$$a_k = \frac{1}{N} e^{jk\frac{2\pi N}{N}} \frac{e^{-j\frac{k2\pi}{N}(N_1+\frac{1}{2})}\left[e^{j\frac{k2\pi}{N}(N_1+\frac{1}{2})} - e^{-j\frac{k2\pi}{N}(N_1+\frac{1}{2})}\right]}{e^{-jk\frac{\pi}{N}}\left[e^{jk\frac{\pi}{N}} - e^{-jk\frac{\pi}{N}}\right]}$$

$$a_k = \frac{1}{N} \frac{\cancel{2j} \sin \frac{k2\pi}{N}(N_1+\frac{1}{2})}{\cancel{2j} \sin \frac{k\pi}{N}}$$

(FS coefficients for discrete-time train of rectangular pulses of width $2N_1$ and period N)

When $\boxed{k=0}$ this gives ∞ → needs a different equation:

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} \underbrace{x[n]}_{1} \underbrace{e^{-jk\frac{2\pi}{N}n}}_{1}$$

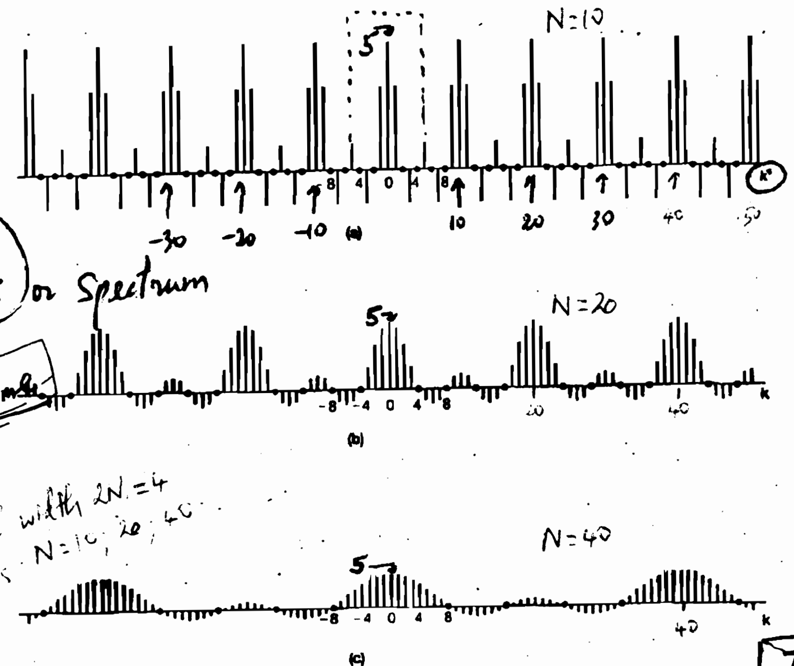
when $k=0$ or a multiple of N ($k = pN$; $p = 0, 1, 2, \ldots$)

$$\rightarrow e^{-jp\cancel{N}\frac{2\pi}{\cancel{N}}n} = 1$$

$$= \frac{1}{N} \sum_{n=-N_1}^{N_1} 1 = \frac{1}{N}(2N_1+1)$$

$$\rightarrow a_k = \begin{cases} \frac{1}{N} \frac{\sin \frac{k2\pi}{N}(N_1+\frac{1}{2})}{\sin \frac{k\pi}{N}} & k \neq 0 \\ \frac{1}{N}(2N_1+1) & k = 0 \text{ or multiple of } N \ (k = pN;\ p = 0, 1, 2, 3 \ldots) \end{cases}$$

$Na_k = 2N_1+1 = 5$
when $k = pN$, $p = 0, 1, 2, \ldots$



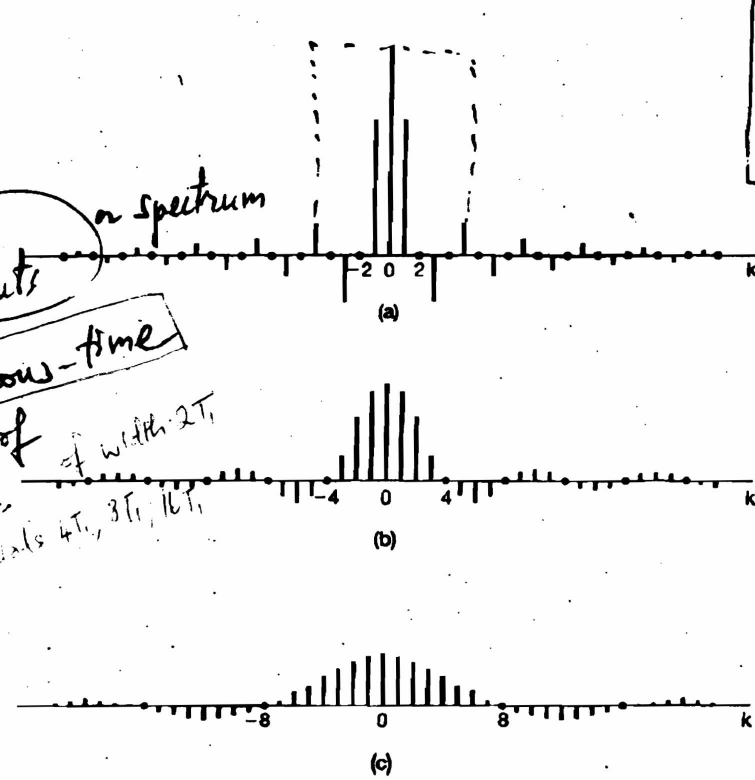
F.S. Coefficients or Spectrum of discrete-time train of pulses of width $2N_1=4$ and periods $N=10, 20, 40$

$$Na_k = \begin{cases} \frac{1}{N}\,\dfrac{\sin \frac{k\pi}{N}(N_1+\frac{1}{2})}{\sin \frac{k\pi}{N}} & k \neq 0 \\ 2N_1+1 & k = pN \\ & (p=0,1,2,\ldots) \end{cases}$$

Figure 3.17 Fourier series coefficients for the periodic square wave of Example 3.12; plots of Na_k for $2N_1 + 1 = 5$ and (a) $N = 10$; (b) $N = 20$; and (c) $N = 40$.

The effect of discretizing a signal is the repetition of the original spectrum in frequency

Reconstruction of a discrete-time signal requires prior application of a low-pass filter



F.S. Coefficients or spectrum of continuous-time train of pulses of width $2T_1$ and periods $4T_1, 8T_1, 16T_1$

$$Ta_k = \begin{cases} \dfrac{2\sin k\omega_0 T_1}{k\omega_0} & k \neq 0 \\ 2T_1 & k = 0 \end{cases}$$

Figure 3.7 Plots of the scaled Fourier series coefficients Ta_k for the periodic square wave with T_1 fixed and for several values of T: (a) $T = 4T_1$; (b) $T = 8T_1$; (c) $T = 16T_1$. The coefficients are regularly spaced samples of the envelope $(2\sin\omega T_1)/\omega$, where the spacing between samples, $2\pi/T$, decreases as T increases.

3.12

$x_1[n]$, period $N=4$; F.S. coefficients a_k: $a_0 = a_3 = 1$ & $a_1 = a_2 = 2$

$x_2[n]$, period $N=4$; F.S coefficients b_k: $b_0 = b_1 = b_2 = b_3 = 1$

Multiplication property in Table 3.1 (continuous time).

$$x(t)\,y(t) \longrightarrow \text{F.S coeff. } c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

$\left(x(t) \to \text{F.S. coeff } a_k\text{'s} \quad y(t) \to \text{F.S. coeff } b_k\text{'s} \right)$

Multiplication property in Table 3.2 (discrete-time)

$$x[n]\,y[n] \longrightarrow \text{F.S coeff} \quad c_k = \sum_{l=\langle N\rangle} a_l b_{k-l}$$

in this problem $N=4 \rightarrow \sum_{l=\langle N\rangle} = \sum_{l=0,1,2,3}$

$$c_k = a_0 b_k + a_1 b_{k-1} + a_2 b_{k-2} + a_3 b_{k-3}$$

F.S coeff for $x_1[n]\,x_2[n]$

$$c_0 = a_0 b_0 + a_1 b_{-1} + a_2 b_{-2} + a_3 b_{-3}$$

$b_{-1} = b_{-1+4} = b_3 = 1$

$b_{-2} = b_{-2+4} = b_2 = 1$

$b_{-3} = b_{-3+4} = b_1 = 1$

$$= 1\cdot 1 + 2\cdot 1 + 2\cdot 1 + 1\cdot 1 = 6$$

$$c_1 = a_0 b_1 + a_1 b_0 + a_2 b_{-1} + a_3 b_{-2} = 1\cdot 1 + 2\cdot 1 + 2\cdot 1 + 1\cdot 1 = 6$$

$$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0 + a_3 b_{-1} = 1\cdot 1 + 2\cdot 1 + 2\cdot 1 + 1\cdot 1 = 6$$

$\vdots$

$$\Rightarrow \boxed{c_k = 6 \quad \forall k}$$

Fourier Series & Linear Systems

$$x(t) \longrightarrow \boxed{\begin{matrix}\text{Linear}\\ h(t)\end{matrix}} \longrightarrow y(t)$$

$x(t)$ ↕ periodic signal ↕ a_k's & fundamental freq ω_0

$y(t)$ ↕ periodic ↕ b_k's & same fund. freq ω_0

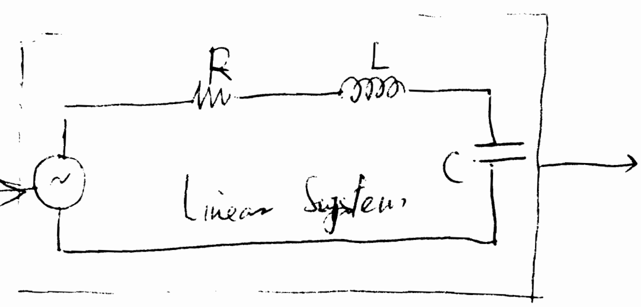
* Linear systems won't change frequency of input signal!

Using the convolution we will see that $\boxed{b_k = a_k \hat{H}(jk\omega_0)}$ (1)

Example:

$v_s(t)$

Sinusoid w/ freq. ω_s



$v_c(t)$ has fundamental freq ω_0

Derive of (1):

a) Instead of $x(t)$, we $a_k e^{jk\omega_0 t}$:

$$a_k e^{jk\omega_0 t} \longrightarrow \boxed{\begin{matrix}\text{Linear}\\ h(t)\end{matrix}} \longrightarrow a_k e^{jk\omega_0 t} * h(t)$$

$$a_k e^{jk\omega_0 t} * h(t) = a_k \int_{-\infty}^{\infty} d\tau \; h(\tau)\, e^{jk\omega_0 (t-\tau)}$$

$$= a_k e^{jk\omega_0 t} \underbrace{\int_{-\infty}^{\infty} d\tau \; h(\tau)\, e^{-jk\omega_0 \tau}}_{\text{Fourier Transform of } h(t) \,=\, \hat{H}(jk\omega_0)}$$

So

$$a_k e^{jk\omega_0 t} \longrightarrow \boxed{\begin{matrix}\text{Linear}\\ h(t)\end{matrix}} \longrightarrow a_k e^{jk\omega_0 t}\,\hat{H}(jk\omega_0)$$

b) Now use $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \longrightarrow \boxed{\begin{matrix}\text{Linear}\\ h(t)\end{matrix}} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\,\hat{H}(jk\omega_0)$$

Also $\quad y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} \longrightarrow \boxed{b_k = a_k \hat{H}(jk\omega_0)}$

<u>Conclusion</u>. F.S coeff of the output of a linear system are simply the <u>F.S. coeff's of the input multiplied by the Fourier transform of the impulse response</u>.

Fourier Series & Linear Systems (discrete signals)

$$x[n] \longrightarrow \boxed{\begin{array}{c}\text{Linear System}\\ h[n]\end{array}} \longrightarrow y[n]$$

$x[n]$: periodic with period N $\left(w_0 = \frac{2\pi}{N}\right)$

$y[n]$: periodic with period N (linear system)

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n} \qquad y[n] = \sum_{k=\langle N\rangle} b_k e^{jk\frac{2\pi}{N}n}$$

?

In 2 steps.

a)

$$e^{jk\frac{2\pi}{N}n} \longrightarrow \boxed{\begin{array}{c}\text{linear System}\\ h[n]\end{array}} \longrightarrow e^{jk\frac{2\pi}{N}n} * h[n]$$

$$= \sum_{m=-\infty}^{\infty} h[m] \cdot e^{jk\frac{2\pi}{N}(n-m)}$$

$$= e^{jk\frac{2\pi}{N}n} \underbrace{\sum_{m=-\infty}^{\infty} h[m] \cdot e^{-jk\frac{2\pi}{N}m}}_{\substack{\text{Discrete-time Fourier}\\ \text{Transform of } h[n]\\ \hat{H}(e^{jw}),\text{ where}\\ w = kw_0}}$$

$$e^{jk\frac{2\pi}{N}n} \longrightarrow \boxed{\begin{array}{c}\text{linear System}\\ h[n]\end{array}} \longrightarrow e^{jk\frac{2\pi}{N}n}\hat{H}(e^{jw})$$

b) $$\sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n} \longrightarrow \boxed{\begin{array}{c}\text{Linear System}\\ h[n]\end{array}} \longrightarrow \sum_{k=\langle N\rangle} \underbrace{\boxed{a_k \hat{H}(e^{j\omega})}}_{b_k} e^{jk\frac{2\pi}{N}n}$$

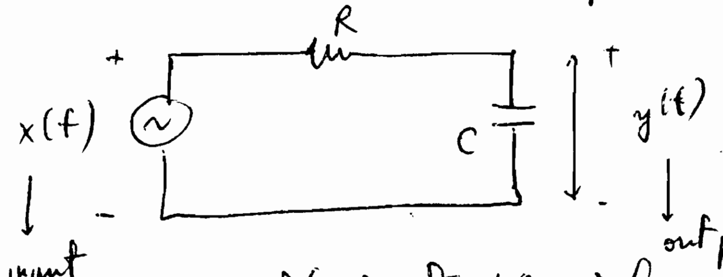
$$\Rightarrow \boxed{b_k = a_k \underbrace{\hat{H}(e^{j\omega})}}$$

discrete-time Fourier Transform of $h[n]$

Filters as Linear Systems:

Example: RC circuit acts as a "low-pass" filter, because the Fourier Transform of the impulse response for an RC circuit shows a "low-pass" behavior

voltage at AC source → $x(t)$



voltage at C

input

output

↓ input/output differential equation (Loop analysis)

$$RC\frac{dy}{dt} + y(t) = x(t)$$

↳ solution: $$y(t) = \frac{1}{RC}\int_0^t d\lambda\; x(\lambda)\, e^{-\frac{1}{RC}(t-\lambda)}$$

$$= \frac{1}{RC}\int_0^\infty d\lambda\; x(\lambda)\, e^{-\frac{1}{RC}(t-\lambda)} u(t-\lambda)$$

$$= x(t) * h(t) = \int_0^\infty d\lambda\; x(\lambda) h(t-\lambda)$$

$$\rightarrow \boxed{h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)} = \text{impulse response for RC circuit.}$$

→ Fourier transform of $h(t)$ is:

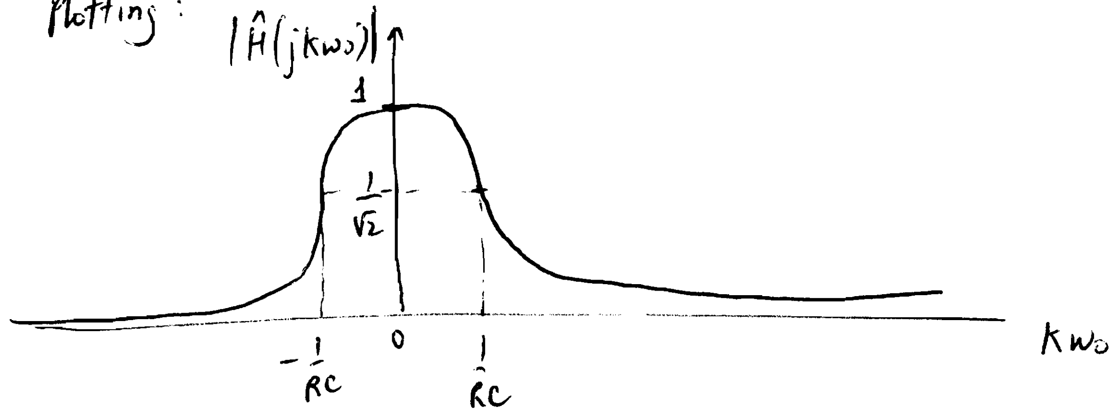
$$\hat{H}(j\omega) = \hat{H}(jk\omega_0) = \int_{-\infty}^{\infty} dt\; h(t)\, e^{-jk\omega_0 t} = \frac{1}{RC}\int_{-\infty}^{\infty} dt\; e^{-\frac{t}{RC}}\, u(t)\, e^{-jk\omega_0 t}$$

$$= \frac{1}{RC}\int_{0}^{\infty} dt\; e^{-(jk\omega_0 + \frac{1}{RC})t}$$

$$= \frac{1}{RC}\left[\frac{e^{-(jk\omega_0+\frac{1}{RC})t}}{-(jk\omega_0+\frac{1}{RC})}\right]_{t=0}^{t=\infty} = \frac{1}{RC}\,\frac{0-1}{-(jk\omega_0+\frac{1}{RC})}$$

$$\boxed{\hat{H}(j\omega) = \frac{1}{1+jk\omega_0 RC}}$$

↓ Plotting:



low-pass behavior = passing low frequencies

→ Conclusion: in general: systems governed by input/output differential equation of type → $\boxed{\frac{dy}{dt} + ay(t) = bx(t)}$ is a low-pass filter!

→ What discrete-time system would behave as a low-pass filter?

If we use finite-difference approximation for time derivative:

$$\frac{dy}{dt} = \lim_{\Delta \to 0} \frac{y(t) - y(t-\Delta)}{\Delta} \rightarrow \text{Finite difference approx:}$$

↓

exact

$$\frac{dy}{dt} = \frac{y(t) - y(t-\Delta)}{\Delta}$$

$$= \frac{y[n] - y[n-1]}{\Delta}$$

$t = n\Delta$ in discrete time
$t - \Delta = n\Delta - \Delta = (n-1)\Delta$
we ignore Δ in time index notation.

$$\frac{dy}{dt} + ay = bx$$

↓

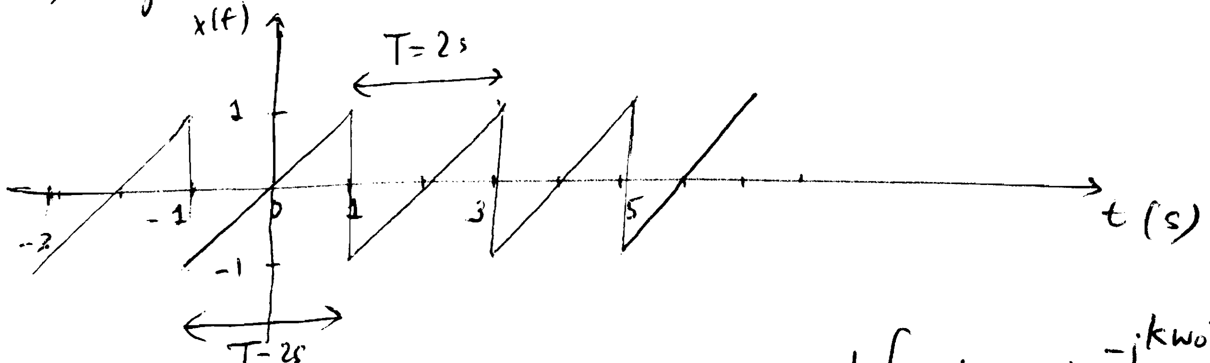
$$\frac{y[n] - y[n-1]}{\Delta} + a y[n] = b x[n]$$

$$\left(\frac{1}{\Delta} + a\right) y[n] - \frac{1}{\Delta} y[n-1] = b x[n]$$

$$\underbrace{\frac{1}{b}\left(\frac{1}{\Delta} + a\right)}_{\beta} y[n] - \underbrace{\frac{1}{b\Delta}}_{\alpha} y[n-1] = x[n]$$

→ $\boxed{\beta y[n] + \alpha y[n-1] = x[n]}$ input/output equation of a discrete-time system that behaves like a low-pass filter.

3.22 a) Figure P3.22a



Find a_k's for $x(t)$ $\rightarrow$ $a_k = \frac{1}{T}\int_T dt\, x(t)\, e^{-jk\omega_0 t}$ centered period.

period of $x(t)$: $T = 2s$; $\omega_0 = \frac{2\pi}{T} = \pi$ (from the figure above)

$$a_k = \frac{1}{2}\int_{-1}^{1} dt\, t\, e^{-jk\omega_0 t}$$ $\rightarrow$ { integration by parts; derivative trick $\leftarrow$; table

$$a_k = \frac{1}{2}\,\frac{1}{-jk}\,\frac{d}{d\omega_0}\left[\int_{-1}^{1} dt\, e^{-jk\omega_0 t}\right]$$

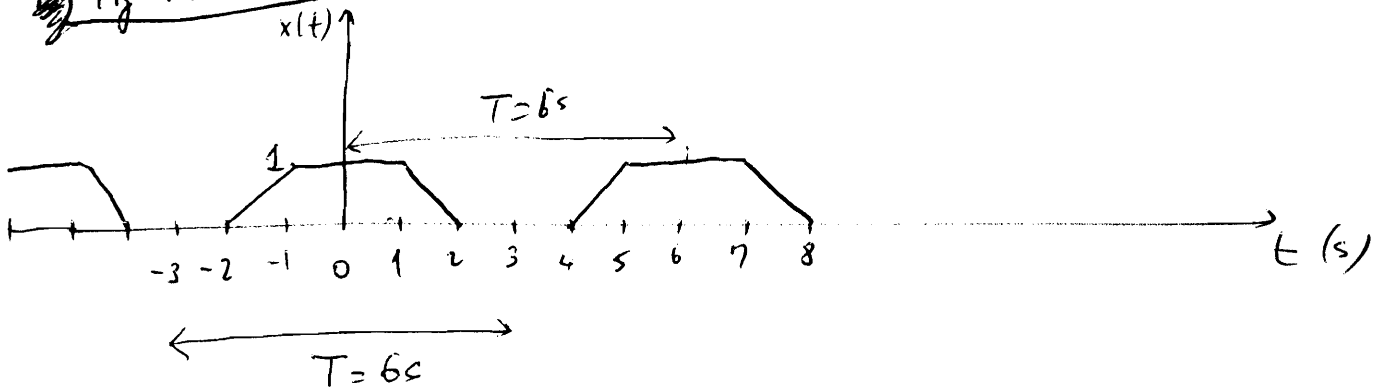
$$\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_{-1}^{1} = \frac{e^{-jk\omega_0} - e^{+jk\omega_0}}{-jk\omega_0} = \frac{\cancel{-}2\cancel{j}\sin k\omega_0}{\cancel{+j}k\omega_0}$$

$$a_k = \frac{1}{2}\,\frac{1}{-jk}\,\frac{d}{d\omega_0}\left(\frac{2\sin k\omega_0}{k\omega_0}\right) = \frac{1}{-jk}\left[\frac{\cancel{k}\cos k\omega_0}{\cancel{k}\omega_0} - \frac{\sin k\omega_0}{k\,\omega_0^2}\right]$$

$$= \frac{1}{-jk\omega_0}\left[\cos k\omega_0 - \frac{\sin k\omega_0}{k\omega_0}\right] \underset{\omega_0=\pi}{=} \frac{1}{-jk\pi}\left[\underbrace{\cos k\pi}_{(-1)^k} - \underbrace{\frac{\sin k\pi}{k\pi}}_{0}\right]$$

$$\Rightarrow a_k = \begin{cases} \frac{j}{k\pi}(-1)^k & k \neq 0 \\ \frac{1}{2}\int_{-1}^{1} dt\, t = \left[\frac{t^2}{4}\right]_{-1}^{1} = 0 & k = 0 \end{cases}$$

Fig P3.22b:



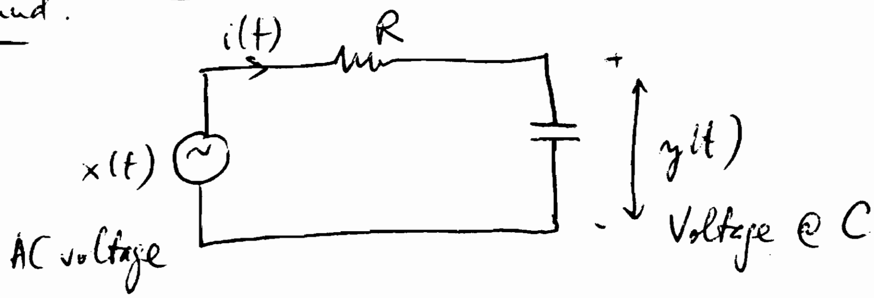
$$a_k = \frac{1}{T}\int_T dt \; x(t)\, e^{-jk\omega_0 t}$$

$\rightarrow$ centered period of $x(t)$ $\rightarrow$ $x(t) = \begin{cases} t+2 & -2<t<-1 \\ 1 & -1<t<1 \\ -t+2 & 1<t<2 \end{cases}$

$$\rightarrow a_k = \frac{1}{6}\left\{ \int_{-2}^{-1} dt\; (t+2)\, e^{-jk\omega_0 t} + \int_{-1}^{1} dt\; e^{-jk\omega_0 t} + \int_{1}^{2} dt\; (-t+2)\, e^{-jk\omega_0 t} \right\}$$

use from previous Fig P.3.22a: $\displaystyle\int_a^b dt\; t\, e^{-jk\omega_0 t} = \frac{1}{-jk}\frac{d}{d\omega_0}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_a^b$

Background. → for pg 65



Loop equation: $x(t) = i(t)R + y(t)$

↕

input/output equation (involving x & y) → eliminate $i(t)$

$$y(t) = \frac{1}{C}\int dt\, i(t) \longrightarrow \frac{dy}{dt} = \frac{1}{C} i(t) \rightarrow i(t) = C\frac{dy}{dt}$$

$$\boxed{x(t) = CR\frac{dy}{dt} + y(t)}$$ 1st order differential equation. (non-homogeneous)

$$\rightarrow \boxed{\frac{dy}{dt} + ay = bx}$$ (general form) → RC circuit $\begin{cases} a = \frac{1}{RC} \\ b = \frac{1}{RC} \end{cases}$

↳ solution: $y(t) = y(t_0)\,e^{-a(t-t_0)} + b\int_{t_0}^{t} d\lambda\, x(\lambda)\, e^{-a(t-\lambda)}$

Why?: using the integrating factor be^{at}: multiplying it to both sides of the D.E:

$$\underbrace{\frac{dy}{dt}be^{at} + aybe^{at}}_{\cancel{b}\frac{d}{dt}\left(ye^{at}\right)} = b^2xe^{at}$$

$$= \cancel{b}xe^{at} \rightarrow \left. ye^{at}\right]_{t_0}^{t} = b\int_{t_0}^{t} dt\, xe^{at}$$

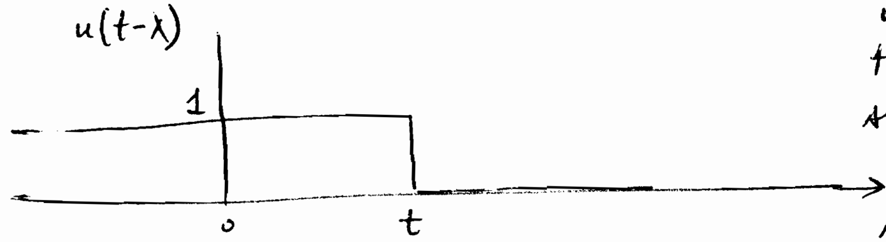
$$y(t)e^{at} - y(t_0)e^{at_0} = b\int_{t_0}^{t} d\lambda\, x(\lambda)\, e^{a\lambda}$$

Then dividing both sides by e^{at}:

$$y(t) = y(t_0)\,e^{-a(t-t_0)} + b\int_{t_0}^{t} d\lambda\, x(\lambda)\, e^{-a(t-\lambda)}$$

For our RC circuit: $\left\{\begin{array}{l} t_0 = 0 \\ y(0) = 0 \\ a = \frac{1}{RC} = b \end{array}\right\} \rightarrow y(t) = \frac{1}{RC}\int_0^t d\lambda\; x(\lambda)\, e^{-\frac{1}{RC}(t-\lambda)}$ ①

Now I can rewrite the upper integration limit t to ∞ if a factor of $u(t-\lambda)$ is incorporated into the integrand since:



$u(t-\lambda)$ is $u(\lambda)$ time reversed & shifted by $t > 0$

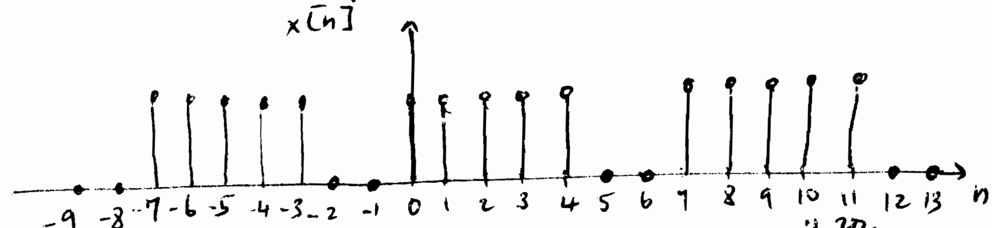
$$\Rightarrow \quad y(t) = \frac{1}{RC}\int_0^{\infty} d\lambda\; x(\lambda)\, e^{-\frac{1}{RC}(t-\lambda)}\, u(t-\lambda) \quad ②$$

① & ② are equal since the integrand in ② is nontrivial only b/w 0 & t

---

3.28 Find F.S. coefficients (a_k) for different discrete-time periodic signals.

a)

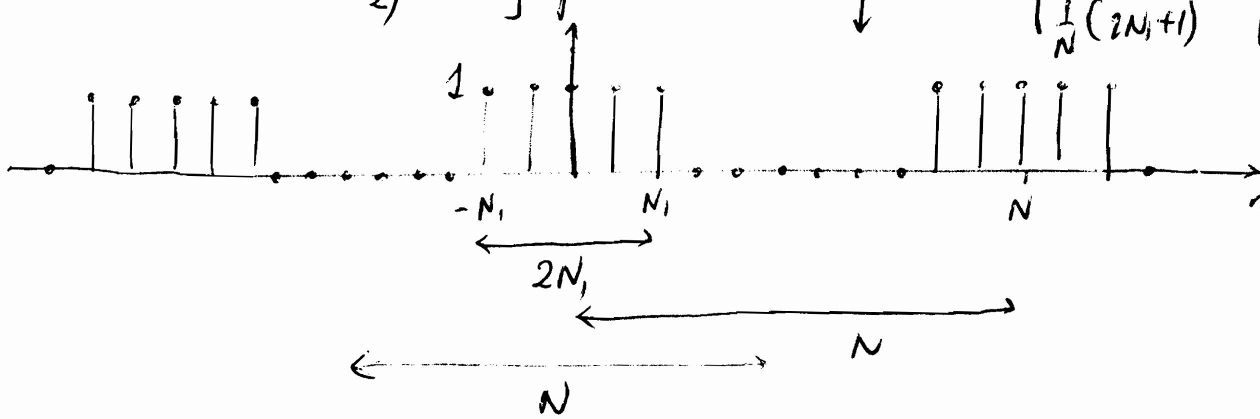


$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n}$

N: period of signal

Alternatives: $\left\{\begin{array}{l} 1)\quad a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jk\frac{2\pi}{N}n} \\ 2)\quad \text{Using previous results: (pg 59)} \end{array}\right.$

$$a_k = \begin{cases} \frac{1}{N}\,\dfrac{\sin \frac{\pi k}{N}(2N_1+1)}{\sin \frac{k\pi}{N}} & k \neq 0 \\ \frac{1}{N}(2N_1+1) & k = 0 \end{cases}$$



Alternative 2) (shorter) . we do not need to do a new geometric sum but will need to incorporate a time shift into the a_k's

Comparing with the original signal $\begin{cases} N_1 = 2 \\ N = 7 \\ \text{time shift } n_0 = +2 \end{cases}$ ← $\boxed{\text{Table 3.2}}$

$$x[n-n_0] \longleftrightarrow a_k e^{-jk\frac{2\pi}{N}n_0}$$

$$\rightarrow a_k = \begin{cases} \dfrac{1}{7} \dfrac{\sin\left[\frac{\pi k}{7}(5)\right]}{\sin\left[\frac{k\pi}{7}\right]} e^{-jk\frac{2\pi}{7}2} & \underline{k \neq 0} = \dfrac{1}{7}\dfrac{\sin\left(\frac{5\pi k}{7}\right)}{\sin\left(\frac{\pi k}{7}\right)} e^{-jk\frac{4\pi}{7}} \\ \dfrac{1}{7}(5)\, e^{-jk\frac{2\pi}{7}2} & \underline{k=0} = \dfrac{5}{7} e^{-jk\frac{4\pi}{7}} \end{cases}$$

3.32

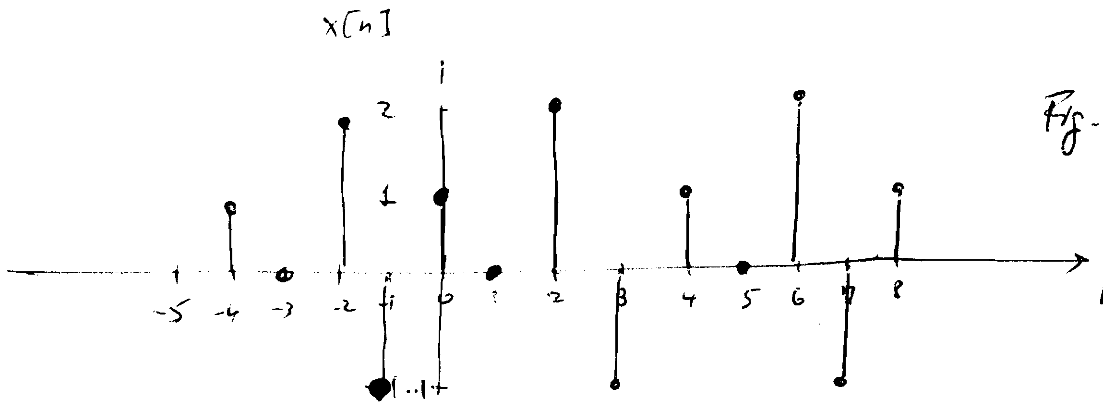


Fig. P3.32

$x[n]$: periodic with $N = 4$ $\longrightarrow$ $x[n] = \sum_{k=\langle 4 \rangle} a_k e^{jk\frac{2\pi}{4}n}$

$$\Rightarrow x[n] = \sum_{k=0}^{3} a_k e^{jk\frac{2\pi}{4}n}$$

To find a_k's (4) → set the system of 4 equations:

$$n=0 \rightarrow x[0] = 1 = a_0 + a_1 + a_2 + a_3$$

$$n=1 \rightarrow x[1] = 0 = a_0 + a_1 e^{j\frac{\pi}{2}} + a_2 e^{j\pi} + a_3 e^{j\frac{3\pi}{2}} = a_0 + ja_1 - a_2 - ja_3$$

$$n=2 \rightarrow x[2] = 2 = a_0 + a_1 e^{j\pi} + a_2 e^{j2\pi} + a_3 e^{j3\pi} = a_0 - a_1 + a_2 - a_3$$

$$n=3 \rightarrow x[3] = -1 = a_0 + a_1 e^{j\frac{3\pi}{2}} + a_2 e^{j3\pi} + a_3 e^{j\frac{9\pi}{2}} = a_0 - ja_1 - a_2 + ja_3$$

$$\rightarrow \begin{cases} 1 = a_0 + a_1 + a_2 + a_3 & (1) \\ 0 = a_0 + ja_1 - a_2 - ja_3 & (2) \\ 2 = a_0 - a_1 + a_2 - a_3 & (3) \\ -1 = a_0 - ja_1 - a_2 + ja_3 & (4) \end{cases}$$

$$(2)+(4) \rightarrow -1 = 2a_0 - 2a_2$$

$$(1)+(3) \rightarrow 3 = 2a_0 + 2a_2 \longrightarrow a_2 = \frac{3-2a_0}{2} \rightarrow \boxed{a_2 = 1}$$

$$2 = 4a_0 \rightarrow \boxed{a_0 = \frac{1}{2}}$$

$$2)\quad 0 = \frac{1}{2} + ja_1 - 1 - ja_3 \rightarrow \frac{1}{2} = ja_1 - ja_3$$

$$4)\quad -1 = \frac{1}{2} - ja_1 - 1 + ja_3 \rightarrow -\frac{1}{2} = -ja_1 + ja_3$$

$$1)\quad 1 = \frac{1}{2} + a_1 + 1 + a_3 \rightarrow -\frac{1}{2} = a_1 + a_3$$

$$3)\quad 2 = \frac{1}{2} - a_1 + 1 - a_3 \rightarrow \frac{1}{2} = -a_1 - a_3$$

$$\begin{cases} \frac{1}{2}j = -a_1 + a_3 \\ \frac{1}{2} = -a_1 - a_3 \end{cases} \rightarrow \begin{array}{l} \boxed{a_1 = -\frac{1}{4}(1+j)} \\ \boxed{a_3 = -\frac{1}{4}(1-j)} \end{array}$$

Try to obtain same results using formula for FS coefficients instead of solving a system of 4 equations.

$$a_k = \frac{1}{4} \sum_{n=0}^{3} x[n]\, e^{-jk\left(\frac{2\pi}{4}\right)n}.$$

$k=0 \quad a_0 = \frac{1}{4}\left\{ \underset{x[0]}{\underset{\downarrow}{1}} + \underset{x[1]}{\underset{\downarrow}{0}} + \underset{x[2]}{\underset{\downarrow}{2}} \underset{x[3]}{\underset{\downarrow}{-1}} \right\} = \frac{1}{2}$ ✓

$k=1 \quad a_1 = \frac{1}{4}\left\{ 1 + 0 + 2 \underbrace{e^{-j\frac{4\pi}{4}}}_{-1} -1 \underbrace{e^{-j\frac{6\pi}{4}}}_{j} \right\} = \frac{1}{4}\{-1-j\}$ ✓

$k=2 \quad a_2 = \frac{1}{4}\left\{ 1 + 0 + 2 \underbrace{e^{-j\frac{8\pi}{4}}}_{1} -1 \underbrace{e^{-j3\pi}}_{-1} \right\} = \frac{1}{4}\{4\} = 1$ ✓

$k=3 \quad a_3 = \frac{1}{4}\left\{ 1 + 0 + 2 \underbrace{e^{-j3\pi}}_{-1} -1 \underbrace{e^{-j\frac{\pi}{2}}}_{-j} \right\} = \frac{1}{4}\{-1+j\}$ ✓

3.46

a) Given $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ & $y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$

(both x & y are periodic with same period $\frac{2\pi}{\omega_0}$)

Prove that, if $z(t) \equiv \underset{\text{product}}{x(t) \cdot y(t)} = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$

$\Rightarrow \boxed{c_k = \sum_{n=-\infty}^{\infty} a_n b_{k-n}}$ c_k's are the convolution of a_k & b_k

<u>Proof</u>:

$$x(t)y(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \cdot \underbrace{\sum_{\ell=-\infty}^{\infty} b_\ell e^{j\ell\omega_0 t}}$$

use ℓ as dummy index for this summation for not missing cross terms in the big product.

$$\overset{\text{regroup}}{=} \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} a_k b_\ell e^{j(k+\ell)\omega_0 t}$$

k & l are dummy indices $\rightarrow$ rename $k \rightarrow k-l$

$$x(t)\,y(t) = \sum_{k=-\infty}^{\infty} \underbrace{\left(\sum_{l=-\infty}^{\infty} a_{k-l}\, b_l \right)}\, \underbrace{e^{j(k-l+l)\omega_0 t}}_{e^{jk\omega_0 t}}$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{\left(\sum_{l=-\infty}^{\infty} a_{k-l}\, b_l \right)}\, e^{jk\omega_0 t}$$

Recall: $z(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{jk\omega_0 t}$

$\rightarrow$ $c_k = \sum_{l=-\infty}^{\infty} a_{k-l}\, b_l$ or $l \rightarrow n$ $c_k = \sum_{n=-\infty}^{\infty} a_{k-n}\, b_n$

$$c_k = b * a \underset{\substack{\downarrow \\ \text{convolution} \\ \text{is commutative}}}{=} a * b = \underbrace{\sum_{n=-\infty}^{\infty} a_n\, b_{k-n}}_{\text{what we wanted!}}$$

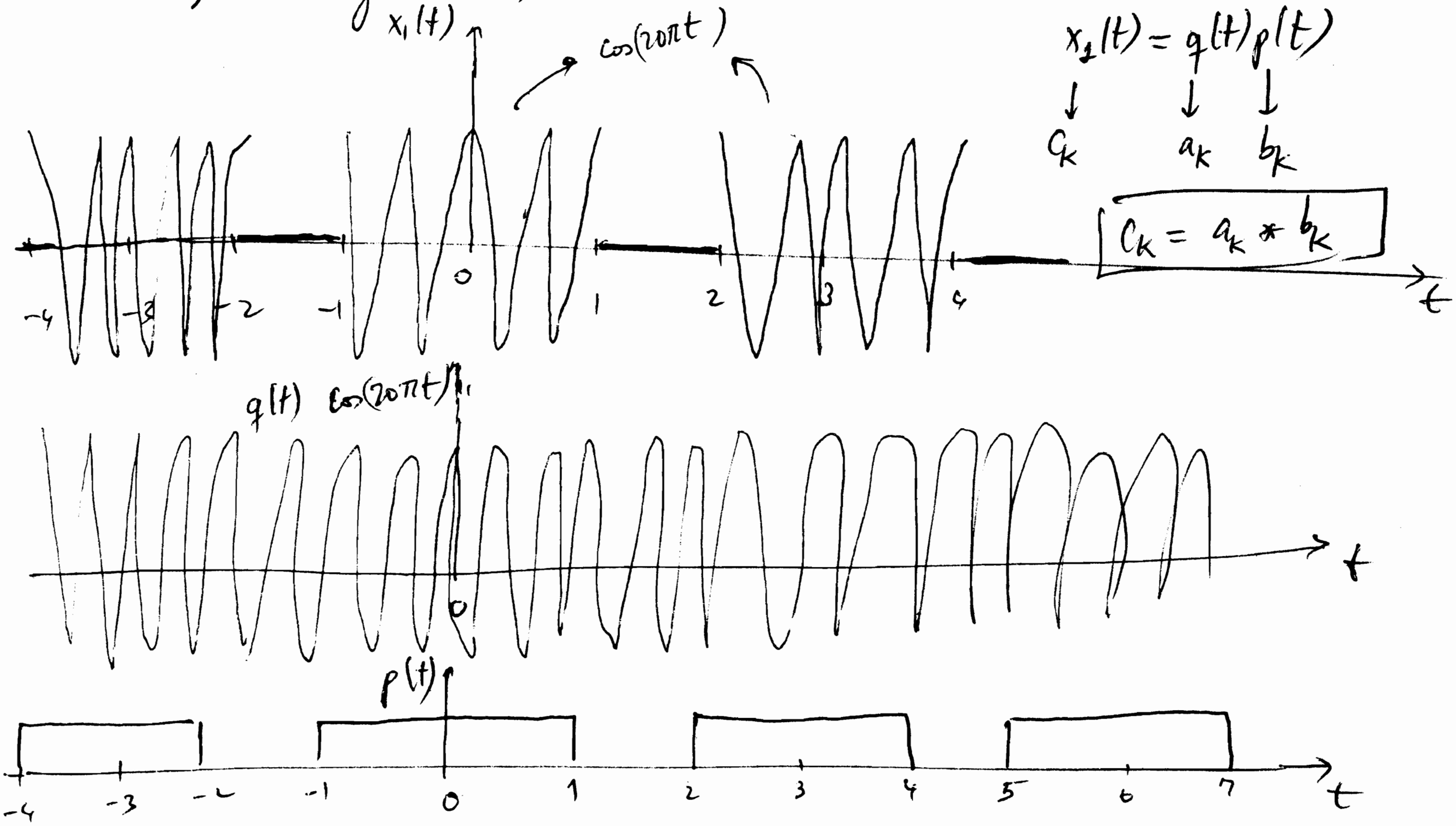
b) Fig P3.46 a)

$x_2(t) = q(t)\,p(t)$

$\downarrow \quad \downarrow \quad \downarrow$

$c_k \quad a_k \quad b_k$

$$\boxed{c_k = a_k * b_k}$$



Strategy: find a_k (F.S. coefficients for $\cos(20\pi t)$)
& b_k (F.S coefficients for $p(t)$: train of rectangular pulses of width 2 ($T_1=1$) & period 3 ($T=3$))

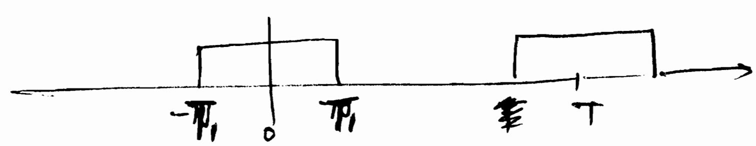
a_k's: $\rightarrow$ $$g(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{where } \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{3}$$
$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{3}t}$$

also $\downarrow$
$$\cos(20\pi t) = \frac{e^{j20\pi t} + e^{-j20\pi t}}{2}$$

$$\longrightarrow a_k = \begin{cases} k=30 \rightarrow a_{30} = \frac{1}{2} \\ k=-30 \rightarrow a_{-30} = \frac{1}{2} \\ \text{otherwise} \quad a_k = 0 \end{cases}$$

Then:
$$\cos(20\pi t) = \frac{1}{2}\delta(k-30)\, e^{jk\frac{2\pi}{3}t} + \frac{1}{2}\delta(k+30)\, e^{jk\frac{2\pi}{3}t}$$
$$= \underbrace{\left[\frac{1}{2}\delta(k-30) + \frac{1}{2}\delta(k+30)\right]}_{a_k} e^{jk\frac{2\pi}{3}t}$$

b_k's:



$$\rightarrow b_k = \begin{cases} \dfrac{2\sin k\omega_0 T_1}{k\omega_0 T} & k \neq 0 \\ \dfrac{2T_1}{T} & k=0 \end{cases}$$

In this problem $\begin{cases} T_1 = 1 \\ T = 3 \end{cases}$ $\rightarrow$ $$b_k = \begin{cases} \dfrac{2\sin k\frac{2\pi}{3}\,1}{k\frac{2\pi}{\cancel{3}}\cancel{3}} = \dfrac{2\sin k\frac{2\pi}{3}}{k2\pi} & k\neq 0 \\ \dfrac{2}{3} & k=0 \end{cases}$$

$\swarrow$ $\omega_0 = \frac{2\pi}{3}$

$$\to \quad c_k = a_k * b_k = \underbrace{\frac{1}{2}\left[\delta(k-30)+\delta(k+30)\right]}_{a_k} * \underbrace{\frac{2\sin k\frac{2\pi}{3}}{k2\pi}}_{b_k}$$

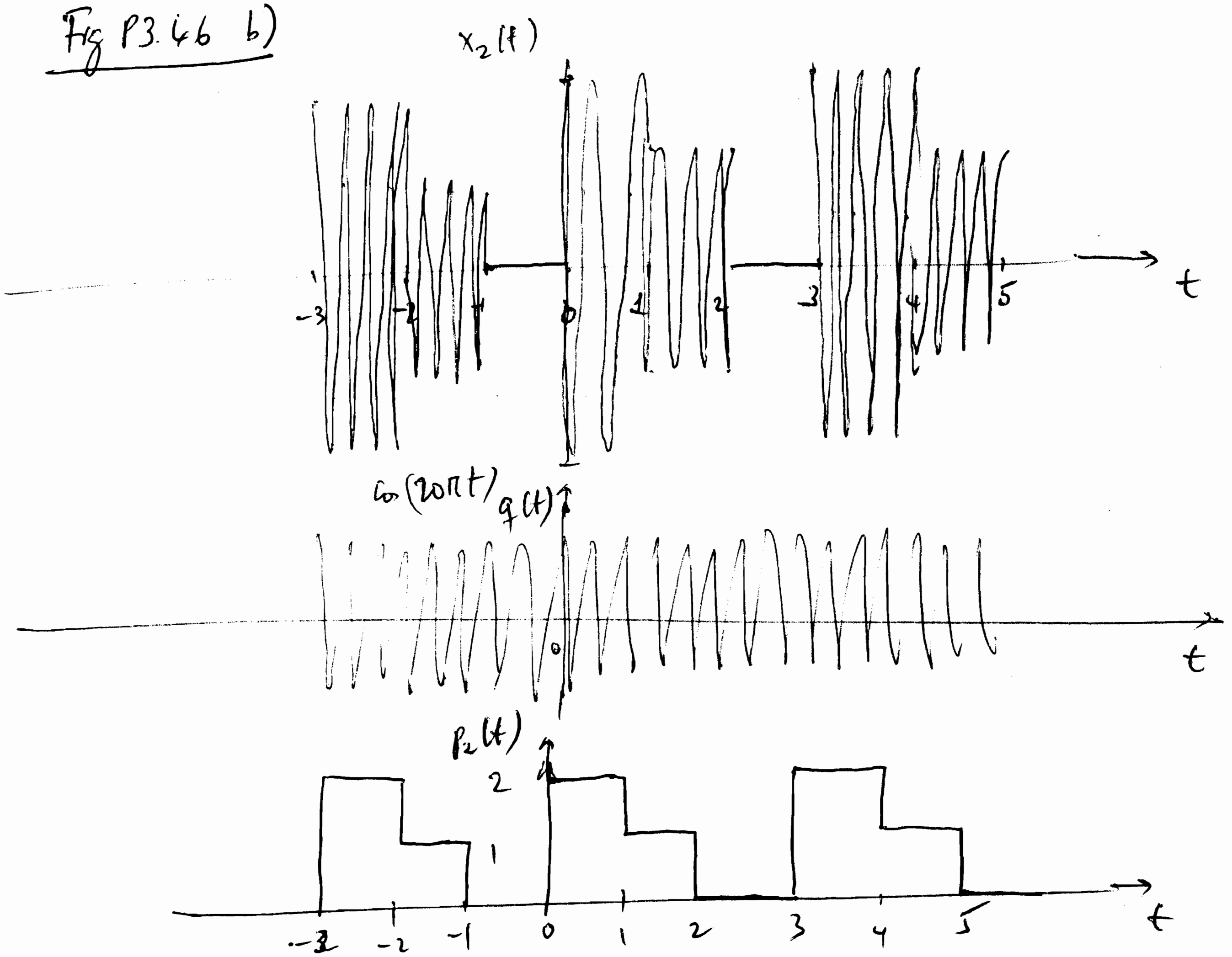
<u>Recall convolution property:</u> $f[n] * \delta[n-j] = f[n-j]$

$$\hookrightarrow c_k = \frac{1}{2}\left[\frac{2\sin(k-30)\frac{2\pi}{3}}{(k-30)\times 2\pi} + \frac{2\sin(k+30)\frac{2\pi}{3}}{(k+30)\times 2\pi}\right]$$

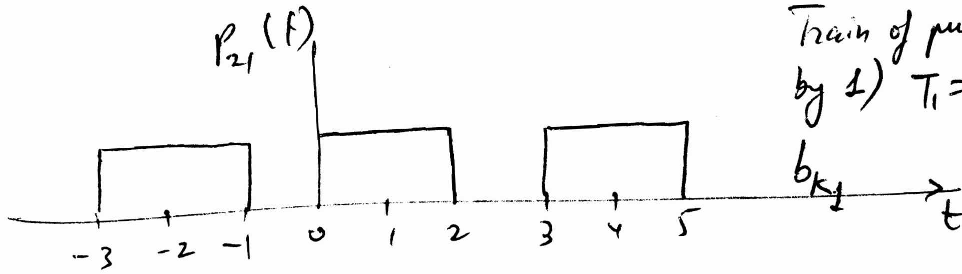
$\downarrow$

F.S. coefficients for $x_1(t)$.

<u>Fig P3.46 b)</u>

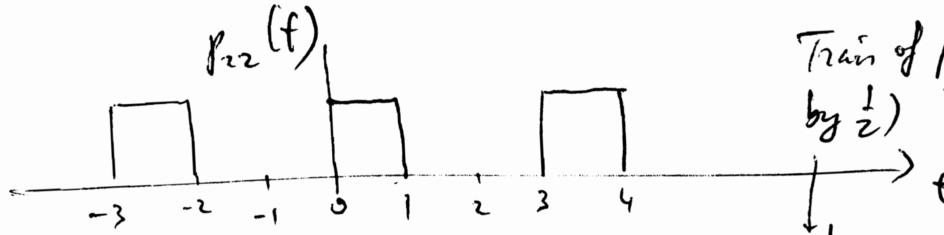


<u>Note</u>: $p_2(t)$ is the sum of 2 signals below:



Train of pulses (shifted by 1) $T_1 = 1$ & $T = 3$

b_{k_1}



Train of pulses (shifted by $\frac{1}{2}$) $T_1 = \frac{1}{2}$, $T = 3$

b_{k_2}

$$p_2(t) = p_{21}(t) + p_{22}(t)$$

$$= \sum_{k=-\infty}^{\infty} b_{k_1} e^{jk\omega_0 t} + \sum_{k=-\infty}^{\infty} b_{k_2} e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} (b_{k_1} + b_{k_2}) e^{jk\omega_0 t}$$

$$b_k = b_{k_1} + b_{k_2}$$

$$\sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$$

$\rightarrow$ b_k's $\rightarrow$

$$b_{k_1} = \frac{2\sin k\frac{2\pi}{3}1}{k\frac{2\pi}{\cancel{3}}\cancel{3}} \underbrace{e^{-jk\frac{2\pi}{3}}}_{\text{shift of } 1} = \frac{2\sin k\frac{2\pi}{3}}{k2\pi} e^{-jk\frac{2\pi}{3}}$$

$$b_{k_2} = \frac{2\sin k\frac{2\pi}{3}\frac{1}{2}}{k\frac{2\pi}{\cancel{3}}\cancel{3}} e^{-jk\frac{2\pi}{3}\frac{1}{2}} = \frac{2\sin k\frac{\pi}{3}}{k2\pi} e^{-jk\frac{\pi}{3}}$$

$$b_k = b_{k_1} + b_{k_2}$$

$$\Rightarrow \quad c_k = a_k * b_k = a_k * (b_{k_1} + b_{k_2})$$

$$= \underbrace{\left[\frac{1}{2}\delta(k-30) + \frac{1}{2}\delta(k+30)\right]}_{\substack{\text{F.S. coefficients for} \\ q(t) = \cos(20\pi t)}} * \underbrace{\left\{ \frac{2\sin k\frac{2\pi}{3}}{k2\pi} e^{-jk\frac{2\pi}{3}} + \frac{2\sin k\frac{\pi}{3}}{k2\pi} e^{-jk\frac{\pi}{3}} \right\}}_{\substack{\text{F.S. coefficients for} \\ p_2(t) = p_{21}(t) + p_{22}(t)}}$$

$$= \frac{\sin(k-30)\frac{2\pi}{3}}{(k-30)2\pi} e^{-j(k-30)\frac{2\pi}{3}} + \frac{\sin(k-30)\frac{\pi}{3}}{(k-30)2\pi} e^{-j(k-30)\frac{\pi}{3}}$$

$$+ \frac{\sin(k+30)\frac{2\pi}{3}}{(k+30)2\pi} e^{-j(k+30)\frac{2\pi}{3}} + \frac{\sin(k+30)\frac{\pi}{3}}{(k+30)2\pi} e^{-j(k+30)\frac{\pi}{3}}$$

convolution with a delta is equivalent to shifting to the center of the delta

3.48

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$x[n]$ periodic with period N; a_k → F.S. coefficients of $x[n]$

a) Find b_k for $x[n-n_0]$

$$x[n-n_0] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}(n-n_0)} = \sum_{k=\langle N\rangle} \underbrace{a_k e^{-jk\frac{2\pi}{N}n_0}}_{b_k} e^{jk\left(\frac{2\pi}{N}\right)n} \checkmark$$

b) Find b_k for $x[n] - x[n-1]$

$$b_k = a_k - a_k e^{-jk\frac{2\pi}{N}1} = a_k\left(1 - e^{-jk\frac{2\pi}{N}}\right)$$

($n_0 = 1$, use a)

...

c) → d)

3.52 | $x[n]$ real periodic with period N & complex F.S. coefficients a_k

$$\boxed{x[n] = \sum_{k=\langle N\rangle} a_k\, e^{jk\frac{2\pi}{N}n}}$$ F.S. expansion for $x[n]$

Notation → $a_k = b_k + j c_k$ where $b_k = Re[a_k]$; $c_k = Im[a_k]$

a) Show that $a_{-k} = \underbrace{a_k^*}_{\text{complex conjugate of } a_k} = b_k - j c_k$] when we replace $k \to -k$ we get the complex conjugate of a_k.

Reversed equation: $a_k = \frac{1}{N}\sum_n x[n]\, e^{-jk\frac{2\pi}{N}n}$

$$\left\{\begin{array}{l} \underset{k\to -k}{a_{-k}} = \frac{1}{N}\sum_n x[n]\, e^{+jk\frac{2\pi}{N}n} \\ a_k^* = \underbrace{\frac{1}{N}}_{\text{real}}\left(\sum_n \underbrace{x[n]}_{\text{real}} e^{-jk\frac{2\pi}{N}n}\right)^* = \frac{1}{N}\sum_n x[n]\, e^{jk\frac{2\pi}{N}n} \end{array}\right.$$

$$\Rightarrow \boxed{a_k^* = a_{-k}}$$

$$b_k - j c_k = b_{-k} + j c_{-k} \quad \left\{\begin{array}{l} \boxed{b_k = b_{-k}} \\ \boxed{c_k = -c_{-k}} \end{array}\right.$$

b) Suppose N is even show that $\underbrace{a_{\frac{N}{2}}}_{\text{"middle" F.S. coefficient}}$ is real

$$\underset{k=\frac{N}{2}}{a_{\frac{N}{2}}} = \frac{1}{N}\sum_n x[n]\, \underbrace{e^{-j\frac{N}{2}\frac{2\pi}{N}n}}_{e^{-j\pi n} = (-1)^n} = \frac{1}{N}\sum_n \underbrace{x[n]}_{\text{real}}\underbrace{(-1)^n}_{\text{real}} = \text{real number!}$$

(complex exponential is a real number when $k = \frac{N}{2}$)

Note a_{N} is also real!