Turn in set 3 11/15/08.

Ch 4: Continuous-Time Fourier Transform:

HW4: 4.10; 4.14; 4.19; 4.23, 4.26; 4.28; 4.34; 4.44; 4.46, 4.53

signal $x(t) \longrightarrow$ Fourier Transform of $x(t)$:

$$\hat{X}(j\omega) \equiv \int_{-\infty}^{\infty} dt\; x(t)\, e^{-j\omega t}$$

complex function of frequency ω

* Can compare $\hat{X}(j\omega)$ with $a_k = \frac{1}{T}\int_T dt\; x(t)\, e^{-j(k\omega_0)t}$ = very similar except:

F.S.	F. Transform
only multiple of ω_0: $k\omega_0$	any ω
a_k only applies for periodic signal	$\hat{X}(j\omega)$ applies to any signal

a signal doesn't have to be periodic to have a F. Transform but has to be periodic to have a F.S. expansion.

* As with F.S. expansion, we can reverse the Fourier transform

Fourier Transform $\hat{X}(j\omega) \longrightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\; \hat{X}(j\omega)\, e^{+j\omega t}$

<u>Properties of Fourier Transform</u>: (Table 4.1 pg 328)

<u>Time shift</u>:

$x(t) \longrightarrow \hat{X}(j\omega)$

$x(t-t_0) \longrightarrow e^{-j\omega t_0}\hat{X}(j\omega)$ (property 4.3.2)

<u>Proof</u>: $x(t-t_0) \longrightarrow$ Fourier Transform: $\int_{-\infty}^{\infty} dt\; x(t-t_0)\, e^{-j\omega t}$

$$= \int_{-\infty}^{\infty} dt'\; x(t')\, e^{-j\omega(t'+t_0)}$$

change of variable $t' = t - t_0 \rightarrow \begin{cases} dt' = dt \\ t = t' + t_0 \end{cases}$

$$= e^{-j\omega t_0} \underbrace{\int_{-\infty}^{\infty} dt' \; x(t')\, e^{-j\omega t'}}_{\hat{X}(j\omega)}$$

Summary $\quad x(t-t_0) \longrightarrow$ Fourier Transform: $e^{-j\omega t_0}\,\hat{X}(j\omega)$

Convolution property: (4.4)

$x(t) \longrightarrow$ Fourier Transform $\hat{X}(j\omega)$

$y(t) \longrightarrow \qquad \hat{Y}(j\omega)$

$$\boxed{x(t) * y(t) \xrightarrow{F.T.} \hat{X}\cdot\hat{Y}}$$

Fourier Transform of $x(t) * y(t)$ is $\hat{X}\cdot\hat{Y}$

By def. of convolution: $\quad x(t) * y(t) = \int_{-\infty}^{\infty} d\tau \; x(\tau)\, y(t-\tau)$

Proof: $\quad x * y = \int_{-\infty}^{\infty} d\tau \; x(\tau)\, y(t-\tau) \xrightarrow{F.T.} \int_{-\infty}^{\infty} dt \,(x*y)\, e^{-j\omega t}$

$$= \int_{-\infty}^{\infty} dt \; e^{-j\omega t} \int_{-\infty}^{\infty} d\tau \; x(\tau)\, y(t-\tau)$$

$$= \int_{-\infty}^{\infty} d\tau \; x(\tau) \underbrace{\int_{-\infty}^{\infty} dt \; y(t-\tau)\, e^{-j\omega t}}_{\text{Fourier Transform of } y(t-\tau)}$$

$= e^{-j\omega\tau}\,\hat{Y}(j\omega)$

Time-shift property

$$= \hat{Y}(j\omega) \underbrace{\int_{-\infty}^{\infty} d\tau \; x(\tau)\, e^{-j\omega\tau}}_{\tau \text{ is time}}$$

$\hat{X}(j\omega)$

$$= \hat{Y}\cdot\hat{X} = \hat{X}\cdot\hat{Y}$$

The Convolution Property for Fourier Transform is very important in Linear System Theory!

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

We can calculate the output $y(t)$ w/o doing any convolution by using the convolution property of the Fourier Transform:

$$y(t) = \underbrace{FT^{-1}}_{\text{inverse Fourier Transform of}}\left[\hat{X}(j\omega) \cdot \hat{H}(j\omega)\right]$$

Proof: since $\quad y(t) = x(t) * h(t)$

$$FT\left[\, y(t) = x(t) * h(t) \,\right] \quad \text{(direct Fourier Transform on both sides of equation)}$$

$$\hat{Y}(j\omega) = \hat{X}(j\omega) \cdot \hat{H}(j\omega)$$

$$FT^{-1}\left[\, \hat{Y}(j\omega) = \hat{X}(j\omega) \cdot \hat{H}(j\omega) \,\right] \quad \text{(inverse Fourier Transform on both sides of equation)}$$

$$y(t) = FT^{-1}\left[\hat{X}(j\omega) \cdot \hat{H}(j\omega)\right]$$

Conclusion: alternative to using convolution ($y = x * h$) to calculate the output $y(t)$ is to find Fourier Transform for $x(t)$ & for $h(t)$, then multiply them together; then apply an inverse Fourier Transform on the product:

Find $y(t)$

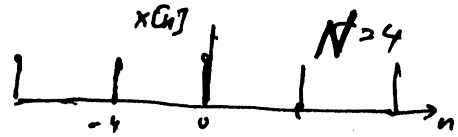
alternatives:

$x * h$

one Fourier transform, one multiplication, one inverse Fourier Transform.

Comparison

Fourier Series coefficients of $x(t)$ a_k	Fourier Transform of $x(t)$ $\hat{X}(j\omega)$
$a_k = \frac{1}{T}\int_T dt\; x(t)\, e^{-jk\omega_0 t}$	$\hat{X}(j\omega) \equiv \int_{-\infty}^{\infty} dt\; x(t)\, e^{-j\omega t}$
→ signal $x(t)$ is periodic with period T $\omega_0 = \frac{2\pi}{T}$	→ signal $x(t)$ can be periodic or non-periodic
→ since integral over a finite period, thus integral always exists	→ since integral is over an infinite interval $(-\infty, \infty)$ → <u>condition of existence</u>: 1) $x(t)$ is absolutely integrable or $\int_{-\infty}^{\infty} dt\, \lvert x(t)\rvert < \infty$ 2) $x(t)$ should have a finite # of max. & min over any finite interval. 3) $x(t)$ should have a finite # of discontinuities over any finite interval.



3.14 $\quad x[n] = \sum_{k=-\infty}^{\infty} \delta[n-4k] = \cdots + \delta[n-8] + \delta[n-4] + \delta[n] + \delta[n+4] + \delta[n+8] + \cdots$

$$x[n] \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ \hat{H}(e^{j\omega})\end{matrix}} \longrightarrow y[n] = \cos\left(\frac{5\pi}{2}n + \frac{\pi}{4}\right)$$

Determine $\hat{H}(e^{jk\frac{\pi}{2}})$ for $k = 0, 1, 2, 3$

$\omega_0 = \frac{\pi}{2}$ (since $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{4} = \frac{\pi}{2}$)

$\omega = k\omega_0$

<u>How do you find $\hat{H}$?</u>

Fourier Series Coefficients & Linear System:

$$\begin{matrix} x[n] & \longrightarrow & \boxed{\hat{H}} & \longrightarrow & y[n] \\ \text{F.S.} \downarrow & & & & \text{F.S.} \downarrow \\ a_k & & & & b_k \end{matrix} \quad \Bigg\} \; b_k = a_k \hat{H}(e^{jk\omega_0})$$

$$x[n] \longrightarrow \quad a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\frac{2\pi}{N}n} \underset{N=4}{=} \frac{1}{4} \sum_{n=0,\dots,3} x[n] e^{-jk\frac{\pi}{2}n}$$

$$= \frac{1}{4} \sum_{n=0}^{3} \underbrace{\sum_{m=-\infty}^{\infty} \delta[n-4m]}_{\text{only nonzero when } n = 4m = \cdots -8, -4, 0, 4, 8, \cdots} e^{-jk\frac{\pi}{2}n} = \frac{1}{4} \quad \forall k$$

$e^{-jk\frac{\pi}{2}n} = 1 \quad \forall 1$

$$y[n] \longrightarrow \quad y[n] = \cos\left(\frac{5\pi}{2}n + \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{2}n + \frac{\pi}{4}\right) = \frac{e^{j\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)} + e^{-j\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)}}{2}$$

(86)

$$y[n] = \frac{1}{2}e^{j\frac{\pi}{4}} \underset{}{\overset{k=1}{e^{j\frac{\pi}{2}n}}} + \frac{1}{2}e^{-j\frac{\pi}{4}} e^{-j\frac{\pi}{2}n} = e^{j3\frac{\pi}{2}n}$$

$$= \sum_{k=0}^{3} b_k e^{jk\frac{\pi}{2}n} \longrightarrow \begin{cases} b_0 = 0 \\ b_1 = \frac{1}{2}e^{j\frac{\pi}{4}} \\ b_2 = 0 \\ b_3 = \frac{1}{2}e^{-j\frac{\pi}{4}} \end{cases}$$

$$= \sum_{k=0}^{3} a_k \hat{H}(e^{jk\frac{\pi}{2}}) e^{jk\frac{\pi}{2}n}$$

In conclusion:

$$\begin{cases} a_0 \hat{H}(e^{j0}) = 0 \rightarrow \hat{H}(e^{j0}) = 0 \\ a_1 \hat{H}(e^{j\frac{\pi}{2}}) = \frac{1}{2}e^{j\frac{\pi}{4}} \underset{a_1 = \frac{1}{4}}{\rightarrow} \hat{H}(e^{j\frac{\pi}{2}}) = 2e^{j\frac{\pi}{4}} \\ a_2 \hat{H}(e^{j2\frac{\pi}{2}}) = 0 \rightarrow \hat{H}(e^{j\pi}) = 0 \\ a_3 \hat{H}(e^{j3\frac{\pi}{2}}) = \frac{1}{2}e^{-j\frac{\pi}{4}} \rightarrow \hat{H}(e^{j3\frac{\pi}{2}}) = 2e^{-j\frac{\pi}{4}} \end{cases}$$

3.25

$T = 0.5$:

$x(t) = \cos 4\pi t$; $y(t) = \sin 4\pi t$; $z(t) = x(t)y(t)$

a) a_k for $x(t)$: $a_k = \frac{1}{T}\int_T x(t)e^{-j\omega t}$ & $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$(\omega \equiv k\omega_0)$

$$x(t) = \cos 4\pi t = \frac{1}{2}e^{j4\pi t} + \frac{1}{2}e^{-j4\pi t} = \frac{1}{2}\underset{k=1}{e^{j\omega_0 t}} + \frac{1}{2}\underset{k=-1}{e^{-j\omega_0 t}}$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{0.5} = 4\pi$$

$a_1 = \frac{1}{2}$; $a_{-1} = \frac{1}{2}$; $a_k = 0$ otherwise

b) $y(t) = \sin 4\pi t = \frac{1}{2j}\underset{k=1}{e^{j4\pi t}} - \frac{1}{2j}\underset{k=-1}{e^{-j4\pi t}}$

$b_1 = \frac{1}{2j}$; $b_{-1} = -\frac{1}{2j} = b_1^*$

c) F.S coefficients for $z(t) = x(t) \cdot y(t)$ using <u>Multiplication Property</u>:

$$C_k = \sum_{l=-\infty}^{\infty} a_l \, b_{k-l}$$

or $\boxed{C_k = a_k * b_k}$

$$C_k = \left(\frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1] \right) * \left(\frac{1}{2j}\delta[k-1] - \frac{1}{2j}\delta[k+1] \right)$$

Convolution with a delta $\leftrightarrow$ shift the argument.

$$= \frac{1}{4j}\delta[k-2] - \cancel{\frac{1}{4j}\delta[k]} + \cancel{\frac{1}{4j}\delta[k]} - \frac{1}{4j}\delta[k+2]$$

$$\begin{cases} C_2 = \frac{1}{4j} \\ C_{-2} = -\frac{1}{4j} = C_2^* \end{cases}$$

d) Find F.S. coefficients for $z(t)$ from its trig. expansion:

$$z(t) = \cos 4\pi t \sin 4\pi t = \frac{1}{2}\sin 8\pi t = \frac{1}{2}\left(\frac{1}{2j} e^{j2\cdot 4\pi t} - \frac{1}{2j} e^{-j2\cdot 4\pi t}\right)$$

$k=2$ $\quad$ $k=-2$

$\sin(\alpha+\beta) = \cos\alpha \sin\beta + \sin\alpha\cos\beta$

$\beta = \alpha \rightarrow \sin 2\alpha = 2\cos\alpha\sin\alpha$

$$C_k = \frac{1}{4j}\delta[k-2] - \frac{1}{4j}\delta[k+2]$$

Basic Pairs of Fourier Transforms :

1) $x(t) = \cos\omega_0 t$

$\downarrow$

$$\hat{X}(j\omega) = \int_{-\infty}^{\infty} dt\, \cos\omega_0 t\, e^{-j\omega t} = \int_{-\infty}^{\infty} dt\, \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} e^{-j\omega t}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} dt\, e^{-j(\omega-\omega_0)t} + \frac{1}{2}\int_{-\infty}^{\infty} dt\, e^{-j(\omega+\omega_0)t}$$

$$= \frac{1}{2} 2\pi\, \delta(\omega-\omega_0) + \frac{1}{2} 2\pi\, \delta(\omega+\omega_0)$$

$\downarrow$

Orthogonality of complex exponentials : $\int_{-\infty}^{\infty} dt\, e^{-j(k-n)\omega t} = 2\pi\delta((k-n)\omega)$

$$= \frac{2\pi}{\omega}\delta(k-n)$$

Scaling property of δ-functions

Conclusion:

$$\cos\omega_0 t \longrightarrow \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

$$\sin\omega_0 t \longrightarrow \frac{1}{2j} 2\pi\,\delta(\omega-\omega_0) - \frac{1}{2j} 2\pi\,\delta(\omega+\omega_0)$$

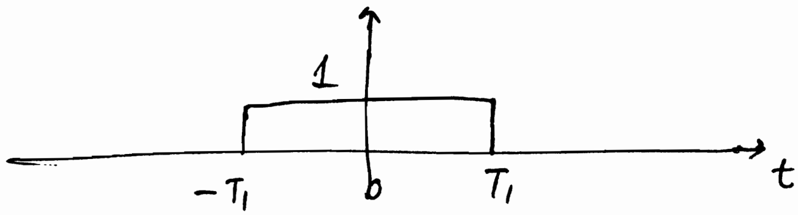
$$= \pi j\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$$

2) $x(t) = 1$

$\downarrow$

$$\hat{X}(j\omega) = \int_{-\infty}^{\infty} dt\, e^{-j\omega t} = 2\pi\,\delta(\omega)$$

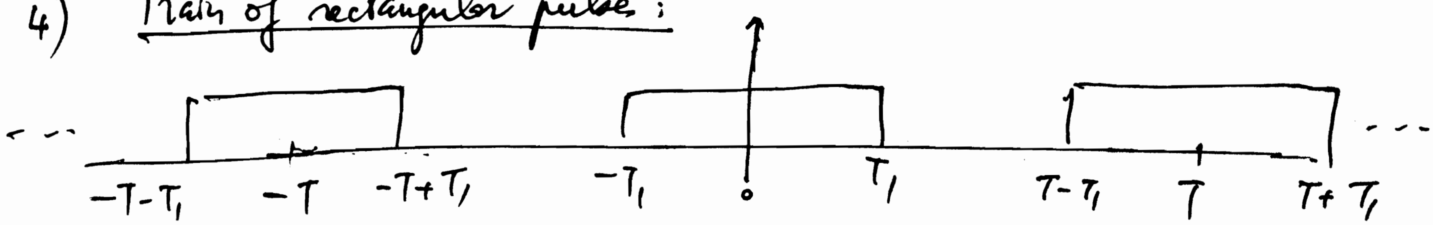
3) Rectangular pulse: $x(t) = \begin{cases} 1 & |t| \leq T_1 \rightarrow \text{half width of pulse} \\ 0 & |t| > T_1 \end{cases}$



$$\hookrightarrow \hat{X}(j\omega) = \int_{-T_1}^{T_1} dt \; 1 \; e^{-j\omega t} = \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{-T_1}^{T_1} = \frac{e^{j\omega T_1} - e^{-j\omega T_1}}{j\omega}$$

$$= 2 \, \frac{\sin \omega T_1}{\omega}$$

4) Train of rectangular pulse:



Fourier Transform is linear $\rightarrow \displaystyle\sum_{k=-\infty}^{\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$

$$\omega_0 = \frac{2\pi}{T}$$

5) $x(t) = e^{-at} u(t)$; $\text{Re}[a] > 0$ or decayed exponential.

$$\hookrightarrow \hat{X}(j\omega) = \int_0^{\infty} dt \; e^{-at} e^{-j\omega t} = \int_0^{\infty} dt \; e^{-(a+j\omega)t}$$

$$= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_0^{\infty} = \frac{0 - 1}{-(a+j\omega)} = \frac{1}{a+j\omega}$$

6) $x(t) = te^{-at}u(t) \quad ; \quad Re[a] > 0$

$$\hookrightarrow \hat{X}(j\omega) = \int_0^\infty dt\; t\underbrace{e^{-at}e^{-j\omega t}}_{e^{-(a+j\omega)t}} = -\frac{d}{d(a+j\omega)}\underbrace{\int_0^\infty dt\, e^{-(a+j\omega)t}}_{\frac{1}{a+j\omega}}$$

$$= \frac{1}{(a+j\omega)^2}$$

7) $x(t) = \dfrac{t^{n-1}}{(n-1)!}e^{-at}u(t) \quad ; \quad Re[a] > 0$

$$\hookrightarrow \hat{X}(j\omega) = \frac{1}{(a+j\omega)^n}$$

Basic pairs of Fourier Transform	
$x(t)$	$\hat{X}(j\omega)$
$\cos \omega_0 t$	$\pi(\delta(\omega+\omega_0) + \delta(\omega-\omega_0))$
$\sin \omega_0 t$	$\pi j(\delta(\omega+\omega_0) - \delta(\omega-\omega_0))$
1	$2\pi\delta(\omega)$
Rectangular pulse $\begin{cases} 1 & \lvert t\rvert \le T_1 \\ 0 & \lvert t\rvert > T_1 \end{cases}$	$2\dfrac{\sin \omega T_1}{\omega}$
Train of rectangular pulses of widths $2T_1$ & period T	$\sum_{k=-\infty}^{\infty} \dfrac{2\sin k\omega_0 T_1}{k}\delta(\omega - k\omega_0)$ $\quad \omega_0 = \dfrac{2\pi}{T}$
$e^{-at}u(t)\; ; Re[a]>0$	$\dfrac{1}{a+j\omega}$
$te^{-at}u(t)\; ; Re[a]>0$	$\dfrac{1}{(a+j\omega)^2}$
$\dfrac{t^{n-1}}{(n-1)!}e^{-at}u(t)\; ; Re[a]>0$	$\dfrac{1}{(a+j\omega)^n}$

4.10 a) Find $\hat{X}(j\omega)$ for $x(t) = t\left(\frac{\sin t}{\pi t}\right)^2$ | use Tables 4.1 & 4.2

$x(t) = t\, y(t)$ | $y(t) = \frac{\sin t}{\pi t} \cdot \frac{\sin t}{\pi t}$.

$\hat{Y}(j\omega) = \frac{1}{2\pi} F\left(\frac{\sin t}{\pi t}\right) * F\left(\frac{\sin t}{\pi t}\right)$

Property 4.5 in Table 4.1 $\boxed{x(t)y(t) \longrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} d\theta\, \hat{X}(j\theta)\,\hat{Y}(j\omega - j\theta)}$

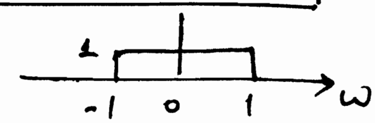
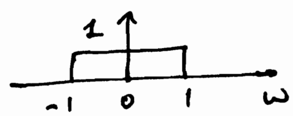
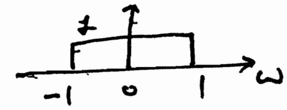
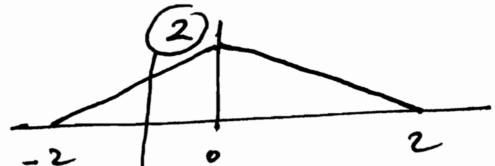
Since $F\left(\frac{\sin t}{\pi t}\right) = \begin{cases} 1 & |\omega| < 1 \\ 0 & |\omega| > 1 \end{cases} =$ 

Table 4.2 $(W=1)$

then $\hat{Y}(j\omega) = \frac{1}{2\pi}\Bigg\{$  $*$  $\Bigg\}$



when two pulses are on top of each other, the overlapped area is that of one pulse : $2 \times 1 = 2$

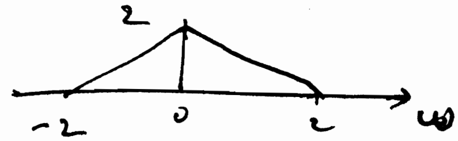
$$= \underbrace{\begin{cases} \frac{1}{2\pi}\left(-|\omega| + 2\right) & \text{if } |\omega| < 2 \\ 0 & \text{if } |\omega| > 2 \end{cases}}_{\text{This is the Fourier transform of } \left(\frac{\sin t}{\pi t}\right)^2}$$

We need: $\hat{X}(j\omega)$ || $x(t) = t\,y(t) = t\left(\frac{\sin t}{\pi t}\right)^2$

Property 4.3-6 in Table 4.1:

$$t\,y(t) \longrightarrow -\frac{d}{d(j\omega)} Y(j\omega)$$

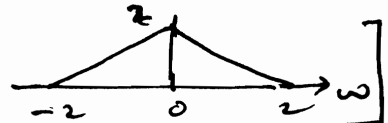
$$\hat{X}(j\omega) = -\frac{d}{d(j\omega)}\Big[\frac{1}{2\pi}$$



$$\Big]$$

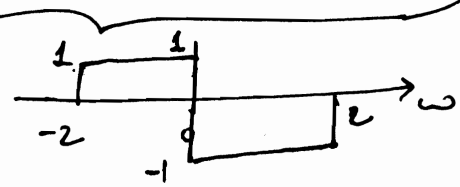
Fourier Transform of $y(t) = \left(\frac{\sin t}{\pi t}\right)^2$

$$= \frac{j}{2\pi}\,\frac{d}{d\omega}\Big[$$



$$\Big]$$

$$= \frac{j}{2\pi}$$



$$= \underbrace{\begin{cases} \frac{j}{2\pi}\left(-\frac{\omega}{|\omega|}\right) & |\omega| < 2 \\ 0 & \text{otherwise} \end{cases}}_{\text{Fourier Transform for } x(t) = t\left(\frac{\sin t}{\pi t}\right)^2}$$

b) Use Parseval relation

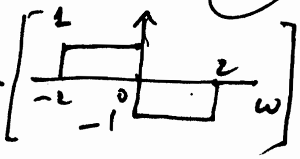
$$\left[\int_{-\infty}^{\infty} dt\, |x(t)|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, |\hat{X}(j\omega)|^2\right]$$

and a) to find: $A = \int_{-\infty}^{\infty} dt\; t^2\left(\frac{\sin t}{\pi t}\right)^4$

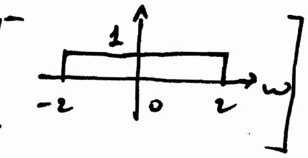
since $x(t) = t\left(\frac{\sin t}{\pi t}\right)^2 \rightarrow A = \int_{-\infty}^{\infty} dt\, |x(t)|^2$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, |\hat{X}(j\omega)|^2$$

$$\hat{X}(j\omega) = \begin{cases} \frac{j}{2\pi}\left(-\frac{\omega}{|\omega|}\right) & |\omega| < 2 \\ 0 & \text{otherwise} \end{cases} = \frac{j}{2\pi}\left[\quad\right]$$



$$|\hat{X}(j\omega)|^2 = \begin{cases} \frac{1}{4\pi^2}\cdot 1 & |\omega| < 2 \\ 0 & \text{otherwise} \end{cases} = \frac{1}{4\pi^2}\left[\quad\right]$$



$$\boxed{A = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, |\hat{X}(j\omega)|^2 = \frac{1}{8\pi^3}\int_{-2}^{2} d\omega = \frac{4}{8\pi^3} = \frac{1}{2\pi^3}}$$

4.14 $x(t) \longrightarrow \hat{X}(j\omega)$, suppose

1) $x(t)$ real & non-negative
2) $F^{-1}\left[(1+j\omega)\hat{X}(j\omega)\right] = A e^{-2t} u(t)$, A independent of t
3) $\int_{-\infty}^{\infty} d\omega\, |\hat{X}(j\omega)|^2 = 2\pi$

$\boxed{\text{Determine expression for } x(t)\,!}$

Information 3) $\Rightarrow$ $\int_{-\infty}^{\infty} dt\, |x(t)|^2 = 1$.

Parseval's relation.

Information 2) $\Rightarrow$ $(1+j\omega)\hat{X}(j\omega) = A\, \underbrace{F\left[e^{-2t}u(t)\right]}_{\frac{1}{2+j\omega}}$

$$\hat{X}(j\omega) = \frac{A}{(1+j\omega)(2+j\omega)} \quad \text{Partial Fraction Exp.}$$

$$\underbrace{F^{-1}\left[\hat{X}(j\omega)\right]}_{x(t)} = F^{-1}\left[\frac{A}{1+j\omega} - \frac{A}{2+j\omega}\right]$$

$$= A e^{-t}u(t) - A e^{-2t}u(t)$$

$$\boxed{x(t) = A\left\{e^{-t} - e^{-2t}\right\}u(t)}$$

Now use information 3) to get A:

$$1 = \int_{-\infty}^{\infty} dt\, |x(t)|^2 = A^2 \int_{-\infty}^{\infty} dt\, \underbrace{u^2(t)}_{u(t)} \left\{ e^{-2t} - 2e^{-3t} + e^{-4t} \right\}$$

$$= A^2 \int_0^{\infty} dt \left(e^{-2t} - 2e^{-3t} + e^{-4t} \right) = A^2 \left[\left(\frac{e^{-2t}}{-2} \right)_0^{\infty} - 2\left(\frac{e^{-3t}}{-3} \right)_0^{\infty} + \left(\frac{e^{-4t}}{-4} \right)_0^{\infty} \right]$$

$$= A^2 \left[\frac{1}{2} - 2\frac{1}{3} + \frac{1}{4} \right] = A^2 \left[\frac{6 - 8 + 3}{12} \right] = \frac{A^2}{12}$$

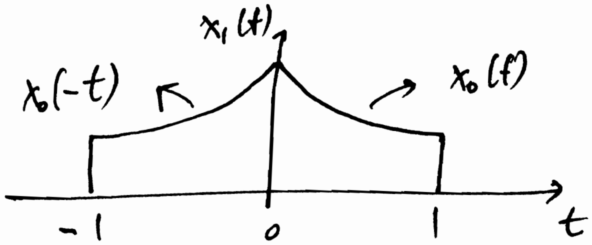
$$\rightarrow A^2 = 12 \quad \rightarrow A = \sqrt{12}$$

<u>Finally</u>: $\boxed{x(t) = \sqrt{12}\, u(t) \left[e^{-t} - e^{-2t} \right]}$

4.23

$$x_0(t) = \begin{cases} e^{-t} & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a)



Find Fourier Transform of $x_1(t)$ using the Fourier Transform $x_0(t)$ & properties of Fourier Transform!

In general: $x(t) \longrightarrow \hat{X}(j\omega) = \int_{-\infty}^{\infty} dt\, x(t)\, e^{-j\omega t}$

$$x_0(t) \longrightarrow \hat{X}_0(j\omega) = \int_0^1 dt\, \underbrace{e^{-t}\, e^{-j\omega t}} = \int_0^1 dt\, e^{-(1+j\omega)t}$$

$$= \left[\frac{e^{-(1+j\omega)t}}{-(1+j\omega)} \right]_0^1 = \frac{1 - e^{-(1+j\omega)}}{1+j\omega}$$

Observation: $x_1(t) = x_0(-t) + x_0(t)$

Time reversal property in Table 4.1

$$\hat{X}_1(j\omega) = \hat{X}_0(-j\omega) + \hat{X}_0(j\omega)$$

$$\hat{X}_1(j\omega) = \frac{1 - e^{-(1-j\omega)}}{1 - j\omega} + \frac{1 - e^{-(1+j\omega)}}{1+j\omega}$$

$$= \frac{\left(1 - e^{-(1-j\omega)}\right)(1+j\omega) + \left(1 - e^{-(1+j\omega)}\right)(1-j\omega)}{1+\omega^2}$$

$$= \frac{1 - e^{-(1-j\omega)} + \cancel{j\omega} - j\omega e^{-(1-j\omega)} + 1 - e^{-(1+j\omega)} - \cancel{j\omega} + j\omega e^{-(1+j\omega)}}{1+\omega^2}$$

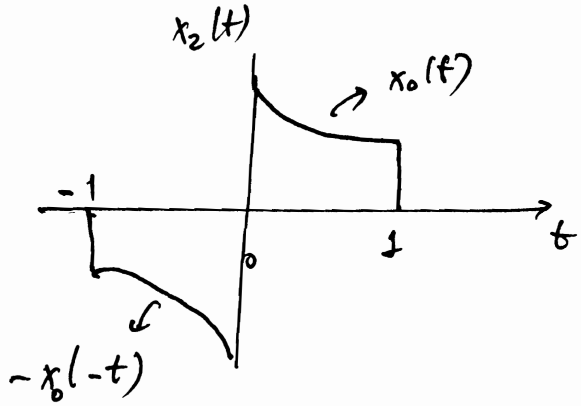
$$= \frac{2 - e^{-1}\left(e^{j\omega} + e^{-j\omega}\right) - j\omega e^{-1}\left(e^{j\omega} - e^{-j\omega}\right)}{1+\omega^2}$$

$$= \frac{2 - e^{-1}2\cos\omega - j\omega e^{-1}2j\sin\omega}{1+\omega^2}$$

$$= \frac{2}{1+\omega^2}\left(1 - e^{-1}\cos\omega + \omega e^{-1}\sin\omega\right)$$

$$\rightarrow \boxed{\hat{X}_1(j\omega) = \frac{2}{1+\omega^2}\left(1 - \frac{\cos\omega}{e} + \frac{\omega\sin\omega}{e}\right)}$$

b)



$$x_2(t) = -x_0(-t) + x_0(t)$$

$$\hat{X}_2(j\omega) = - \hat{X}_0(-j\omega) + \hat{X}_0(j\omega)$$

difference vs. part a)!

$$= \frac{-1 + e^{-(1-j\omega)} - j\omega + j\omega e^{-(1-j\omega)} + 1 - e^{-(1+j\omega)} - j\omega + j\omega e^{-(1+j\omega)}}{1+\omega^2}$$

$$= \frac{e^{-1}(e^{j\omega} - e^{-j\omega}) + j\omega e^{-1}(e^{j\omega} + e^{-j\omega}) - 2j\omega}{1+\omega^2}$$

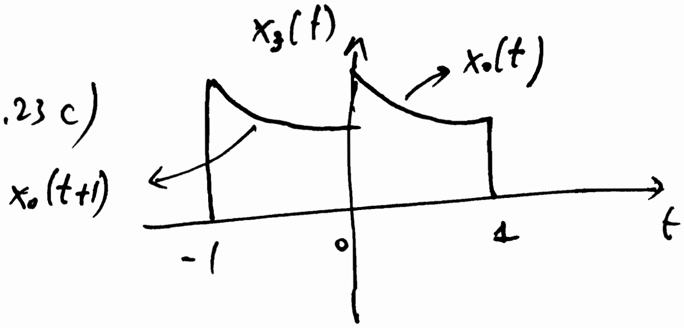
$$= \frac{e^{-1} 2j \sin\omega + j\omega e^{-1} 2\cos\omega - 2j\omega}{1+\omega^2}$$

$$\hat{X}_2(j\omega) = \frac{2}{1+\omega^2}\left(-j\omega + j\omega \frac{\cos\omega}{e} + j\frac{\sin\omega}{e} \right)$$

$$\boxed{\hat{X}_2(j\omega) = \frac{2j}{1+\omega^2}\left(\omega\left(-1 + \frac{\cos\omega}{e}\right) + \frac{\sin\omega}{e} \right)}$$

c) & d).

4.23 c)



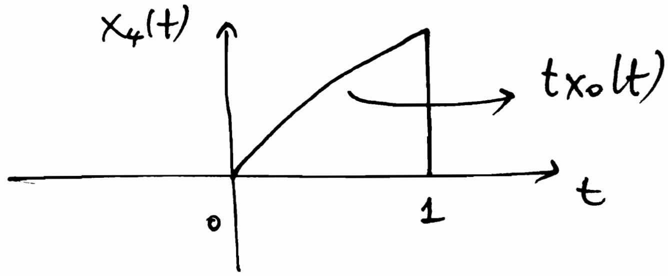
$$x_3(t) = \underbrace{x_o(t+1)}_{\text{time shift}} + x_o(t)$$

Find $\hat{X}_3(j\omega)$:

$$\hookrightarrow \underbrace{e^{j\omega 1}}_{t_o=1} \hat{X}_o(j\omega) + \hat{X}_o(j\omega) = (e^{j\omega}+1)\underbrace{\frac{1-e^{-(1+j\omega)}}{1+j\omega}}_{\hat{X}_o(j\omega)}$$

$$\hat{X}_3(j\omega) = \frac{e^{j\omega} - e^{-1} + 1 - e^{-(1+j\omega)}}{1+j\omega} = \frac{1+e^{j\omega} - e^{-1}(1+e^{-j\omega})}{1+j\omega}$$

4.23 d)



$$\hat{X}_4(j\omega) = -\frac{d}{d(j\omega)}\hat{X}_o(j\omega) = -\frac{d}{d(j\omega)}\frac{1-e^{-(1+j\omega)}}{1+j\omega}$$

Property of Fourier Transform

$$= -\left[\frac{e^{-(1+j\omega)}}{1+j\omega} - \frac{1-e^{-(1+j\omega)}}{(1+j\omega)^2}\right]$$

$$= \frac{1-e^{-(1+j\omega)} - (1+j\omega)e^{-(1+j\omega)}}{(1+j\omega)^2}$$

$$= \frac{1-2e^{-(1+j\omega)} - j\omega e^{-(1+j\omega)}}{(1+j\omega)^2}$$

4.26)

a) ii)

$$x(t) = t e^{-2t} u(t)$$
$$h(t) = t e^{-4t} u(t)$$

Find $x * h = F^{-1}[\hat{X}(j\omega) \cdot \hat{H}(j\omega)]$

convolution property of Fourier Transform
"Fourier transform of a convolution is the product of the Fourier transforms"

$$\hat{X}(j\omega) = -\frac{d}{d(j\omega)}\left[F[e^{-2t}u(t)]\right] = -\frac{d}{d(j\omega)}\left[\frac{1}{2+j\omega}\right] = \frac{1}{(2+j\omega)^2}$$

$$\hat{H}(j\omega) = \frac{1}{(4+j\omega)^2}$$

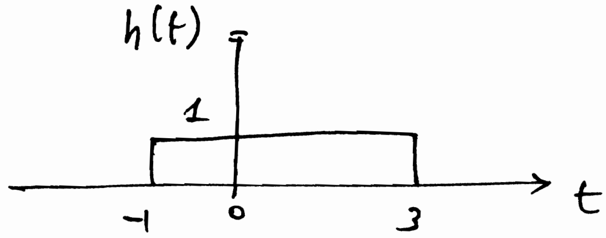
$$x(t) * h(t) = F^{-1}\left[\frac{1}{(2+j\omega)^2} \cdot \frac{1}{(4+j\omega)^2}\right]$$

Partial Fraction Expansion:

$$\left[\underbrace{\frac{1}{(2+j\omega)^2}}_{\text{double pole}} \cdot \underbrace{\frac{1}{(4+j\omega)^2}}_{\text{double pole}}\right] = \left[\frac{\frac{1}{4}}{2+j\omega} + \frac{\frac{1}{4}}{(2+j\omega)^2} - \frac{\frac{1}{4}}{4+j\omega} + \frac{\frac{1}{4}}{(4+j\omega)^2}\right]$$

$$x(t) * h(t) = F^{-1}[\;] = \frac{1}{4}e^{-2t}u(t) + \frac{1}{4}te^{-2t}u(t) - \frac{1}{4}e^{-4t}u(t) + \frac{1}{4}te^{-4t}u(t)$$
$$= \frac{1}{4}e^{-2t}u(t)\left[1 + t - e^{-2t} + te^{-2t}\right]$$

4.26 b) $x(t) = e^{-(t-2)} \underbrace{u(t-2)}_{b/w\ 2\ \&\ \infty}$



(rectangular pulse of width 4 centered @ 1)

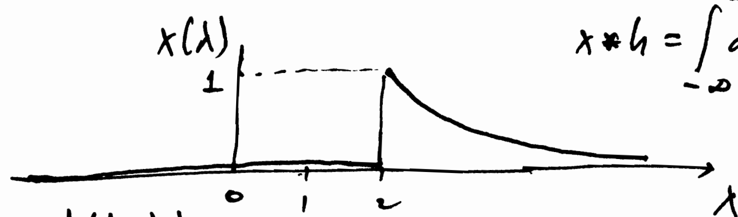
~~Sho~~ Verify the convolution property of Fourier Transform:

1) Calculate $x(t) * h(t)$; then $F[x(t) * h(t)]$

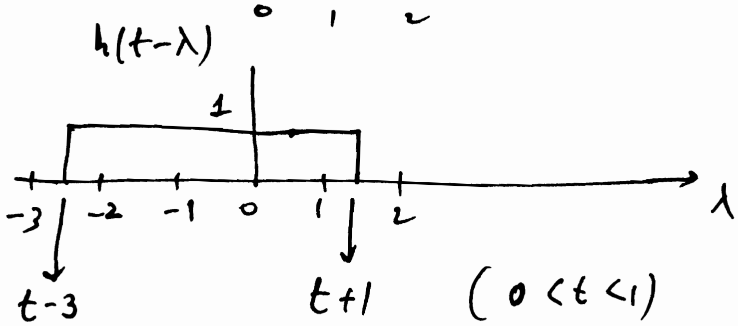
2) Calculate $\underbrace{F[x(t)]}_{\hat{X}(j\omega)}$; $\underbrace{F[h(t)]}_{\hat{H}(j\omega)}$; then $\hat{X} \cdot \hat{H}$

3) Check if 1) & 2) give same result.

1)



$$x * h = \int_{-\infty}^{\infty} d\lambda \; x(\lambda) h(t-\lambda)$$



$-\infty < t < 1 \rightarrow$ No overlap b/w $x(\lambda)$ & $h(t-\lambda) \rightarrow x * h = 0$

$1 < t \leq 5 \rightarrow$ Overlap b/w $2 \rightarrow t+1$

$$x * h = \int_{2}^{t+1} d\lambda \; e^{-(\lambda-2)} \cdot 1 = \frac{e^{-(\lambda-2)}}{-1}\Bigg]_{2}^{t+1} = 1 - e^{-(t-1)}$$

$5 < t < \infty \rightarrow$ Overlap b/w $t-3 \rightarrow t+1$

$$x * h = \int^{t+1} d\lambda \; e^{-(\lambda-2)} = \frac{e^{-(\lambda-2)}}{\quad}\Bigg]^{t+1} = e^{-(t-5)} - e^{-(t-1)}$$

$$\rightarrow \quad x * h = \left\{\begin{matrix} 0 & t<1 \\ 1-e^{-(t-1)} & 1<t\le 5 \\ e^{-(t-5)} - e^{-(t-1)} & 5<t\le\infty \end{matrix}\right\} = y(t)$$

$$\hat{Y}(j\omega) = F[\underbrace{x*h}_{y(t)}] = \int_{-\infty}^{\infty} dt\; y(t)\, e^{-j\omega t} = \int_1^5 dt\,\left(1-e^{-(t-1)}\right)e^{-j\omega t}$$

$$+ \int_5^\infty dt\,\left(e^{-(t-5)} - e^{-(t-1)}\right) e^{-j\omega t}$$

$$= \int_1^5 dt\; e^{-j\omega t} \underbrace{- e\int_1^5 dt\; e^{-(j\omega+1)t}} + e^5\int_5^\infty dt\; e^{-(j\omega+1)t} \underbrace{- e\int_5^\infty dt\; e^{-(j\omega+1)t}}$$

$$= \int_1^5 dt\, e^{-j\omega t} - e\int_1^\infty dt\; e^{-(j\omega+1)t} + e^5\int_5^\infty dt\; e^{-(j\omega+1)t}$$

$$= \left[\frac{e^{-j\omega t}}{-j\omega}\right]_1^5 - e\left[\frac{e^{-(j\omega+1)t}}{-(j\omega+1)}\right]_1^\infty + e^5\left[\frac{e^{-(j\omega+1)t}}{-(j\omega+1)}\right]_5^\infty$$

$$= \frac{e^{-j\omega} - e^{-j\omega 5}}{j\omega} - e\,\frac{e^{-(j\omega+1)}}{j\omega+1} + e^5\,\frac{e^{-(j\omega+1)5}}{j\omega+1}$$

$$= \frac{(j\omega+1)\left(e^{-j\omega}-e^{-j\omega 5}\right) - e^{-j\omega}j\omega + e^{-j\omega 5}j\omega}{j\omega(j\omega+1)}$$

$$= \frac{(j\omega+1)\left(e^{-j\omega}-e^{-j\omega 5}\right) - j\omega\left(e^{-j\omega}-e^{-j\omega 5}\right)}{j\omega(j\omega+1)}$$

$$= \frac{(\cancel{j\omega}+1-\cancel{j\omega})\left(\left(e^{-j\omega}-e^{-j\omega 5}\right)\right)}{j\omega(j\omega+1)} = \frac{e^{-3j\omega}\left(e^{+2j\omega}-e^{-2j\omega}\right)}{j\omega(j\omega+1)}$$

$$\rightarrow \hat{Y}(j\omega) = \frac{e^{-j\omega 3}\, 2\cancel{j} \sin 2\omega}{\cancel{j}\omega(j\omega+1)} = \frac{2\sin 2\omega \; e^{-j\omega 3}}{\omega(j\omega+1)}$$

Fourier Transform of $x * h$

2) $x(t) = e^{-(t-2)} u(t-2) \equiv x_0(t-2)$

where $x_0(t) = e^{-t} u(t)$

$\downarrow$

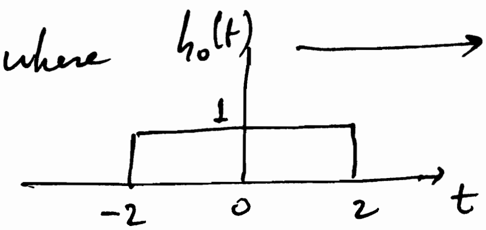
$\hat{X}_0(j\omega) = \dfrac{1}{1+j\omega}$ (Table 4.2)

$\rightarrow \hat{X}(j\omega) = e^{-j\omega 2} \dfrac{1}{1+j\omega}$ (Table 4.1)

$\downarrow$

$t_0 = 2$

$h(t) = h_0(t-1)$ where $h_0(t) \longrightarrow h_0(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & \text{otherwise.} \end{cases}$



rectangular pulse centered @ 0 & width 4

$\boxed{T_1} = 2.$

$\downarrow$

Table 4.1

$\boxed{\text{half of width}}$

$\hat{H}_0(j\omega) = \dfrac{2\sin \omega 2}{\omega}$

$\downarrow$

$T_1 = 2.$

$\rightarrow \hat{H}(j\omega) = e^{-j\omega 1} \dfrac{2\sin \omega 2}{\omega}$

$$\hat{X} \cdot \hat{H} = e^{-j\omega 3} \frac{2\sin \omega 2}{\omega(1+j\omega)}$$ ✓ Same as from 1)

$\rightarrow$ Convolution property of Fourier Transform verified in this example!

4.28

a) $p(t)$: periodic , fundamental freq ω_0 $\left(= \frac{2\pi}{T}\right)$

$\hookrightarrow p(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}$

$x(t) \to F[x(t)] = \hat{X}(j\omega)$

Find $F[x(t) p(t)]$

a1) Table 4.1 : $x(t)p(t) \xrightarrow{F} \frac{1}{2\pi} \hat{X}(j\theta) * \hat{P}(j\theta)$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta \, \hat{X}(j\theta) \hat{P}(j\omega - j\theta)$$

a2) Get $\hat{P}(j\omega) = F[p(t)] = \sum_{n=-\infty}^{\infty} a_n \underbrace{F[e^{jn\omega_0 t}]}_{\substack{2\pi \delta(\omega - n\omega_0) \\ \text{Table 4.2}}}$

$$= 2\pi \sum_{n=-\infty}^{\infty} a_n \delta(\omega - n\omega_0)$$

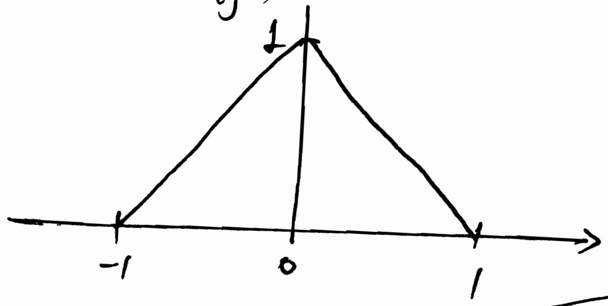
$$x(t)p(t) \xrightarrow{F} \frac{1}{\cancel{2\pi}} \hat{X}(j\omega) * \cancel{2\pi} \sum_{n=-\infty}^{\infty} a_n \delta(\omega - n\omega_0)$$

Note: convolution with a delta (δ) : $f[n] * \delta[n-j] = f[n-j]$

$$x(t)p(t) \xrightarrow{F} \sum_{n=-\infty}^{\infty} a_n \hat{X}(j\omega) * \delta[\omega - n\omega_0]$$

$$= \sum_{n=-\infty}^{\infty} a_n \hat{X}(j(\omega - n\omega_0))$$

b) Suppose $\hat{X}(j\omega)$



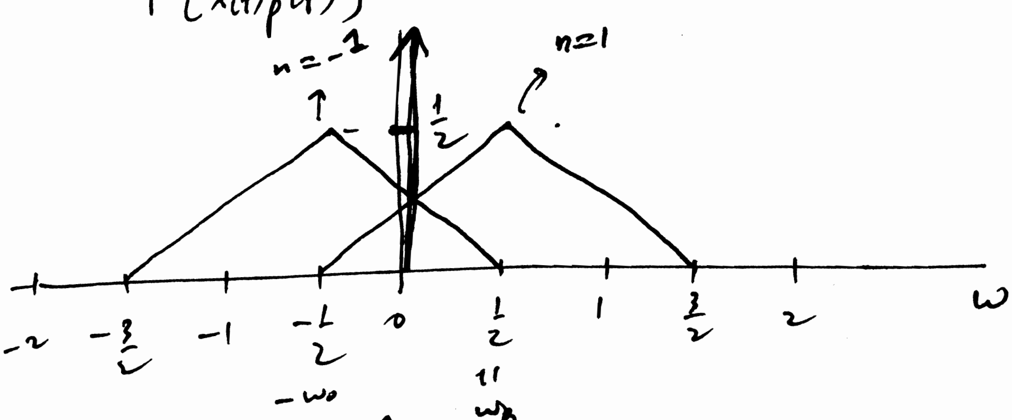
Sketch spectrum of
$y(t) = x(t)p(t)$

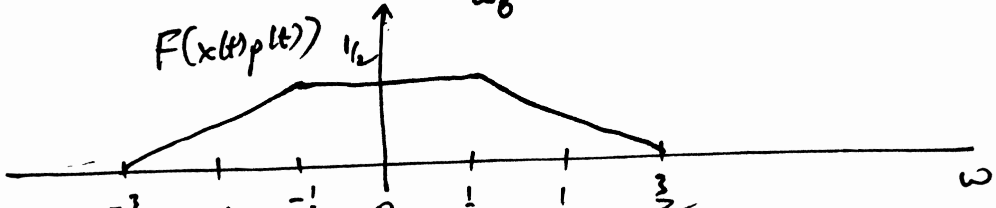
i) $p(t) = \cos\left(\frac{t}{2}\right)$ $\left\{ \begin{array}{l} \boxed{w_0 = \frac{1}{2}} \rightarrow T = \frac{2\pi}{w_0} = 4\pi \\ a_n = \frac{1}{2} \leftrightarrow \cos\frac{t}{2} = \boxed{\frac{e^{j\frac{t}{2}} + e^{-j\frac{t}{2}}}{2}} \end{array} \right.$

$n = 1$ & -1

From a): $x(t)p(t) \xrightarrow{F} \underbrace{\sum_{n=-\infty}^{\infty} a_n \hat{X}(j(\omega - n\omega_0))}$

repetition of the spectrum of x (or Fourier Transform of $x(t)$) centered @ multiple of w_0, weighed by Fourier Series Coefficients of $p(t)$: a_n's





Exam 2 on 12/2.
(HW4 due)