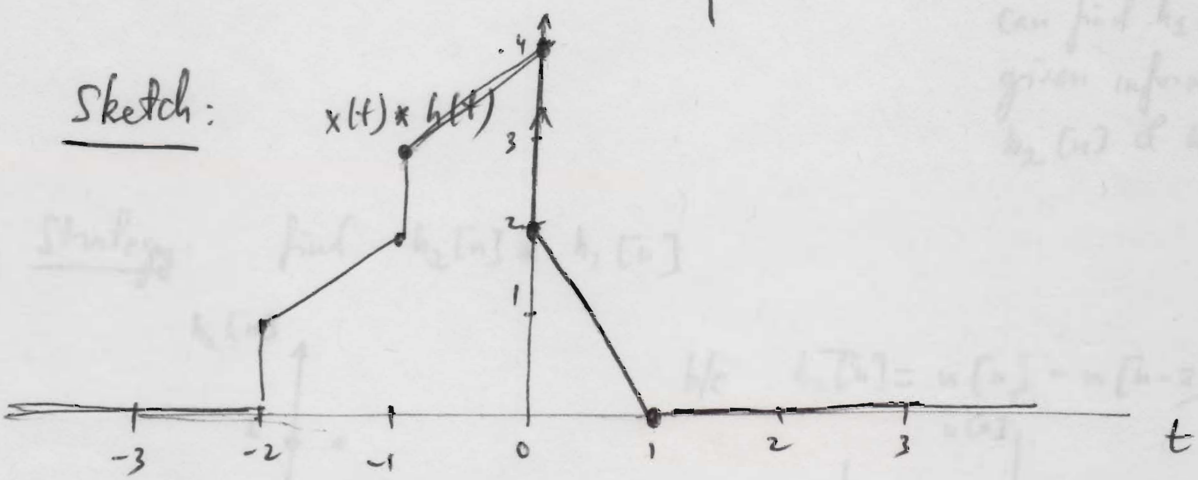
$$x(t) * h(t) = x(t+2) + 2\,x(t+1)$$

$$x(t+2) = \begin{cases} t+3 & 0 \le t+2 \le 1 \leftrightarrow -2 \le t \le -1 \\ 2-t-2 = -t & 1 \le t+2 \le 2 \leftrightarrow -1 \le t \le 0 \\ 0 & \text{elsewhere} \end{cases}$$

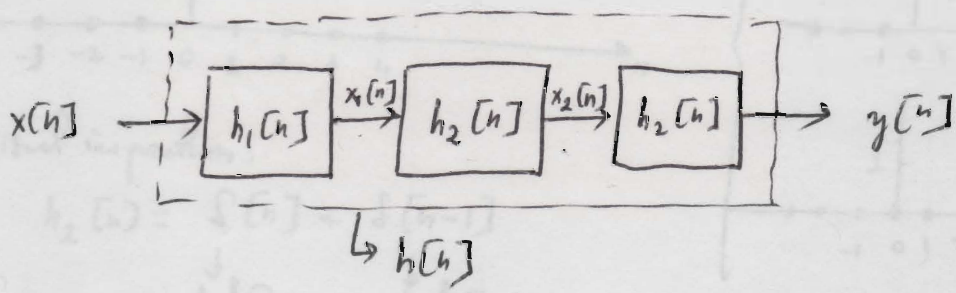
$$x(t+1) = \begin{cases} t+2 & 0 \le t+1 \le 1 \leftrightarrow -1 \le t \le 0 \\ 1-t & 1 \le t+1 \le 2 \leftrightarrow 0 \le t \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$\rightarrow x(t) * h(t) = x(t+2) + 2x(t+1) = \begin{cases} t+3 & -2 \le t \le 1 \\ -t+2(t+2) = t+4 & -1 \le t \le 0 \\ 2(1-t) = 2-2t & 0 \le t \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

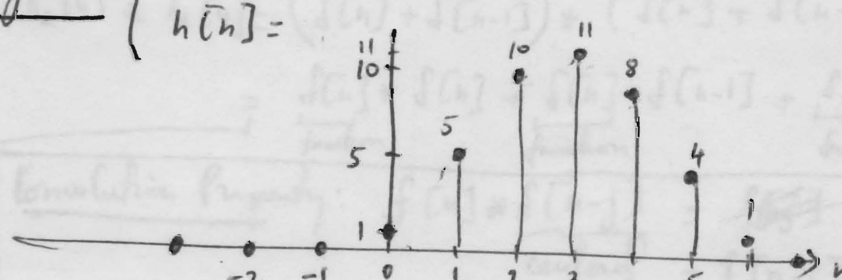
Sketch:



(2-24)



given: $\begin{cases} h_2[n] = u[n] - u[n-2] & \text{(rect. pulse b/w 0 \& 2)} \\ h[n] = & \end{cases}$



$\boxed{\text{Find } h_1[n]\ ?}$

→ <u>Strategy</u>: write $h[n]$ ~~at~~ in terms of h_1, h_2:

$$x_1[n] = x[n] * h_1[n]$$

$$x_2[n] = x[n] * h_1[n] * h_2[n]$$

$$y[n] = x[n] * \underbrace{h_1[n] * h_2[n] * h_2[n]}$$

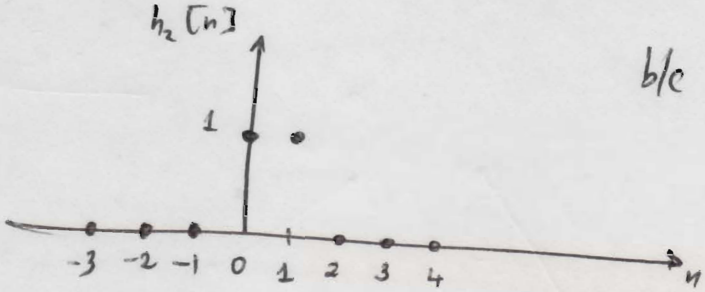
also $$y[n] = x[n] * h[n]$$

$$\boxed{h[n] = h_1[n] * h_2[n] * h_2[n]}$$

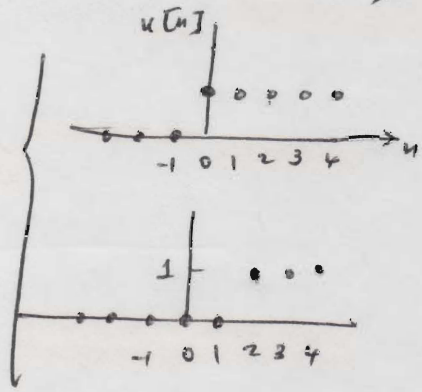
↓

can find $h_1[n]$ w/ given information on $h_2[n]$ & $h[n]$.

→ <u>Strategy</u>: find $h_2[n] * h_2[n]$



b/c $h_2[n] = u[n] - u[n-2]$



visual inspection:

$$h_2[n] = \delta[n] + \delta[n-1]$$

↓ dot @ 0 ↓ dot @ 1

→ $h_2[n] * h_2[n] = (\delta[n] + \delta[n-1]) * (\delta[n] + \delta[n-1])$

$$= \underbrace{\delta[n]}_{\text{function}} * \delta[n] + \underbrace{\delta[n]}_{\text{function}} * \delta[n-1] + \underbrace{\delta[n-1]}_{\text{function}} * \delta[n] + \underbrace{\delta[n-1]}_{\text{function}} * \delta[n-1]$$

<u>Convolution Property</u>: $f[n] * \underbrace{\delta[n-j]}_{\text{centered @ } j} = \cancel{f[n]} = f[n-j]$ → function shifted by j

$$= \delta[n] + \delta[n-1] + \delta[n-1] + \delta[n-2]$$

$$\boxed{h_2[n] * h_2[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]}$$

$$h[n] = h_1[n] * h_2[n] * h_2[n] = \underbrace{h_1[n]}_{\text{function}} * \left(\delta[n] + 2\delta[n-1] + \delta[n-2]\right)$$

$$h[n] = h_1[n] + 2h_1[n-1] + h_1[n-2]$$

→ Strategy: Find $h_1[n]$ ~~point by point~~ from this equation

Observation: h_1 represents a causal LTI system (given in problem) → $h_1[n] = 0 \;\; \forall n < 0$

$h[-2]=0$
$h[-1]=0$
$h[0]=1$
$h[1]=5$
$h[2]=10$
$h[3]=11$
$h[4]=8$
$h[5]=4$
$h[6]=1$
$h[7]=0$
$h[8]=0$
⋮

$n=0$ $\quad h[0] = h_1[0] + 2\underbrace{h_1[-1]}_{0} + \underbrace{h_1[-2]}_{0} \Rightarrow \boxed{h_1[0]=1}$

$n=1$ $\quad \underbrace{h[1]}_{5} = h_1[1] + 2\underbrace{h_1[0]}_{1} + \underbrace{h_1[-1]}_{0} \Rightarrow \boxed{h_1[1]=3}$

$n=2$ $\quad \underbrace{h[2]}_{10} = h_1[2] + 2\underbrace{h_1[1]}_{3} + \underbrace{h_1[0]}_{1} \Rightarrow \boxed{h_1[2]=3}$

$n=3$ $\quad \underbrace{h[3]}_{11} = h_1[3] + 2\underbrace{h_1[2]}_{3} + \underbrace{h_1[1]}_{3} \Rightarrow \boxed{h_1[3]=2}$

$n=4$ $\quad \underbrace{h[4]}_{8} = h_1[4] + 2\underbrace{h_1[3]}_{2} + \underbrace{h_1[2]}_{3} \Rightarrow \boxed{h_1[4]=1}$

$n=5$ $\quad \underbrace{h[5]}_{4} = h_1[5] + 2\underbrace{h_1[4]}_{1} + \underbrace{h_1[3]}_{2} \Rightarrow \boxed{h_1[5]=0}$

$n=6$ $\quad \underbrace{h[6]}_{1} = h_1[6] + 2\underbrace{h_1[5]}_{0} + \underbrace{h_1[4]}_{1} \Rightarrow \boxed{h_1[6]=0}$

⋮

Conclude:
$$h_1[n] = \begin{cases} 0 & n<0 \\ 1 & n=0 \\ 3 & n=1 \,\&\, 2 \\ 2 & n=3 \\ 1 & n=4 \\ 0 & n \geq 5 \end{cases}$$

b) Find response $y[n]$ of overall system to the input

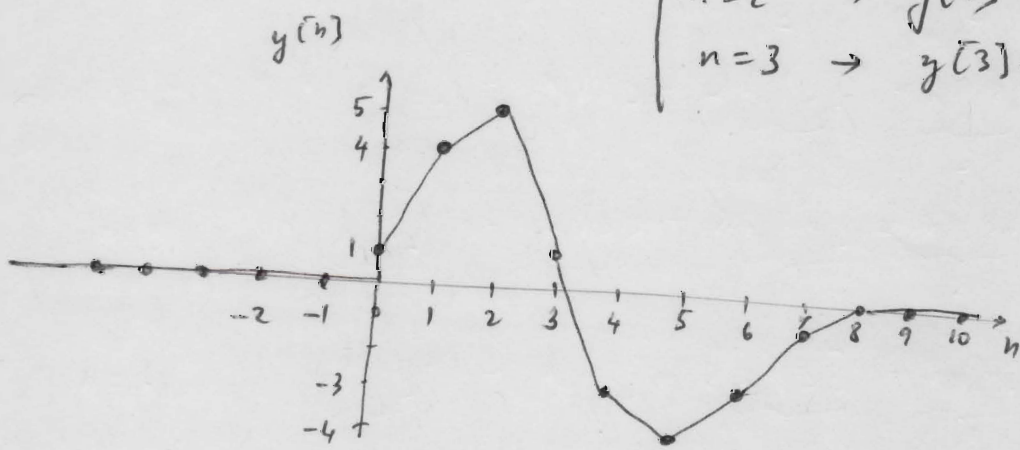
$$x[n] = \delta[n] - \delta[n-1]$$

$$x[n] \rightarrow \boxed{\quad h[n] \quad} \rightarrow y[n]$$

$$y[n] = x[n] * h[n] = \left(\delta[n] - \delta[n-1]\right) * h[n]$$

$$= \underbrace{h[n]}_{\text{function}} * \left(\underbrace{\delta[n]}_{\substack{\text{centered}\\ @\,0}} - \underbrace{\delta[n-1]}_{\substack{\text{centered}\\ @\,1}} \right)$$

$$= h[n] - h[n-1] \quad \begin{cases} n=-2 \rightarrow y[-2]=0 \\ n=-1 \rightarrow y[-1]=0 \\ n=0 \rightarrow y[0]=1 \\ n=1 \rightarrow y[1]=4 \\ n=2 \rightarrow y[2]=5 \\ n=3 \rightarrow y[3]=1 \end{cases}$$



2-17

$$x(t) \longrightarrow \boxed{\text{System LTI}} \longrightarrow y(t)$$

$$\begin{cases} \dfrac{dy}{dt} + 4y = x \\ \text{initial rest} : \text{ no activity } \forall t < 0 \end{cases}$$

a) If $x(t) = e^{(-1+3j)t} u(t) \rightarrow y = ?$

Solve for this 1st order non homogeneous D.E.:

$$\frac{dy}{dt} + 4y = e^{(-1+3j)t} u(t) \Rightarrow y(t) = y_h(t) + y_p(t)$$

$$\begin{cases} \dfrac{dy_h}{dt} + 4y_h = 0 \rightarrow y_h = Ae^{-4t} \rightarrow \text{Find } A \text{ from initial conditions } y(0) = 0 \\ y_p = Ce^{(-1+3j)t} \end{cases}$$

$\rightarrow$ Find C by plugging y_p into the D.E.

$$\frac{dy_p}{dt} + 4y_p = C(-1+3j)\cancel{e^{(-1+3j)t}} + 4C\cancel{e^{(-1+3j)t}} = \cancel{e^{(-1+3j)t}} u(t)$$

$$\forall t \geq 0 \rightarrow (-1+3j)C + 4C = 1$$

$$\boxed{C = \frac{1}{3(1+j)}}$$

$$\rightarrow y(t) = Ae^{-4t} + \frac{1}{3(1+j)} e^{(-1+3j)t}, \quad \forall t \geq 0$$

$\rightarrow$ Find A:

$$y(0) = A + \frac{1}{3(1+j)} = 0 \rightarrow \boxed{A = -\frac{1}{3(1+j)}}$$

$$\rightarrow \boxed{y(t) = \frac{1}{3(1+j)}\left[-e^{-4t} + e^{(-1+3j)t}\right] \underbrace{u(t)}_{\text{no initial activity } \forall t<0}}$$

b) Find $y(t)$ if $x(t) = e^{-t}\cos(3t)\,u(t)$

↳ From part a): $x(t) = e^{-t} e^{3jt} u(t) = e^{-t}(\cos 3t + j\sin 3t)\,u(t)$

$$\rightarrow x_b(t) = Re\left[x_a(t)\right]$$

System is linear $\rightarrow$ $y_b(t) = Re\left[\frac{1}{3(1+j)}\left(-e^{-4t} + e^{-t}e^{3jt}\right)\right]u(t)$

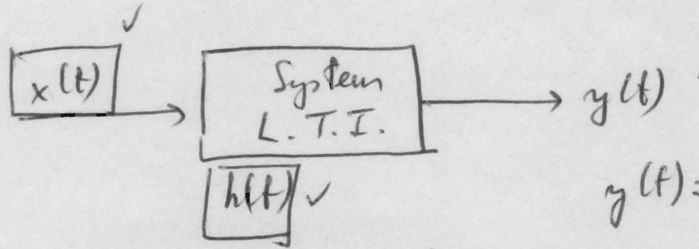
$$= \left\{ Re\left[\frac{-e^{-4t}}{3(1+j)}\right] + Re\left[\frac{e^{-t}}{3}\,\frac{e^{3jt}}{1+j}\right]\right\} u(t)$$

$$= \left\{ -\frac{e^{-4t}}{3}\, Re\Bigg[\underbrace{\frac{1}{1+j}}_{\frac{1-j}{2}}\Bigg] + \frac{e^{-t}}{3}\, Re\Bigg[\underbrace{\frac{e^{3jt}}{1+j}}_{\frac{1-j}{2}(\cos 3t + j\sin 3t)}\Bigg]\right\} u(t)$$

$$= \left\{ -\frac{e^{-4t}}{3}\,\frac{1}{2} + \frac{e^{-t}}{3}\left(\frac{\cos 3t}{2} + \frac{\sin 3t}{2}\right)\right\} u(t)$$

$$= \frac{1}{6}\left\{ e^{-t}\cos 3t + e^{-t}\sin 3t - e^{-4t}\right\} u(t).$$

2.22



$$y(t) = x(t) * h(t)$$

a) $x(t) = e^{-\alpha t} u(t)$; $h(t) = e^{-\beta t} u(t)$ ($\alpha \neq \beta$ & $\alpha = \beta$)

$$y(t) = x(t) * h(t) \equiv \int_{-\infty}^{\infty} d\lambda \; x(\lambda)\, h(t-\lambda)$$

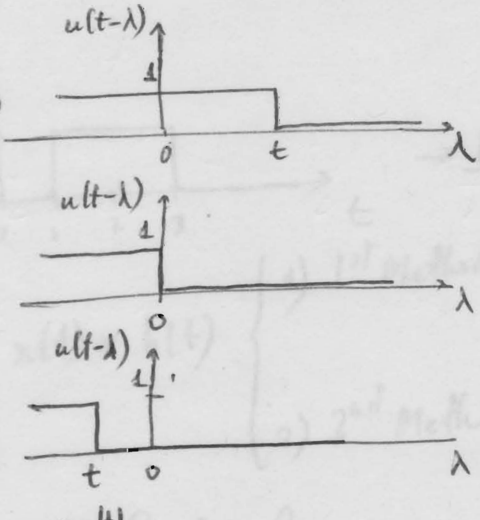
$$= \int_{-\infty}^{\infty} d\lambda \; e^{-\alpha\lambda} u(\lambda) \; e^{-\beta(t-\lambda)} \; u(t-\lambda)$$

integral vs wrt λ

$$= e^{-\beta t} \int_{-\infty}^{\infty} d\lambda \; e^{-(\alpha-\beta)\lambda} \; \underbrace{u(\lambda)}_{0 \text{ if } \lambda<0} u(t-\lambda)$$

$$= e^{-\beta t} \int_{0}^{\infty} d\lambda \; e^{-(\alpha-\beta)\lambda} \; \underbrace{u(t-\lambda)}$$

$u(t-\lambda) =$ $\begin{cases} t>0 \\ t=0 \\ t<0 \end{cases}$



$\rightarrow y(t) = e^{-\beta t} \int_0^t d\lambda\, e^{-(\alpha-\beta)\lambda} = \dfrac{e^{-\beta t} - e^{-\alpha t}}{\alpha - \beta}$ $\boxed{\alpha \neq \beta}$

$\rightarrow y(t) = e^{-\beta t} \int_0^0 d\lambda = 0$

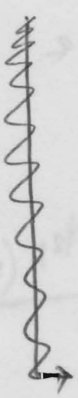
$\rightarrow y(t) = 0$

when $\boxed{\alpha = \beta}$

$t>0 \rightarrow y(t) = e^{-\beta t} \int_0^t d\lambda = t e^{-\beta t}$

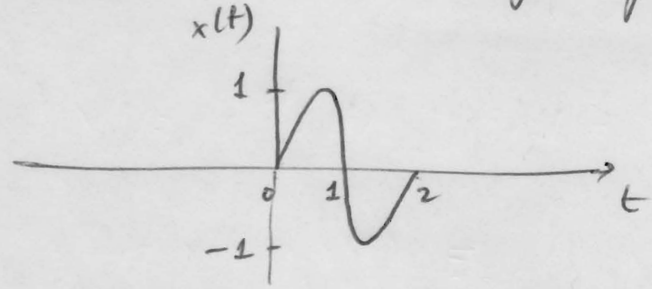
$t=0 \Rightarrow y(t) = 0$

$t<0 \rightarrow y(t) = 0$



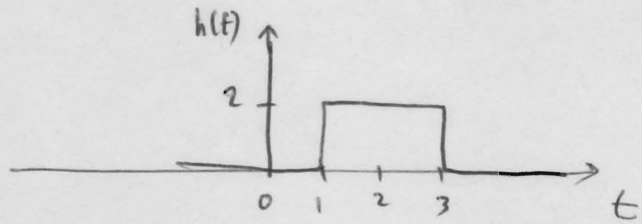
Do 2.22 b) 1) & e)

2.22 c x & h are given graphically



one period of a sinusoid
$T=2 \rightarrow \omega = \frac{2\pi}{T} = \pi$

Mathematically: $x(t) = \sin(\pi t)\underbrace{[u(t)-u(t-2)]}_{\text{rect. pulse b/w 0 \& 2}}$



→ Mathematically: $h(t) = 2[u(t-1) - u(t-3)]$

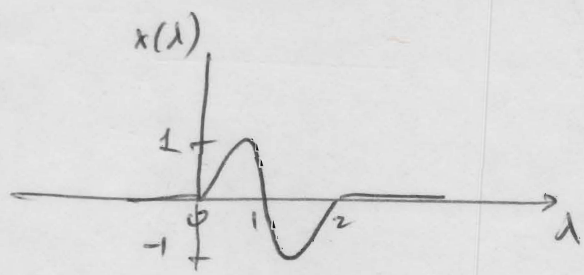
→ Find $y(t) = x(t) * h(t)$
1) 1st Method: using graphics & integrals
2) 2nd Method: using only mathematics

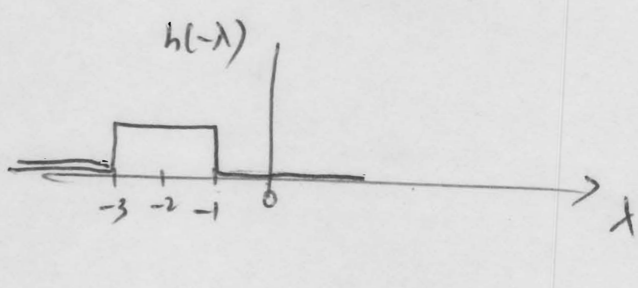
1) 1st Method: graphics & integrals:

$$y(t) = \int_{-\infty}^{\infty} d\lambda \; \underbrace{x(\lambda)\; h(t-\lambda)}_{\text{look for overlap for these two factors.}}$$

t=0
t<0
t>0

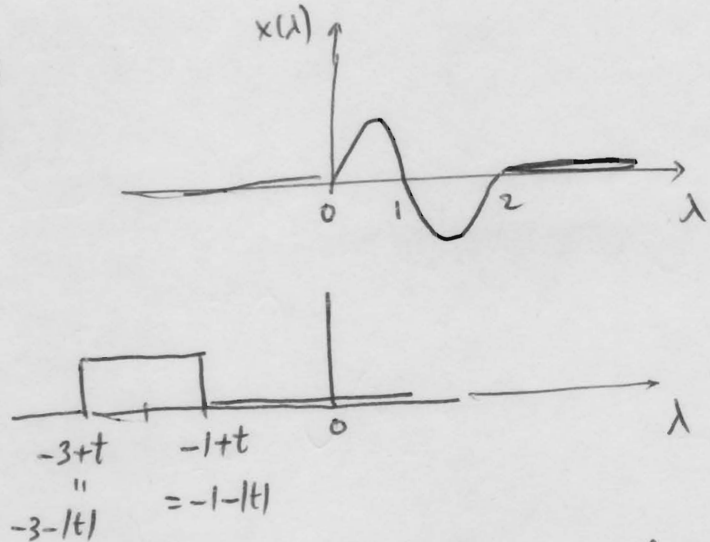
$\boxed{t=0}$





No overlap
↓
$y(t) = \int_{-\infty}^{\infty} d\lambda \; 0 = 0$

$\boxed{t<0}$



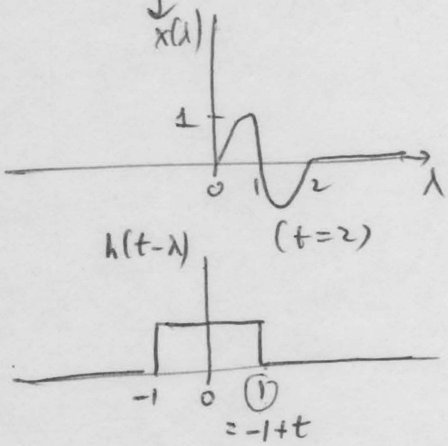
No overlap

↓

$y(t) = 0$

$\boxed{t>0}$

- $0 < t < 1$ → still no overlap → $y(t) = 0$
- $\boxed{1 \le t \le 3}$ → overlap b/w 0 & t
- $\boxed{3 < t \le 5}$ → overlap b/w $-3+t$ & 2



$$y(t) = \int_0^{-1+t} d\lambda \; 2 \sin \pi \lambda$$

$$= -2 \left[\frac{\cos \pi \lambda}{\pi} \right]_0^{t-1}$$

$$= \frac{2}{\pi} \left(1 - \cos \pi (t-1) \right)$$