm{I}) = P_r(\bar{B} - A \cap \bar{B}) = P_r(\bar{B}) - P_r(A \cap \bar{B}) =$$

$$= P_r(\bar{B}) - P_r(A) \cdot P_r(\bar{B}) = P_r(\bar{B}) \underbrace{\left[1 - P_r(A)\right]}_{P_r(\bar{A})}$$

$$= P_r(\bar{B}) \cdot P_r(\bar{A})$$

2-15-07

(4.5) Daniel

HP	120V AC	240V AC	240V 3Ø
0.1	900	400	0
0.5	200	500	100
1	100	200	600

a) $Pr(HP = 0.5) = \frac{200+500+100}{3000} = \frac{8}{30} = \frac{4}{15} = 26.7\%$

b) $Pr(240V \text{ single phase}) = \frac{400+500+200}{3000} = \frac{11}{30} = 36.7\%$

c) $Pr(1HP,\ 240V\ 3\phi) = \frac{600}{3000} = 20\%$

d) $Pr(0.1HP,\ 120V) = \frac{900}{3000} = \frac{3}{10} = 30.0\%$

(6.1) William

$S = \{1,3,5,7,9,11\}$

$A = \{1,3,5\}$; $B = \{7,9,11\}$; $C = \{1,3,9,11\}$

a) $Pr(A) = \frac{3}{6} = 50\%$

b) $Pr(B) = \frac{3}{6} = 50\%$

c) $Pr(C) = \frac{4}{6} = 66.7\%$

d) $Pr(A \cup B) = Pr(S) = 100\%$

e) $Pr(A \cup C) = Pr\{1,3,5,9,11\} = \frac{5}{6} = 83.3\%$

$= Pr(A) + Pr(C) - Pr(A \cap C) = \frac{1}{2} + \frac{2}{3} - \frac{1}{3} = \frac{1}{2} + \frac{1}{3} = \frac{3+2}{6} = \frac{5}{6}$

f) $Pr[(A-C) \cup B] = Pr(A-C) + Pr(B) - Pr((A-C) \cap B)$

$= \frac{1}{6} + \frac{1}{2} - 0 = \frac{4}{6} = 66.7\%$

$A - C = \{5\}$

8.4 Emmanuel 90%

A & B are independent. or $P_r(A \cap B) = P_r(A) \cdot P_r(B)$

a) Prove A & $\bar{B}$ are independent or $P_r(A \cap \bar{B}) = P_r(A) \cdot P_r(\bar{B})$

$\bar{B} = S - B$

$P_r(A \cap \bar{B}) = P_r(A \cap (S - B)) = P_r(A \cap S - A \cap B)$

$= P_r(A) - P_r(A \cap B) = P_r(A)\left[1 - P_r(B)\right] = P_r(A) P_r(\bar{B})$

b) Prove $\bar{A}$ & $\bar{B}$ are independent or $P_r(\bar{A} \cap \bar{B}) = P_r(\bar{A}) \cdot P_r(\bar{B})$

$P_r(\bar{A} \cap \bar{B}) = P_r((S-A) \cap \bar{B}) = P_r(S \cap \bar{B}) - P_r(A \cap \bar{B})$

$= P_r(\bar{B}) - P_r(A) \cdot P_r(\bar{B})$

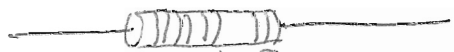
$= P_r(\bar{B})(1 - P_r(A))$

$= P_r(\bar{B}) \cdot P_r(\bar{A})$ ✓

Ch. 2: Random Variable. Distribution Function. Density Function

Random variable:

For example: a box of 10^6 resistors, the actual value of each is a random variable R (upper case)



color code → value, e.g. 100Ω; precision, eg 1%, 10%

e.g. $100\Omega \pm 1\%$ these resistors can have actual values between 99Ω & 101Ω.

R is a continuous random variable

The probability of a random variable can be described by a distribution function (probability distribution function), which associates a probability to certain range of values of the random variable.

Values are written the same way as the random variable, but in lower case.

$$\underbrace{F(x)}_{\substack{\text{lower case}\\\text{(value)}}} \equiv Pr(\underbrace{X}_{\substack{\text{upper case}\\\text{(variable)}}} \le \underbrace{x}_{\substack{\text{lower case}\\\text{(value)}}})$$

; F is the probability distribution function for the random variable X

→ Why not $F(x) = Pr(X = x)$?

(e.g. $Pr(R = 100.000\Omega) = \frac{1}{\infty} \approx 0$.)

ok for a discrete random variable, but not for a continuous random variable.

Let's check the three axioms to see if F defined that way is a "good" probability: $F(x) = \Pr(X \le x)$

1) $0 \le F(x) \le 1$?

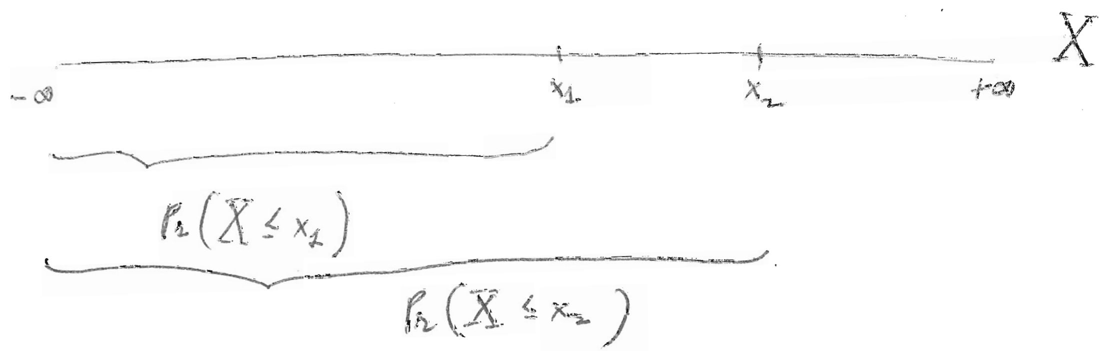
Yes, since $0 \le \Pr(X \le x) \le 1$

2) $\Pr(\Omega) = 1$ Here the certain event is $X \le \infty$

$\rightarrow F(\infty) = 1$, clearly.

$F(-\infty) = 0$ (impossible event is $X < -\infty$)

3)



$\Pr(x_1 < X \le x_2) = \underbrace{\Pr(X \le x_2)}_{F(x_2)} - \Pr(X \le x_1)$

$F(x_2) - F(x_1)$

Discrete random variable:

Four coins, random variable X is the number of heads.

a) Sketch the distribution function for X:

$F(0) = \Pr(X \le 0) = \Pr(X = 0) = \Pr(TTTT) = \frac{1}{16}$

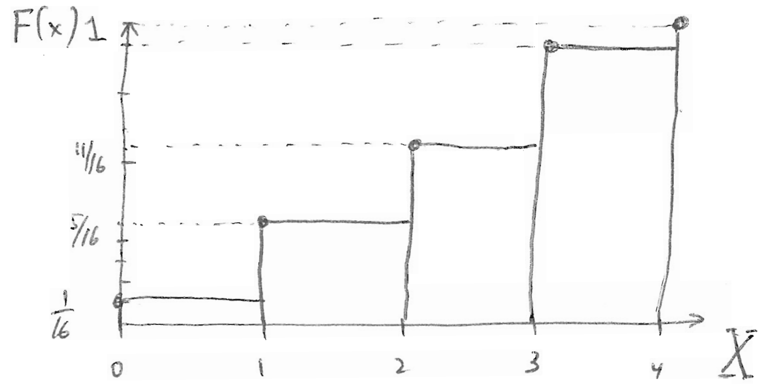
$F(1) = \Pr(X \le 1) = \Pr(X = 0) + \Pr(X = 1) = \Pr(TTTT) + \Pr(\text{one } H) = \frac{1}{16} + \frac{4}{16} = \frac{5}{16}$

$F(2) = \Pr(X \le 2) = \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2) = \frac{5}{16} + \frac{6}{16} = \frac{11}{16}$

$F(3) = F(2) + P_r(\text{one } T) = \frac{11}{16} + \frac{4}{16} = \frac{15}{16}$

$F(4) = F(3) + \frac{1}{16} = 1$

- How many possible events (or combinations of 4 elements when each can be of two types : H or T)? 16 $= 2 \times 2 \times 2 \times 2$
- There are 6 possible combinations of 4 coins with 2 H's



F(x) is increasing with x ✓ (general)

b) $F(3.5) = P_r(X \leq 3.5) = P_r(X \leq 3) = F(3)$
(we used this fact in the sketch above)

c) $P_r(X > 2.5) = \underbrace{P_r(X=3)}_{P_r(\text{one } T)} + \underbrace{P_r(X=4)}_{P_r(HHHH)} = \frac{4}{16} + \frac{1}{16} = \frac{5}{16} = F(1)$

or $P_r(X > 2.5) = 1 - \underbrace{P_r(X \leq 2.5)}_{\text{complement of } X > 2.5} = 1 - \underbrace{F(2.5)}_{F(2)} = 1 - F(2) = 1 - \frac{11}{16} = \frac{5}{16}$

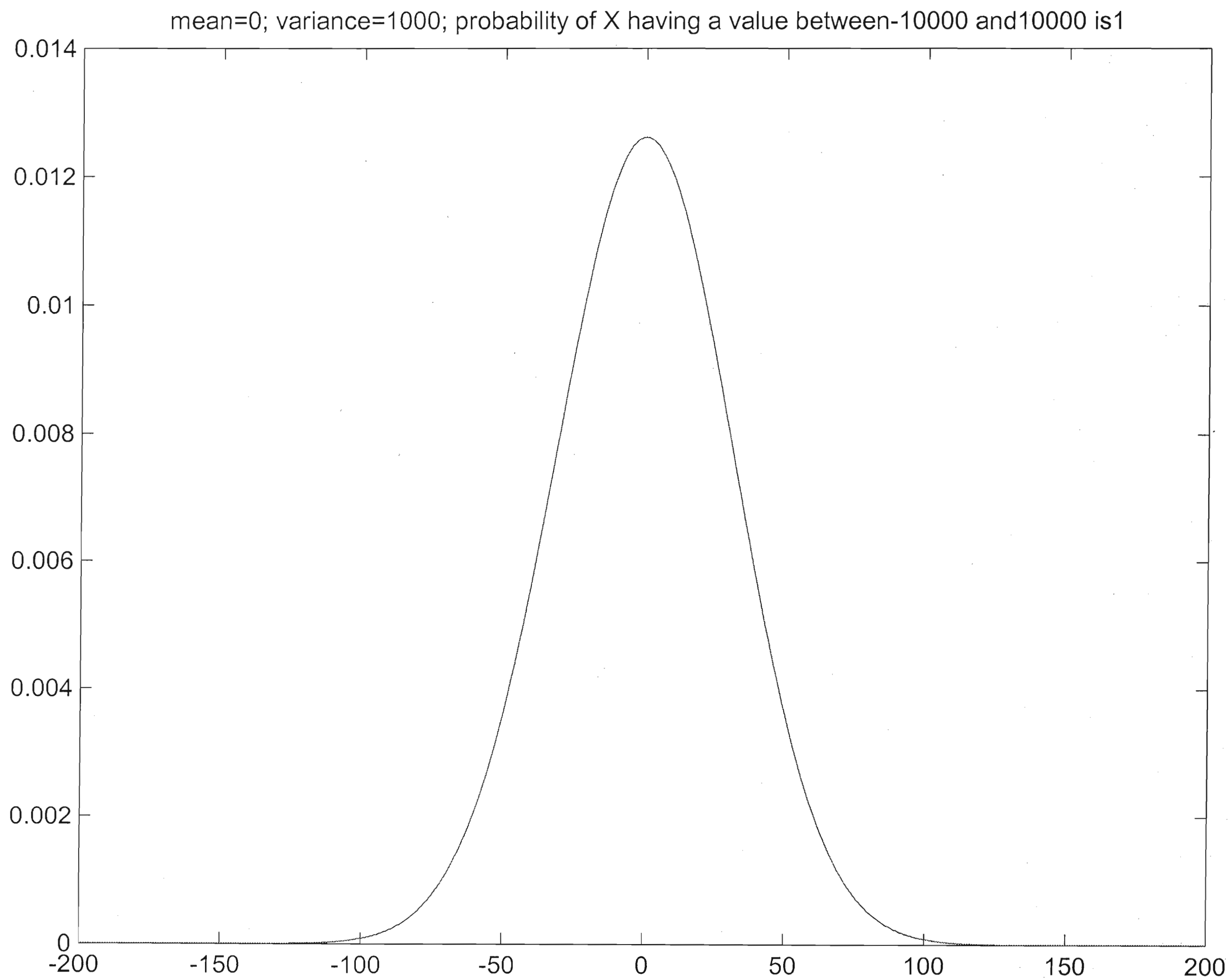
d) $P_r(0.5 < X \leq 3) = F(3) - F(0.5) = F(3) - F(0) = \frac{15}{16} - \frac{1}{16} = \frac{14}{16}$

```
%Gaussian distribution
clear all
close all
syms fx Fdist x;
var=1000;%variance
sigma=sqrt(var);%standard deviation
xbar=0; %mean

fx=1/(sqrt(2*pi)*sigma)*exp(-(x-xbar)^2/(2*var));%This is a probability density function
(to get the probability we need to integrate)
int(fx,x)
x1=-10000;
x2=10000;
Fdist=eval(int(fx,x,x1,x2))

%Gaussian distribution

clear fx x
x=-200:200;
fx=1/(sqrt(2*pi)*sigma)*exp(-(x-xbar).^2/(2*var));
plot(x,fx), title(strcat('mean=',num2str(xbar),'; variance=',num2str(var),'; probability
of X having a value between ',num2str(x1),' and ',num2str(x2),' is ',num2str(Fdist)))
```


mean=0; variance=1000; probability of X having a value between-10000 and10000 is1
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
-200
-150
-100
-50
0
50
100
150
200