Linear System Theory II Sp' 07

- Incorporate probability into the study of linear systems (to include noise)

- LST I:

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

$x(t)$ & $y(t)$ are signals (function of time)

If $x(t) = \delta(t) \longrightarrow y(t) = h(t)$

→ impulse response
→ use this to characterize our system.

$$y(t) = \underbrace{x(t) * h(t)}_{\text{convolution}} \equiv \int_{-\infty}^{\infty} d\tau \; x(\tau)\, h(t-\tau)$$

$$= \int_{-\infty}^{\infty} d\tau \; x(t-\tau)\, h(\tau) = h(t) * x(t)$$

- LST II:
  - Introduce probability & distributions
  - Incorporate this into $y(t) = x(t) * h(t)$

Ch 1 Introduction to Probability

Random Experiment: rolling a dice : outcome is uncertain.

Matlab: use "rand" function : results change from run to run

Event: the event of getting a four has different relative-frequency probability depending on the number of times we rolle the dice.

Event : get 4	# times	Relative - frequency probability
120	27	22.5 %
1200	185	15.4 %
12000	1972	16.4 %
1200000	200000	16.67 %

← meaningful experiment (sufficient # of samples)

$0.1667 = \frac{1}{6} = P_r(\text{"event of getting a 4"})$

Random expt: rolling a dice → there are six possible events

Event could be composite : Event A = "getting a one or a two"

$P_r(A) = \frac{2}{6} = \frac{1}{3}$

$P_r(E) = \frac{n}{N} \leq 1$ E could be composit

$\begin{cases} N : \# \text{ of possible single event or simple} \\ n : \# \text{ of simple events in } E \end{cases}$

Certain event

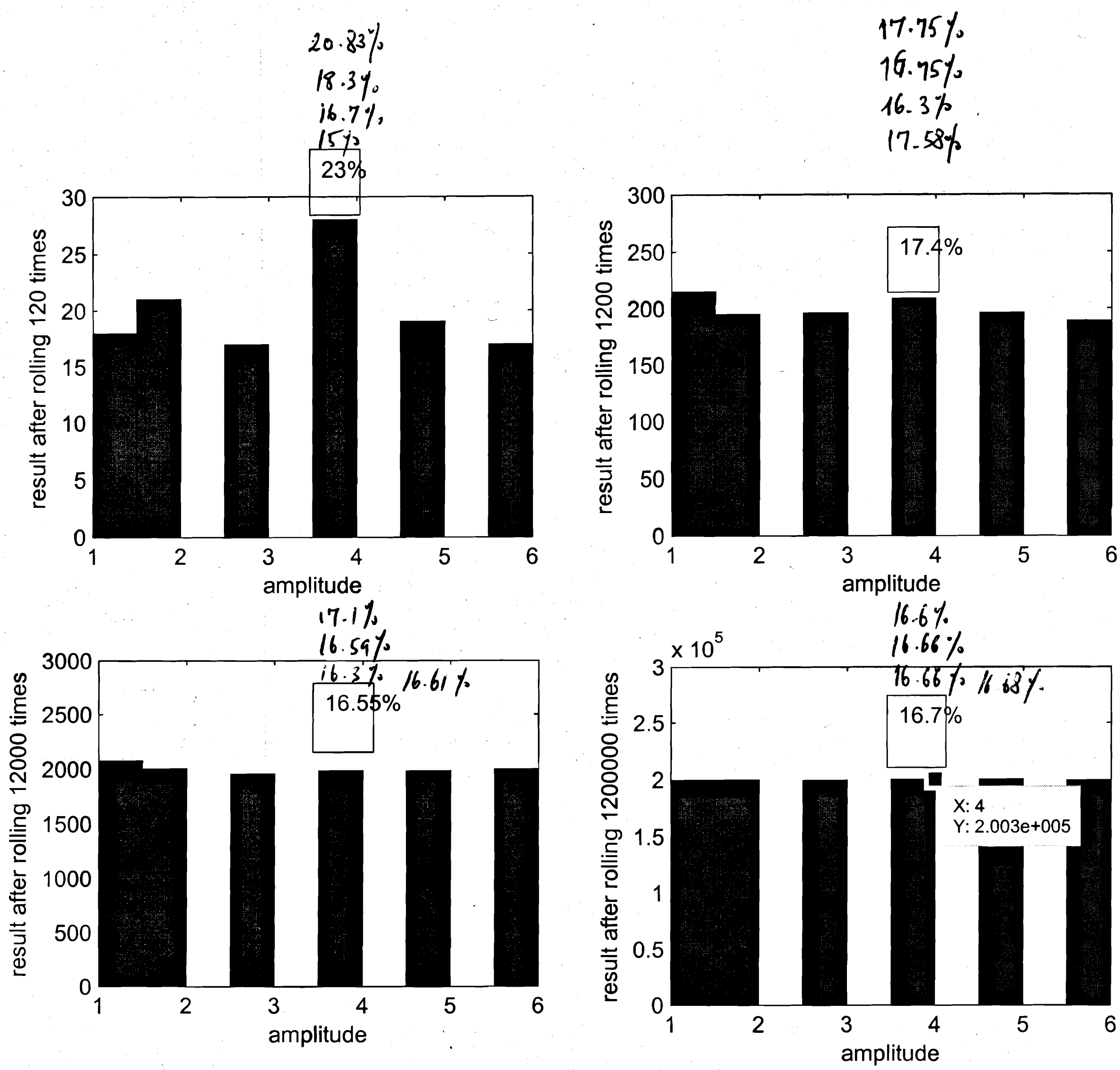
Event Ω will have $P_r(\Omega) = 1$

For this random expt Ω is the event of getting a 1 or 2 or 3 or 4 or 5 or 6.

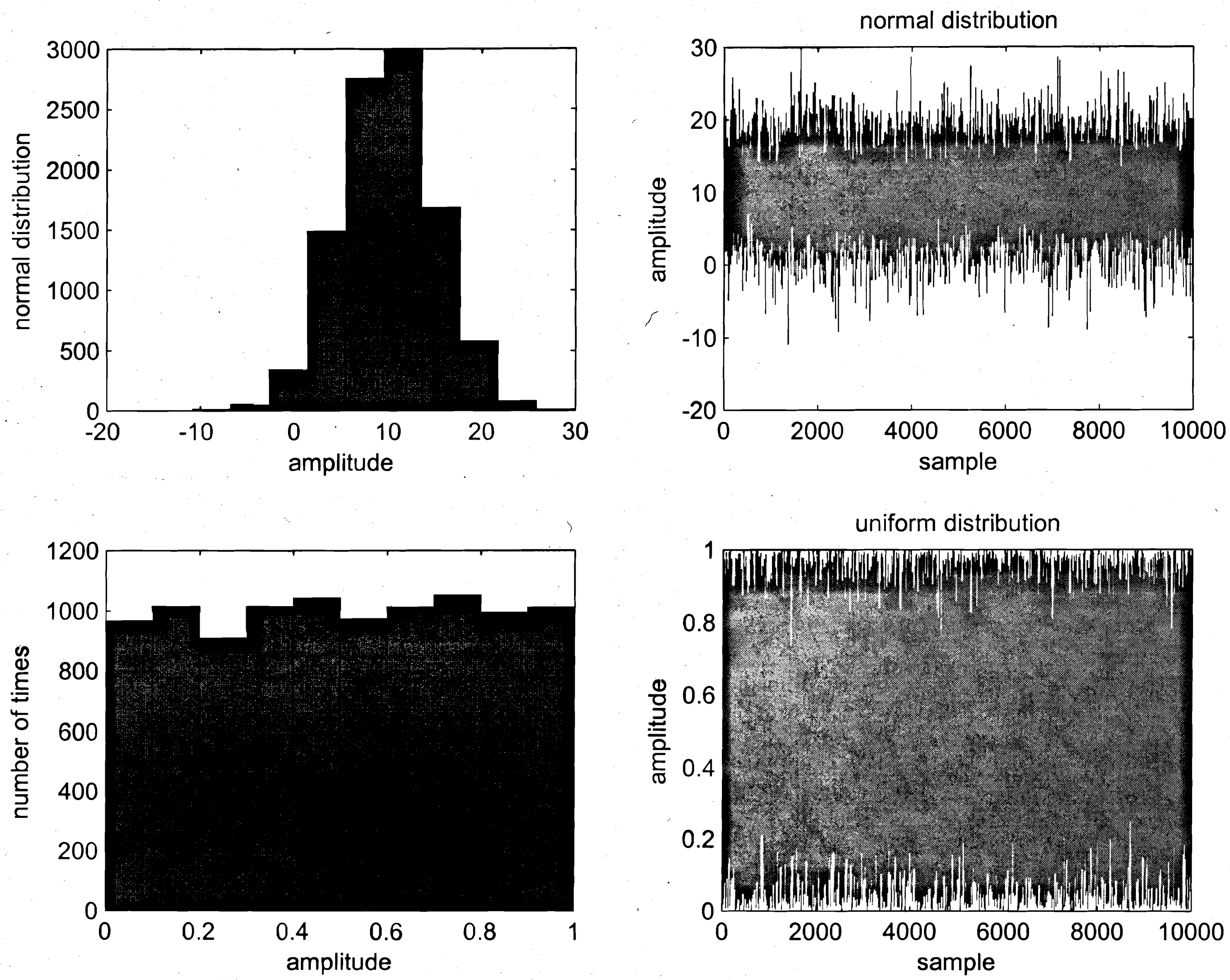
Event $\emptyset$ will have $P_r(\emptyset) = 0$

impossible event.

```
% Modeling the normal and uniform distributions

n=10000;
%Normal distribution

subplot(2,2,1)
variance=25;
mean=10;
x=sqrt(variance)*randn(1,n) + mean*ones(1,n);
hist(x)
xlabel('amplitude')
ylabel('normal distribution')

subplot(2,2,2)
plot(x)
xlabel('sample')
ylabel('amplitude')
title('normal distribution')


% Uniform distribution

subplot(2,2,3)
y=rand(1,n);
hist(y)
xlabel('amplitude')
ylabel('number of times')

subplot(2,2,4)
plot(y)
xlabel('sample')
ylabel('amplitude')
title('uniform distribution')
```


20.83%
18.3%
16.7%
15%
23%
result after rolling 120 times
amplitude
17.75%
16.75%
16.3%
17.58%
17.4%
result after rolling 1200 times
amplitude
17.1%
16.59%
16.3%
16.61%
16.55%
result after rolling 12000 times
amplitude
16.6%
16.66%
16.68%
16.63%
16.7%
x 10^5
X: 4
Y: 2.003e+005
result after rolling 1200000 times
amplitude

```
% Dice experiment (a random experiment)
% Feb 1, 2007

nplot=4;
nx=2;
ny=2
% nroll=[120 1200 12000 1200000];
nroll=[200 2000 20000 2000000];
for i=1:nplot
    subplot(nx,ny,i)
    y1=ones(1,nroll(i)) +floor(6*rand(1,nroll(i)));
    hist(y1)
    axis([1 6 0 nroll(i)/4]);
    xlabel('amplitude')
    ylabel(strcat('result after rolling', num2str(nroll(i)), 'times'))
end
```



normal distribution
amplitude
sample
uniform distribution
amplitude
sample
normal distribution
amplitude
number of times
amplitude

$$\boxed{\overbrace{P_r(B|A)\cdot P_r(A)}^{P_r(B,A)} = P_r(A,B)}$$

$P_r(B|A) \rightarrow \frac{150}{500} = 0.3$

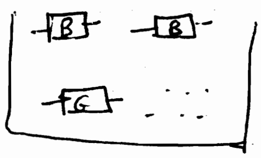
$P_r(A) \rightarrow 0.5$

$P_r(A,B) \rightarrow 0.15$

$$P_r(B,A) = 0.15 = P_r(A,B)$$

Example 1-4-1:

a) Box with 50 diodes



10 bad
40 good.

Event A: getting a bad diode from the box $P_r(A) = \frac{10}{50} = 0.2$

b) First diode drawn is good, what is the probability the second pick will be good?

Relative-frequency approach:

After 1st diode: 10 bad, 39 good $\rightarrow \frac{39}{49} = 0.795$

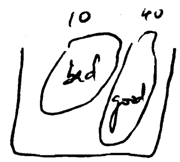
→ c) Combinatorics: $\binom{n}{a} = \frac{n!}{a!\,(n-a)!}$

Number of possible combinations of a elements out of an universe of n elements.

If 2 diodes are drawn, what is the probability they are both good?

1-4.1 c): * Using the relative-frequency approach:

$$Pr(\text{getting 2 good at once}) = \frac{\#\text{ combinations of two out of 40 good diodes}}{\#\text{ combinations of two out of 50 diodes.}}$$



50 diodes.

$$= \frac{\binom{40}{2}}{\binom{40}{2} + \binom{10}{2} + 40.10}$$

include 2 types of diodes.

$\binom{40}{2}$: ~~two of~~ two out of 40 good

$\binom{10}{2}$: two out of 10 bad

40.10: one good and one bad

$$= \frac{\frac{40.39}{2}}{\frac{40.39}{2} + \frac{10.9}{2} + 400} = \frac{780}{1225}$$

$$= \frac{156}{245} = 0.6367$$

* Using the joint probability formula:

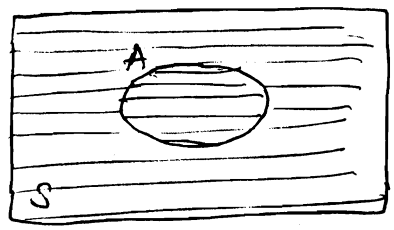
Event A: getting a good diode

$$\underbrace{Pr(A,A)} = \underbrace{Pr(A|A)}.\underbrace{Pr(A)} = \frac{39}{49} \cdot \frac{40}{50} = \frac{39\times 4}{49\times 5} = \frac{156}{245}.$$

Probability of getting two good at once (Joint)

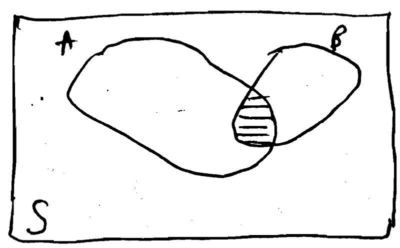
Prob. of getting a good diode given the first pick was good (Conditional)

Prob. of getting a good diode (Marginal)



Shade $A \cup S = S$

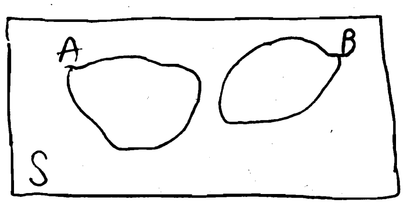
<u>Intersection or product</u>: of A and B is $A \cap B$ "A intersecting B" : contains elements that are common to <u>both</u> A and B.



Shade $A \cap B$

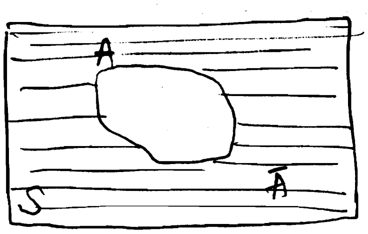
<u>Properties</u>:

$$A \cap B = B \cap A \quad \text{(commutative)}$$
$$A \cap S = A \; ; \quad A \cap \phi = \phi$$
$$A \cap A = A$$
$$(A \cap B) \cap C = A \cap (B \cap C) \quad \text{(associative)}$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad \text{(distributive)}$$



$A \cap B = \phi$ "A and B are mutually exclusive" or "disjoint"

<u>Complement</u> of A is $\overline{A}$: what belongs to S but not to A



Shade $\overline{A}$

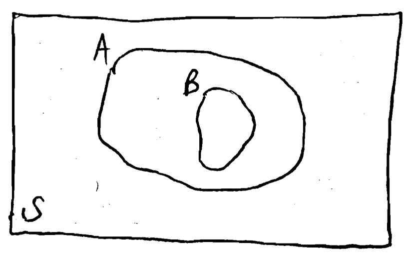
Elementary Set Theory:

Set: a collection of elements $A = \{\alpha_1, \alpha_2, \dots \alpha_n\}$

Subset: any set B whose elements are also elements of another set A. In this case: $B \subset A$
"B is included in A"

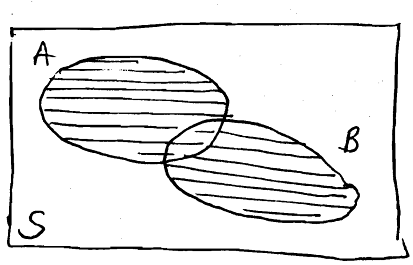
Largest set S or space: if S has N elements, there are 2^N subsets:
$S = \{x, y, z\} \rightarrow$ subsets are $\{x\}, \{y\}, \{z\}, \{xy\}, \{xz\}, \{yz\}, \{xyz\}, \phi$
($2^3 = 8$ subsets)

Venn diagrams:



Equality: $A = B$ iff (if and only if) $A \subset B$ and $B \subset A$

Sum or Union: of two sets will contain elements of the 1st set or element of the 2nd set, or of both sets.



Shade the union of A and B or $A \cup B$

Properties:
$A \cup B = B \cup A$ (commutative)
$A \cup \phi = A$; $A \cup A = A$; $A \cup S = S$
$(A \cup B) \cup C = A \cup (B \cup C)$ (associative)

Complement

↳ Properties:

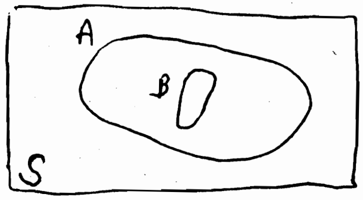
$$\overline{\phi} = S \quad ; \quad \overline{S} = \phi$$

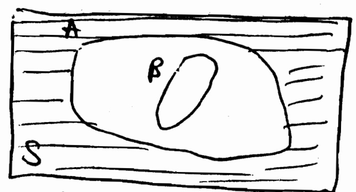
$$\overline{\overline{A}} = A$$

$$A \cup \overline{A} = S$$

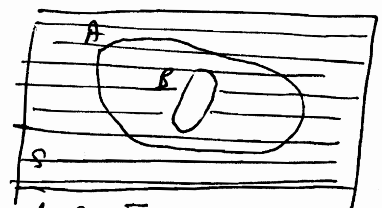
$$A \cap \overline{A} = \phi$$

* If $B \subset A \Rightarrow \overline{A} \subset \overline{B}$





Shade $\overline{A}$

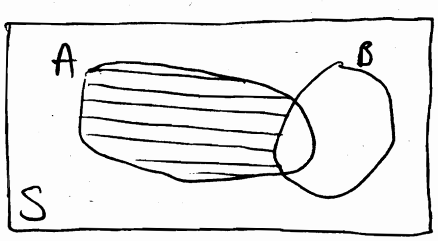


Shade $\overline{B}$

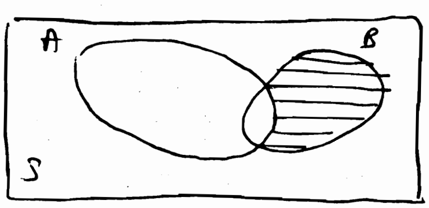
* If $\overline{B} = \overline{A} \Longrightarrow A = B$

* $\overline{A \cup B} = \overline{A} \cap \overline{B}$
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ } De Morgan's laws

Difference of A and B: $A - B$: consists of elements that belong to A but not to B

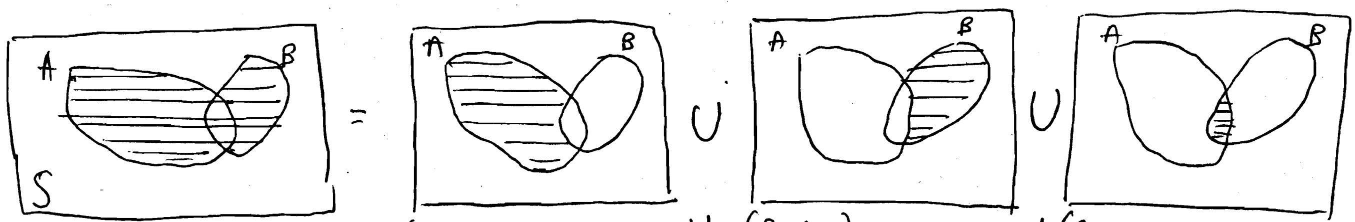


Shade $A - B$



Shade $B - A$

Write $A \cup B$ in terms of $A - B$; $B - A$; $A \cap B$



$A \cup B = (A-B) \cup (B-A) \cup (A \cap B)$

Properties: $\bar{A} = S - A$

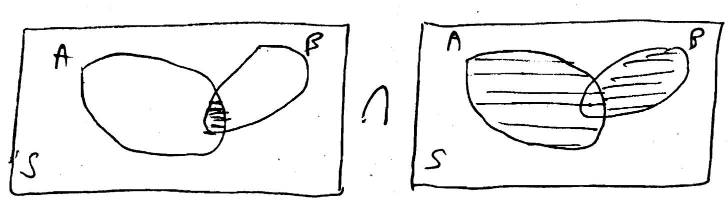
$A - B = A \cap \bar{B}$

$\quad = A - (A \cap B)$

Example 1-5.1 : a) $(A \cap B) \cup (A - B) = A$

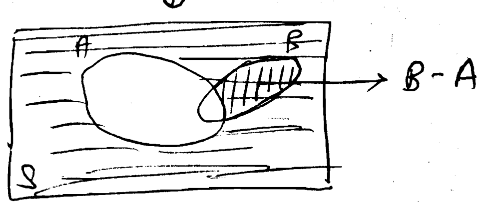
b) $\bar{A} \cap (A - B) = \emptyset$

c) $(A \cap B) \cap (B \cup A) = A \cap B$

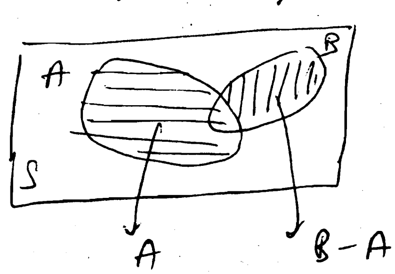


Example 1-5.2 : a) $A \cup (A \cap B) = A$ Since $(A \cap B) \subset A$

b) $A \cup \underbrace{(\bar{A} \cap B)} = A \cup B$ ✓



$A \cup (B - A) = A \cup B$



1.6 The Axiomatic Approach to Probability:

Probability space: contains all outcomes of a random experiment

Subsets of this space: events (S: certain event; ϕ: impossible event)

Subset A or event A is assigned a probability $Pr(A)$ that satisfies three axioms:

- $Pr(A) \geq 0$ (non-negative)
- $Pr(S) = 1$
- If $A \cap B = \phi$ (A and B are disjoint or mutually exclusive) $\rightarrow Pr(A \cup B) = Pr(A) + Pr(B)$

axioms → postulates, can't be proved.

Corollaries (consequences):

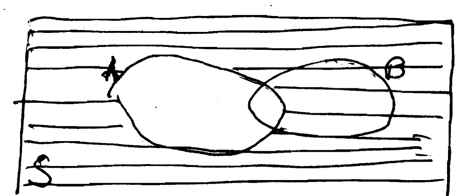
1) $A \cap \bar{A} = \phi$: $\underbrace{Pr(\underbrace{A \cup \bar{A}}_{S})}_{1} = Pr(A) + Pr(\bar{A})$

$\Rightarrow Pr(\bar{A}) = 1 - Pr(A)$

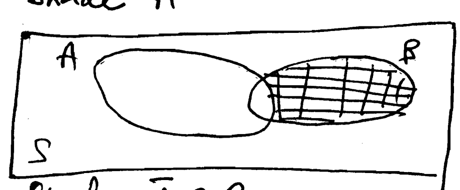
2) If $A \cap B \neq \phi \rightarrow Pr(A \cup B) = Pr(A) + Pr(B) - \underline{Pr(A \cap B)}$

$A \cup B = A \cup (\bar{A} \cap B)$ ✓

3rd axiom → $Pr(A \cup B) = Pr(A) + Pr(\bar{A} \cap B)$ (1)

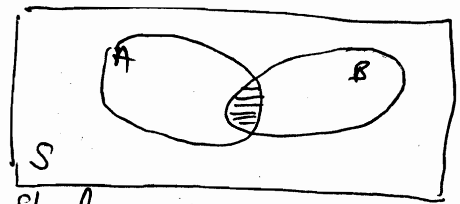


Shade $\bar{A}$



Shade $\bar{A} \cap B$

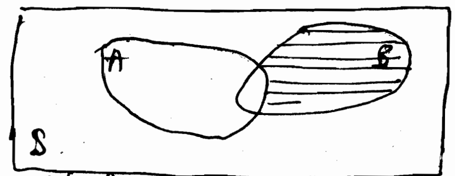
$$B = \underbrace{(A\cap B)} \cup \underbrace{(\bar{A}\cap B)} \checkmark$$



Shade $A\cap B$

3rd axiom $\rightarrow$ $Pr(B) = Pr(A\cap B) + Pr(\bar{A}\cap B)$ (2)

$\rightarrow$ $Pr(\bar{A}\cap B) = Pr(B) - Pr(A\cap B)$



Shade $\bar{A}\cap B$

Back in (1) $\boxed{Pr(A\cup B) = Pr(A) + Pr(B) - Pr(A\cap B)}$

$\downarrow$

Prob. of $A\cup B$ would be proportional to the total area of $A\cup B$ (using Venn Diagrams to represent A & B): if you just add $Pr(A)$ and $Pr(B)$ we are double counting the area of $A\cap B$. To be correct we need to take out one time $A\cap B$.

Throwing a single dice: $S = \{1, 2, 3, 4, 5, 6\}$

Random experiment

event of getting a one, etc.

All of these single events are equally probable.

Let's define event $A = \{1, 3\}$; event $B = \{3, 5\}$

getting a one or a three

getting a three or a five

From the axiomatic approach: $Pr(A\cup B) = Pr(A) + Pr(B) - Pr(A\cap B)$

$$\stackrel{?}{=} \frac{2}{6} + \frac{2}{6} - \frac{1}{6} = \frac{3}{6} = 0.5$$

$\updownarrow$

$$Pr(\{1, 3, 5\}) = Pr(\{1, 3\}) + Pr(\{3, 5\}) - Pr(\{3\})$$

1-6.1 Roulette wheel with 37 slots $\begin{cases} \text{slot } 37^{th} \text{ green, } 0 \\ \text{slots } 1 \to 36 \text{ : black and red alternately,} \end{cases}$

Bets $\begin{cases} \text{or select one number b/w 1 and 36 ; pays } 35:1 \\ \text{or select two adjacent numbers , pays } 17:1 \text{ if either wins} \end{cases}$

A: event of getting number 1 ; B : event of getting number 2

a) $Pr(A) = \frac{1}{37}$

Probable return of \$1 bet on number 1. → $(35+1) \times Pr(A) = \frac{36}{37}$

If you play 1000 times : your investment (total) = \$1000,
total return is \$972

b) $Pr(\underbrace{A \cup B}) = Pr(\{1,2\}) = \frac{2}{37}$

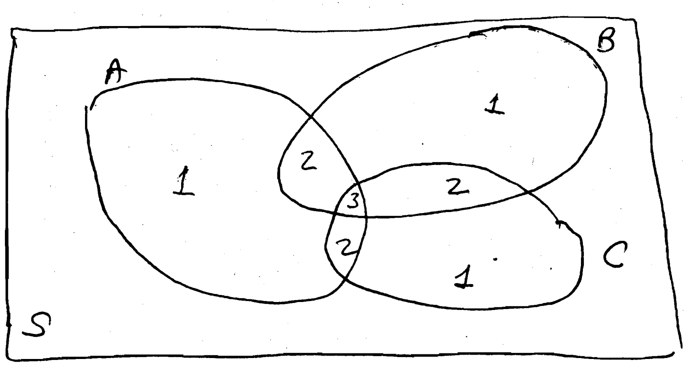
Probable return on a \$1 bet on $(A \cup B)$ = $(17+1) \times \frac{2}{37} = \frac{36}{37}$

Event of getting either 1 or 2

1-6.2 A, B, C not mutually exclusive : derive:

$$Pr(A \cup B \cup C) = Pr(A) + Pr(B) + Pr(C) - Pr(A \cap B) - Pr(B \cap C) - Pr(A \cap C) + Pr(A \cap B \cap C)$$

Using axioms
or use elimination
of double countings



Next meeting: will finish this chapter 1, will start looking at HW 1: if you bring in a solved problem to present.

→ get 100% credit for that set

→ get 50% credit if solution is not correct.