D.E background:

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

Try $y = e^{st} \longrightarrow s^2 e^{st} + bse^{st} + ce^{st} = 0$

$\rightarrow s^2 + bs + c = 0 \rightarrow s = s_1, s_2$

$\rightarrow$ General sol: $y(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$

$S_1 = j\omega_1$ & $S_2 = j\omega_2 \rightarrow$ find S_1, S_2 to specify ω_1 & ω_2

In our circuit: $s^2 + 1 = 0$

$$\frac{d^2y}{dt^2} + \frac{1}{LC} y(t) = \frac{1}{LC} x(t) \rightarrow \text{homogeneous eq:}$$

$$\frac{d^2y}{dt^2} + \frac{1}{LC} y(t) = 0 \qquad \begin{cases} c = \frac{1}{LC} = 1 \\ b = 0 \end{cases}$$

$\rightarrow s^2 + 0s + 1 = 0 \rightarrow s = \pm\sqrt{-1} = \pm j \begin{cases} s_1 = j \\ s_2 = -j \end{cases}$

$\rightarrow \omega_1 = 1$ & $\omega_2 = -1$

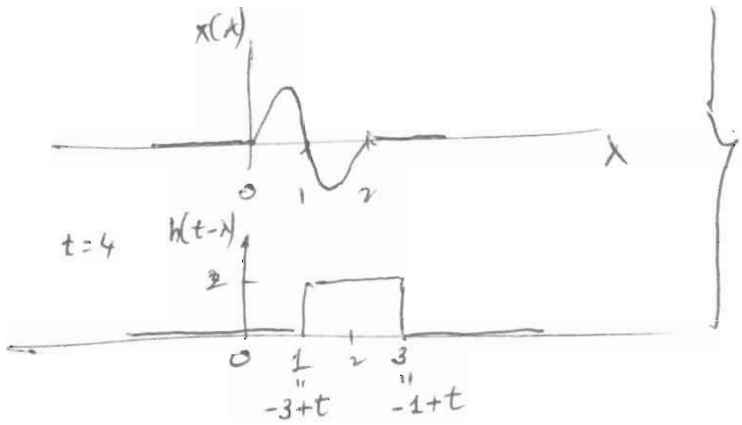
3) $\underbrace{v \,\&\, i \text{ are real}}$ $\xrightarrow{?}$ $y(t)$ be sinusoidal

$\hookrightarrow$ $y(t)$ has to be real

$y(t) = K_1 e^{jt} + K_2 e^{-jt} \rightarrow K_1 = K_2 \equiv K$

$= K\, 2\cos t = 2K \sin(t + \frac{\pi}{2})$

$3 < t \le 5$:



overlap b/w $-3+t$ & 2

$$y(t) = \int_{-3+t}^{2} d\lambda\, 2\sin\pi\lambda = -2\left[\frac{\cos\pi\lambda}{\pi}\right]_{t-3}^{2} = \frac{2}{\pi}\left[\cos\pi(t-3) - 1\right]$$

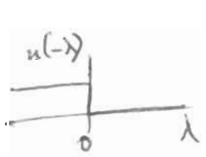
$$\rightarrow \quad y(t) = \begin{cases} 0 & -\infty < t < 1 \ \& \ t > 5 \\ \frac{2}{\pi}\left[1 - \cos\pi(t-1)\right] & 1 \le t \le 3 \\ \frac{2}{\pi}\left[\cos\pi(t-3) - 1\right] & 3 < t \le 5 \end{cases}$$

2) Now solving 2.22c) using only integrals

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} d\lambda\ x(\lambda)\, h(t-\lambda) = \int_{-\infty}^{\infty} d\lambda\ \underbrace{\sin\pi\lambda\left[u(\lambda) - u(\lambda-2)\right]}_{x(\lambda)} \cdot 2\left[u(t-\lambda-1) - u(t-\lambda-3)\right]$$

$$= 2\int_{-\infty}^{\infty} d\lambda\ \sin\pi\lambda\left[u(\lambda)u(t-\lambda-1) - u(\lambda)u(t-\lambda-3) - u(\lambda-2)u(t-\lambda-1) + u(\lambda-2)u(t-\lambda-3)\right]$$

$$= 2\int_{0}^{\infty} d\lambda\ \sin\pi\lambda\ u(t-\lambda-1) - 2\int_{0}^{\infty} d\lambda\ \sin\pi\lambda\ u(t-\lambda-3) - 2\int_{2}^{\infty} d\lambda\ \sin\pi\lambda\ u(t-\lambda-1) + 2\int_{2}^{\infty} d\lambda\ \sin\pi\lambda\ u(t-\lambda-3)$$



$$= 2\int_{0}^{t-1} d\lambda\ \sin\pi\lambda - 2\int_{0}^{t-3} d\lambda\ \sin\pi\lambda - 2\int_{2}^{t-1} d\lambda\ \sin\pi\lambda + 2\int_{2}^{t-3} d\lambda\ \sin\pi\lambda$$

By looking @ the limits of integrals :

$y(t) =$

$t<1$: No contributions from any of the 4 terms $\rightarrow$ $y(t)=0$

$1<t<3$: contribution from 1^{st} & $\underline{3^{rd}}$ terms

actually not ! since $0<t-1<2$
& lower limit here is already 2 !

$$y(t) = 2\int_0^{t-1} d\lambda \sin\pi\lambda = -2\left[\frac{\cos\pi\lambda}{\pi}\right]_{\lambda=0}^{\lambda=t-1} = \frac{2}{\pi}\left[1-\cos\pi(t-1)\right]$$

$\boxed{3<t<5}$ $\rightarrow$ $\begin{cases} 2<t-1<4 & \rightarrow \text{ now contributions from } 1^{st} \,\&\, 3^{rd} \text{ terms} \\ 0<t-3<2 & \rightarrow \, 2^{nd} \text{ term also (but not } 4^{th} \text{ term)} \end{cases}$

$$y(t) = \underbrace{2\int_0^{t-1} d\lambda \sin\pi\lambda} - 2\int_2^{t-1} d\lambda \sin\pi\lambda \underbrace{- 2\int_0^{t-3} d\lambda \sin\pi\lambda}$$

$$= 2\int_{t-3}^{t-1} d\lambda \sin\pi\lambda - 2\int_2^{t-1} d\lambda \sin\pi\lambda = 2\int_{t-3}^{2} d\lambda \sin\pi\lambda$$

$\boxed{\begin{matrix} 2<t-1<4 \\ 0<t-3<2 \end{matrix}}$

$\boxed{t-3<2<t-1}$

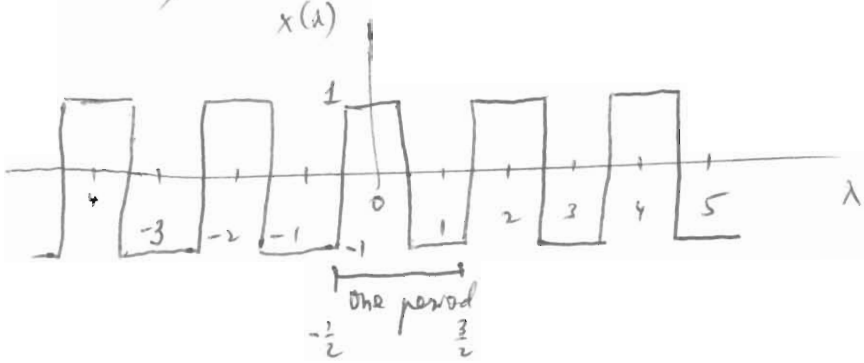
$$= -2\left[\frac{\cos\pi\lambda}{\pi}\right]_{\lambda=t-3}^{\lambda=2} = \frac{2}{\pi}\left[\cos\pi(t-3) - \underbrace{\cos 2\pi}_{1}\right]$$

$t>5$ $\begin{cases} t-1>4 & \rightarrow 1^{st} \,\&\, 3^{rd} \\ t-3>2 & \rightarrow 2^{nd} \,\&\, 4^{th} \end{cases}$

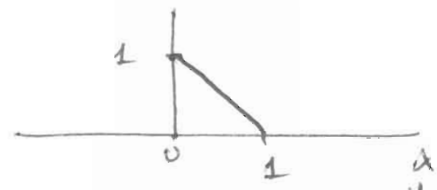
$$y(t) = \underbrace{2\int_{t-3}^{t-1} d\lambda \sin\pi\lambda}_{1^{st} \,\&\, 2^{nd}} \underbrace{- 2\int_{t-3}^{t-1} d\lambda \sin\pi\lambda}_{3^{rd} \,\&\, 4^{th}} = 0$$

$\rightarrow$ Summary $\left\{ y(t) = \begin{cases} 0 & t<1 \text{ or } t>5 \\ \frac{2}{\pi}\left[1-\cos\pi(t-1)\right] & 1<t<3 \\ \frac{2}{\pi}\left[\cos\pi(t-3) - 1\right] & 3<t<5 \end{cases} \right.$

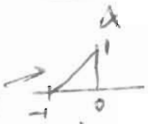
2.22 e) $y(t) = x(t) * h(t)$



$h(\lambda) = 1-\lambda$



$h(-\lambda) = 1+\lambda$



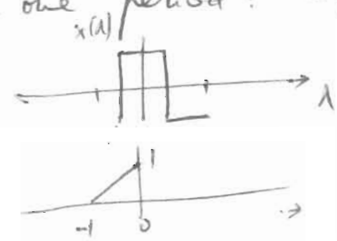
$h(t-\lambda) = 1-t+\lambda$

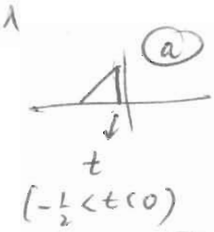
$$y(t) = \int_{-\infty}^{\infty} d\lambda \; x(\lambda)\, h(t-\lambda)$$

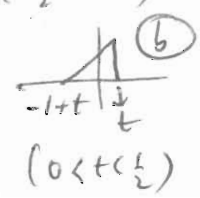
: since $x(\lambda)$ is periodic so is $y(t)$

→ just need to calculate $y(t)$ for one period: $-\frac{1}{2} < t < \frac{3}{2}$

$$y(t) = \begin{cases} \int_{-\frac{1}{2}}^{t} d\lambda \, 1(1-t+\lambda) \; (a) & -\frac{1}{2} < t < \frac{1}{2} \\ + \int_{-1+t}^{\frac{1}{2}} d\lambda (1-t+\lambda) \; (b) & \frac{1}{2} < t < \frac{3}{2} \end{cases}$$





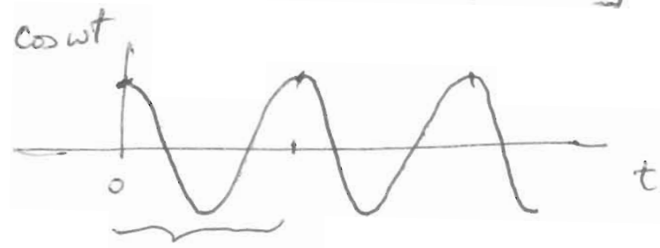


Ch3: Fourier Series Representation of Periodic Signals:

HW3: 3.5; 3.12; 3.14; 3.22; 3.25; 3.28; 3.32;
3.46; 3.48; 3.52 a) b) c) d)

Continuous-time periodic signals:

$$\cos \omega t = \mathrm{Re}\left[e^{j\omega t} \right]$$



One period

$$T = \frac{2\pi}{\omega}$$

$$x(t) = \underbrace{2\cos 10t}_{\text{period } T_1} + \underbrace{\sin 4t}_{\text{period } T_2} \longrightarrow \text{periodic?}$$

Yes ✓ → T = when both signals repeat at the same time for the first time

No

$T_1 = \frac{\pi}{5}$ $\qquad$ $T_2 = \frac{\pi}{2}$

multiples: $\frac{2\pi}{5}$, $\frac{3\pi}{5}$, $\frac{4\pi}{5}$, π, $\frac{6\pi}{5}$, ⋮

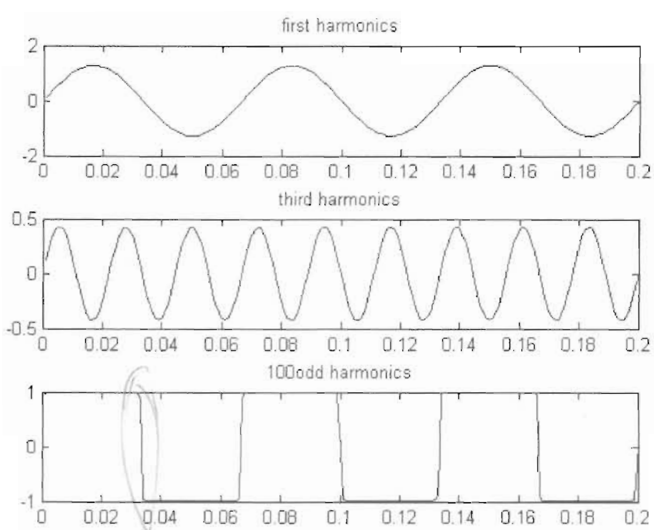
multiples: π, $\frac{3\pi}{2}$, ⋮

$$\boxed{T = \pi}$$

x(t), is a periodic signal w/ period π, is a combination of two periodic signals.: which are sinusoids

```
%Engin 321, October 16, 2008
%Will any periodic signal be a combination of sinusoids?

clear all
close all
nharm=99; %this is the number of odd harmonics to add to the first harmonics of frequency f0
f0=15;
sr=1000;
deltat=1/sr;
ns=5000;
nt=ns*deltat/5;
sq=zeros(1,ns);
sq1=zeros(1,ns);
n=deltat:deltat:nt;
nr=deltat:deltat:nt/5;
clear sq;
sq=4/pi*sin(2*pi*f0*n);
sq0=sq;
subplot(311), plot(nr,sq(1:length(nr))); title('first harmonics')
sq1=4/(3*pi)*sin(2*pi*3*f0*n);
subplot(312),plot(nr,sq1(1:length(nr))),title('third harmonics')
for i=1:1:nharm;
   sq=sq+(4/((2*i+1)*pi))*sin(2*pi*(2*i+1)*f0*n+1/nharm);
end
subplot(313), plot(nr,sq(1:length(nr))),title(strcat(num2str(nharm+1),'odd harmonics'))
```



→ Can we generalize this: <u>will any periodic signal be a combination of sinusoids?</u>

Using Matlab we could see that a periodic square wave which is very different compared to a sinusoid, is actually a combination of sinusoids. In fact, any periodic signal, no matter how strange it looks, can be written as a combination of sinusoids:

For example: $x(t)$ is a periodic signal $\leftrightarrow$ $\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}$

Fourier Series Representation of periodic signal $x(t)$

where a_k's complex amplitudes & ω_0 is the fundamental angular frequency.

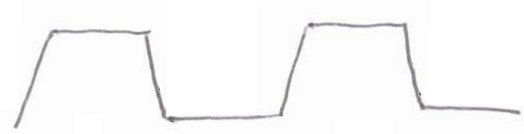
<u>For a periodic square wave</u>: (see Matlab code)

$$\begin{cases} k = 2i \quad (i = 0, 1, 2, \dots) \rightarrow a_k = 0 \\ k = 2i+1 \quad (i = 0, 1, 2, \dots) \rightarrow a_k = \dfrac{4}{(2i+1)\pi} \end{cases}$$

<u>Questions</u>:

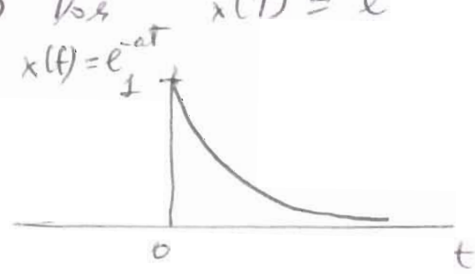
1) Why do I need ∞ terms in Fourier Series representation?

With 100 harmonics: we get slopes on sides of squares



To get perfect vertical sides we need ∞ terms in the Fourier series.

2) Does $x(t) = e^{-at}$ have any F.S. representation?



<u>No</u>. Does this contradict our statement that any periodic signal will have a F.S. representation? No, since it is not periodic to start with.

3) If any periodic signal will have a F.S. representation can we identify it by its complex amplitudes a_k's and fundamental freq w_0? Yes!

$x(t)$ periodic $\longleftrightarrow$ uniquely identified by its a_k's & w_0

How do I find a_k's for a periodic signal?

↳ Use math: orthogonality of complex exponentials (plane wave) e^{jkw_0t}

$$\underbrace{\int_0^T dt\, e^{jkw_0t} \cdot e^{jnw_0t}}_{\substack{\text{This is the product}\\ \text{of 2 complex exponentials}\\ \text{of frequencies } kw_0 \text{ \& } nw_0}} = \left.\begin{cases} T & \text{if } k=n \\ 0 & \text{if } k \neq n \end{cases}\right\} = T\delta[k-n]$$

(complex exponentials of different frequencies are orthogonal)

To find a_k for periodic $x(t)$:

F.S.: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0t}$

Apply $\int_0^T dt\, e^{-jnw_0t}$ to both sides of F.S. equation

$$\int_0^T dt\, e^{-jnw_0t} x(t) = \sum_{k=-\infty}^{\infty} a_k \underbrace{\int_0^T dt\, e^{-jnw_0t} e^{jkw_0t}}_{T\delta[k-n]}$$

$$= T \underbrace{\sum_{k=-\infty}^{\infty} a_k \delta[k-n]}_{a_n} = Ta_n$$

$$\Rightarrow \quad a_n = \frac{1}{T}\int_0^T dt\; e^{-jn\omega_0 t}\, x(t) \quad \text{or} \quad \boxed{a_k = \frac{1}{T}\int_0^T dt\; e^{-jk\omega_0 t}\, x(t)}$$

T: fundamental period $= \frac{2\pi}{\omega_0}$

Using this equation we can calculate a_k's for any periodic $x(t)$

Properties of F.S. representation: (Table 3.1, pg 206)

Linearity:

If $x(t)$ & $y(t)$ are periodic with same fundamental frequency ω_0 then they are identified by complex amplitudes a_k's & b_k's:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \& \quad y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$$

$A\,x(t) + B\,y(t)$ will also be periodic and its complex amplitudes are: $\quad A a_k + B b_k$.

Proof: this is obvious from linear combination of $x(t)$ & $y(t)$:

$$A x(t) + B y(t) = \sum_{k=-\infty}^{\infty} A a_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{\infty} B b_k e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{\left(A a_k + B b_k \right)}_{\text{These are the complex amplitudes of } Ax+By} e^{jk\omega_0 t}$$

Time-shift:

Periodic $x(t) \longleftrightarrow$ complex amplitude a_k's

↳ Periodic $x(t-t_0) \longleftrightarrow$ complex amplitude : $a_k e^{-jk\omega_0 t_0}$

Proof: just need to apply time shift to the F.S. representation for $x(t)$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x(t-t_0) = \sum_{k=-\infty}^{\infty} a_k \underbrace{e^{jk\omega_0(t-t_0)}}_{e^{jk\omega_0 t} \cdot e^{-jk\omega_0 t_0}} = \sum_{k=-\infty}^{\infty} \underbrace{\left(a_k e^{-jk\omega_0 t_0}\right)}_{\text{complex amplitude of } x(t-t_0)} e^{jk\omega_0 t}$$

Other properties: frequency shift, time-reversal; time scaling; ~~multiplation~~, multiplication; differentiation; integration

Parseval's relation: $$\boxed{\underbrace{\frac{1}{T}\int_T dt\, |x(t)|^2}_{\text{operation in time domain}} = \underbrace{\sum_{k=-\infty}^{\infty} |a_k|^2}_{\text{operation in frequency domain}}}$$

operation in time domain ⟷ operation in frequency domain

↓

related !!!

(3.5): periodic $x_1(t)$ with fundamental frequency ω_1 and complex amplitudes a_k (Fourier series coefficients)

↳ what is the fundamental frequency of

$$x_2(t) = \underbrace{x_1(1-t)}_{\text{time reversed \& shifted}} + \underbrace{x_1(t-1)}_{\text{time shifted}}$$

→ same as that for $x_1(t)$ since time reversal & shift don't change ω_1

What are the complex amplitude or Fourier series coefficients b_k for $x_2(t)$ in terms of a_k for $x_1(t)$

Strategy: write $x_1(1-t)$ and $x_1(t-1)$ in term of their F.S.'s then combine to determine b_k in term of a_k's

$$x_2(t) = \underbrace{x_1(1-t)}_{\substack{F.S \rightarrow \\ \text{time reversed} \\ \& \\ \text{shifted by } 1}} + \underbrace{x_1(t-1)}_{\substack{F.S \rightarrow \\ \text{time shifted} \\ \text{by } 1}} = \sum_{k=-\infty}^{\infty} a_k e^{jkw_1(1-t)} + \sum_{k=-\infty}^{\infty} a_k e^{jkw_1(t-1)}$$

$$= \sum_{k=-\infty}^{\infty} \left[\underbrace{a_k e^{jkw_1} e^{-jkw_1 t}}_{k \rightarrow -k} + a_k e^{-jkw_1} e^{jkw_1 t} \right]$$

$$= \sum_{k=\infty}^{-\infty} a_{-k} \underbrace{e^{-jkw_1} e^{jkw_1 t}} + \sum_{k=-\infty}^{\infty} a_k \underbrace{e^{-jkw_1} e^{jkw_1 t}}$$

$$'' \sum_{k=-\infty}^{\infty}$$ since summation is commutative (order doesn't matter)

$$= \sum_{k=-\infty}^{\infty} \left[a_{-k} e^{-jkw_1} e^{jkw_1 t} + a_k e^{-jkw_1} e^{jkw_1 t} \right]$$

$$\Rightarrow x_2(t) = \sum_{k=-\infty}^{\infty} (a_{-k} + a_k) e^{-jkw_1} e^{jkw_1 t}$$

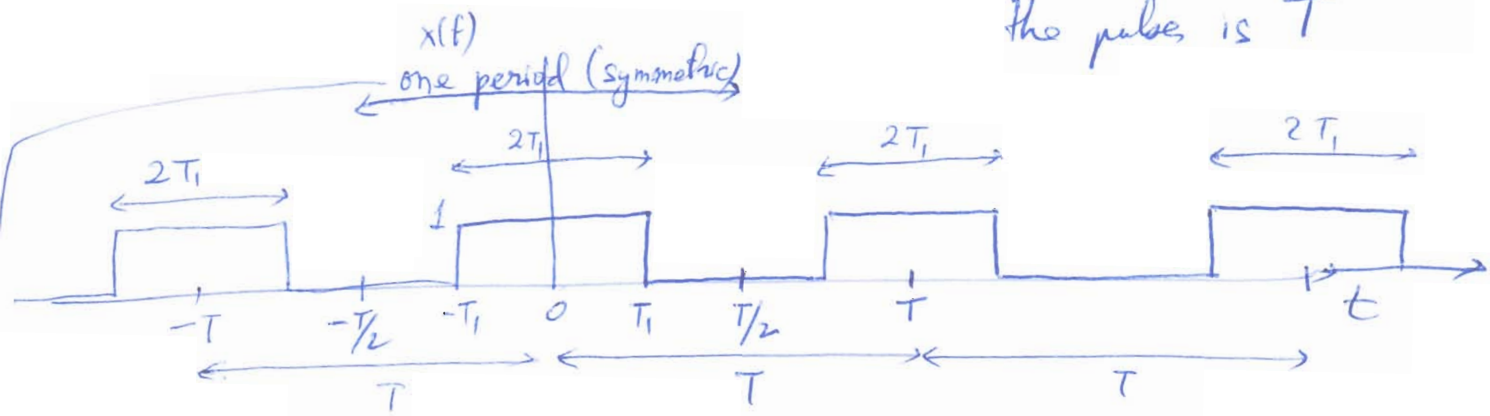
Since $x_2(t)$ is periodic with fundamental frequency w_1:

$$x_2(t) = \sum_{k=-\infty}^{\infty} b_k e^{jkw_1 t} \rightarrow \boxed{b_k = (a_{-k} + a_k) e^{-jkw_1}}$$

Fourier Series Representation for a continuous-time periodic signal:

Book example 3.5: Determine a_k's for $x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < \frac{T}{2} \end{cases}$

Train of rectangular pulses being $2T_1$ width of the pulses, and period of the pulses is T



$$a_k = \frac{1}{T}\int_T dt\, e^{-jk\omega_0 t}\, x(t)$$

T: period of $x(t)$

$\omega_0 = \frac{2\pi}{T}$

a_k = Fourier Series coefficient for $x(t)$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} dt\, e^{-jk\omega_0 t}\, x(t) = \frac{1}{T}\int_{-T_1}^{T_1} dt\, e^{-jk\omega_0 t}\, 1$$

apply to our signal $x(t)$

$$= \frac{1}{T}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_{t=-T_1}^{t=T_1} = \frac{e^{-jk\omega_0 T_1} - e^{+jk\omega_0 T_1}}{-jk\omega_0 T}$$

$$a_k = \frac{2\cancel{j}\sin k\omega_0 T_1}{\cancel{j}k\omega_0 T} = \frac{\cancel{2}\sin k\omega_0 T_1}{\cancel{2}\pi k} = \left.\frac{\sin(k\omega_0 T_1)}{\pi k}\right|_{k\neq 0}$$

When $k=0$ $\rightarrow a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt\ 1.1 = \frac{2T_1}{T}$

Fourier Series Representation for a discrete-time periodic signal:

There is a big difference when dealing with a discrete-time periodic signal as opposed to a continuous-time periodic signal.

$\cos[w_0 n]$ is periodic discrete-time signal with period $N = \frac{2\pi}{w_0}$ (w_0 needs to carry a factor of π so that N is an integer!)

$$\cos[w_0 n] = \frac{e^{jw_0 n} + e^{-jw_0 n}}{2} = \frac{e^{j(1+N)w_0 n} + e^{-j(1+N)w_0 n}}{2}$$

(since $e^{\pm j N w_0 n} = e^{\pm j 2\pi n} = 1$)

$$= \frac{e^{j(1+5N)w_0 n} + e^{-j(1+5N)w_0 n}}{2} = \text{etc..}$$

however:

$\cos w_0 t$ is periodic continuous-time signal with period $T = \frac{2\pi}{w_0}$

$$\cos w_0 t = \frac{e^{jw_0 t} + e^{-jw_0 t}}{2} \neq \frac{e^{j(1+T)w_0 t} + e^{-j(1+T)w_0 t}}{2} \neq \text{etc.}$$

since $e^{\pm j 2\pi t} \neq 1$ (since t could be any real number)

Implication for Fourier series representation of periodic signals:

Continuous-time

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}$$

Discrete-time

$$x[n] = \underbrace{\sum_{k=\langle N\rangle}}_{} a_k e^{jk\frac{2\pi}{N}n}$$

Only has N terms where N is the integer period of the discrete-time signal.

continuous-time

$$a_k = \frac{1}{T}\int_T dt \;\; x(t)\, e^{-jk\omega_0 t}$$

discrete-time

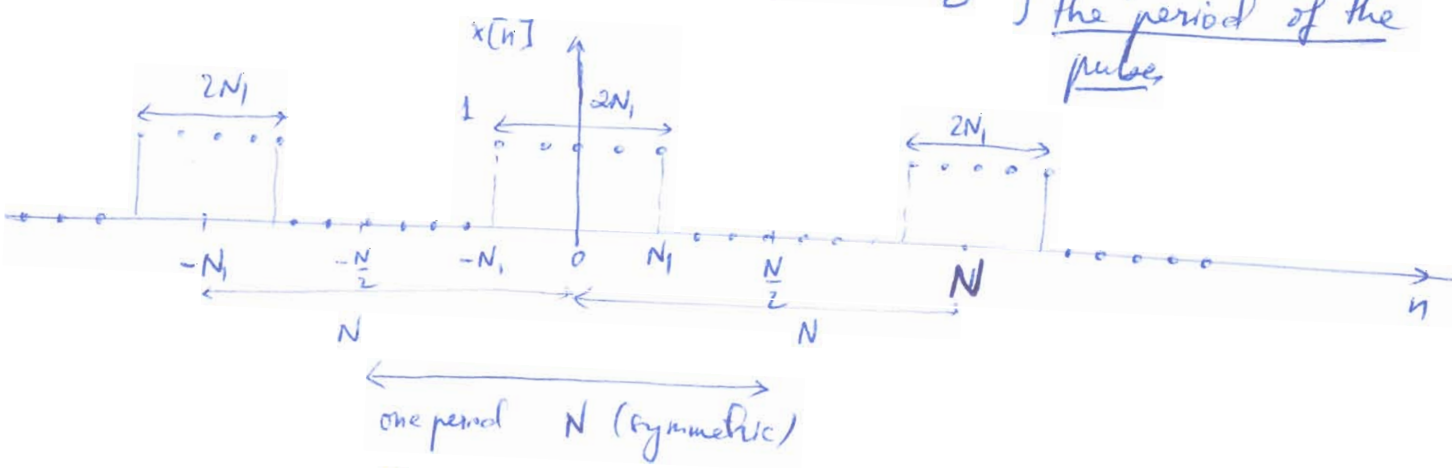
$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jk\frac{2\pi}{N}n}$$

n is a modulus of N. Only has N terms

Book example 3.12: Determine a_k for a discrete-time train of rectangular pulse

$$x[n] = \begin{cases} 1 & |n| < N_1 \\ 0 & N_1 < |n| < \frac{N}{2} \end{cases}$$

$2N_1$ is the width of the pulse and N is the period of the pulse



$$\rightarrow \quad a_k = \frac{1}{N}\sum_{n=-\frac{N}{2}}^{\frac{N}{2}} x[n]\, e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\sum_{n=-N_1}^{N_1} 1\; e^{-jk\frac{2\pi}{N}n}$$

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1}\left(e^{-jk\frac{2\pi}{N}}\right)^n$$

Note: $\sum_{n=-N_1}^{N_1}\left(e^{-jk\frac{2\pi}{N}}\right)^n$ is a geometric sum $\sum_{k=0}^{P}\alpha^k = \frac{1-\alpha^{P+1}}{1-\alpha}$

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1}\alpha^n = \frac{\alpha^{-N_1}}{N}\sum_{m=0}^{2N_1}\alpha^m = \frac{\alpha^{-N_1}}{N}\,\frac{1-\alpha^{2N_1+1}}{1-\alpha}$$

$\alpha \equiv e^{-jk\frac{2\pi}{N}}$

rename $n \rightarrow n+N_1 = m$

$\hookrightarrow \alpha^n = \alpha^m \alpha^{-N_1}$

$$a_k = \frac{1}{N} e^{jk\frac{2\pi N_1}{N}} \frac{1 - e^{-jk\frac{2\pi}{N}(2N_1+1)}}{1 - e^{-jk\frac{2\pi}{N}}}$$

This is the Fourier series coefficients for the discrete-time train of rectangular pulse in example 3.12

This can be written more compactly if we use property:

$$1 - e^{-j\beta} = e^{-j\frac{\beta}{2}}\left(e^{j\frac{\beta}{2}} - e^{-j\frac{\beta}{2}}\right)$$