Convolution example:

Our linear time-inv. system is described by $h[n] = u[n]$ a step function. We would like to find the output $y[n]$ if an input $x[n] = \alpha^n u[n]$ is applied (α : constant), i.e., input applied from $n=0$ and on and is a power of α.

output, input, impulse response

$$y[n] = x[n] * h[n]$$

$$= \underbrace{\sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]}_{\text{regular multiplication.}}$$

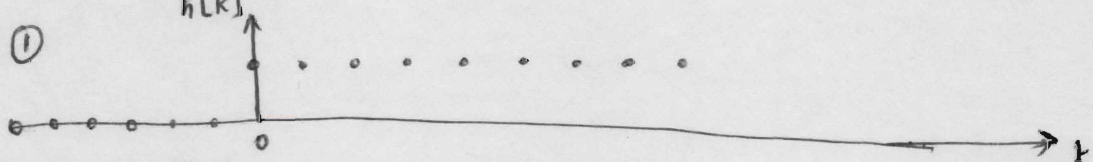
convolution definition

$x[k]$: $x[n]$ when renaming $n \to k$

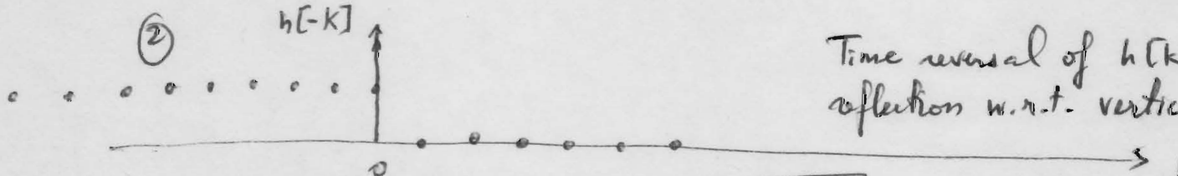
$h[n-k]$ = involves a time reversal on $h[k]$ plus a time shift of n

In our example: $h[k] = u[k]$

① $h[k]$

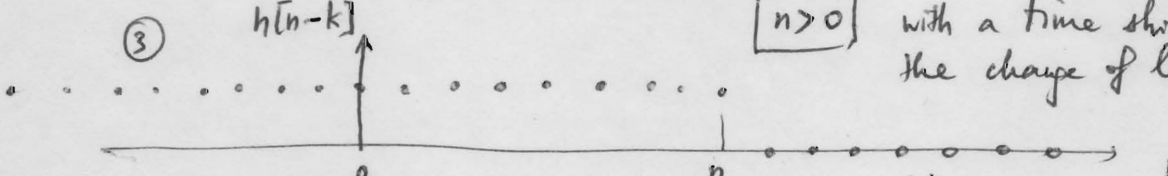


② $h[-k]$



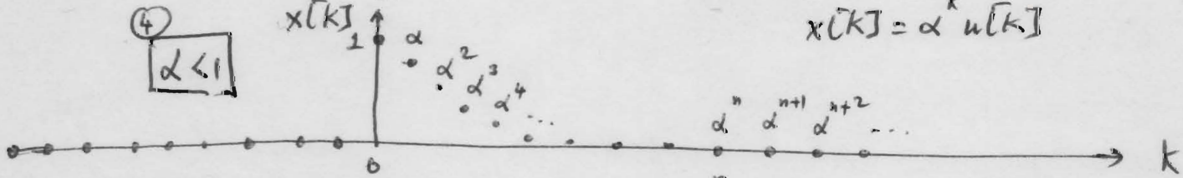
Time reversal of $h[k]$ is the reflection w.r.t. vertical axis

③ $h[n-k]$ $\boxed{n>0}$

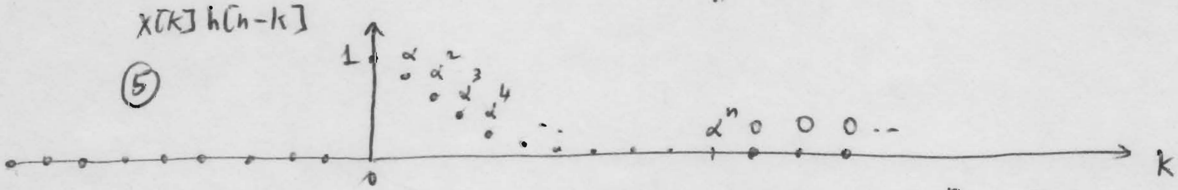


with a time shift of n the change of levels moved from 0 to n

④ $\boxed{\alpha<1}$ $x[k]$ $\quad x[k] = \alpha^k u[k]$



⑤ $x[k]\,h[n-k]$



So: $\left\{\begin{matrix} n>0 \\ \alpha<1 \end{matrix}\right\}$ $\quad y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = \sum_{k=0}^{n} \alpha^k$: sum of a geometric series

Math review: geometric series: $\sum_{k=0}^{n} \alpha^k = S = \frac{1-\alpha^{n+1}}{1-\alpha}$

Proof:

$$\sum_{k=0}^{n} \alpha^k = \underbrace{\alpha^0 + \alpha^1 + \alpha^2 + \cdots \alpha^n}_{n+1 \text{ terms}} = S$$

α^0: 1

$$\Rightarrow \underbrace{\alpha + \alpha^2 + \cdots \alpha^n}_{n \text{ terms}} = S - 1$$

$$\Rightarrow \alpha\big(\underbrace{1 + \alpha + \cdots + \alpha^{n-1}}_{S - \alpha^n}\big) = S - 1$$

$$\alpha(S - \alpha^n) = S - 1$$

$$1 - \alpha^{n+1} = S - \alpha S = S(1-\alpha)$$

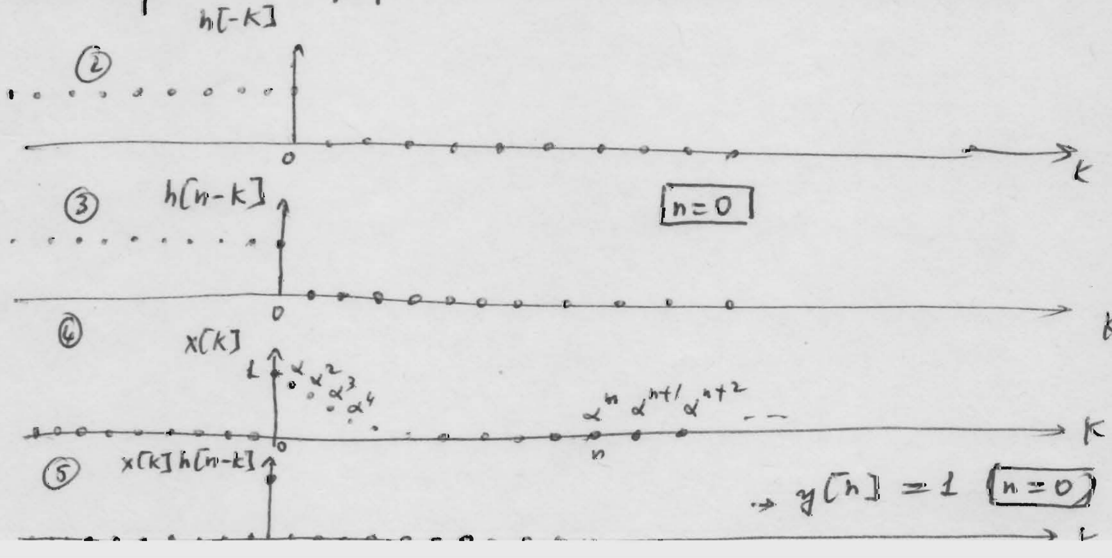
$$\Rightarrow \boxed{S = \frac{1-\alpha^{n+1}}{1-\alpha}}$$

$\Rightarrow$ output is $y[n] = x[n] * h[n] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}$

$\boxed{\begin{array}{l} n>0 \\ \alpha<1 \end{array}}$

Now we would like to have $y[n]$ $\forall n$ $\begin{cases} n<0 & \leftarrow \\ n=0 & \leftarrow \checkmark \\ n>0 & \checkmark \end{cases}$

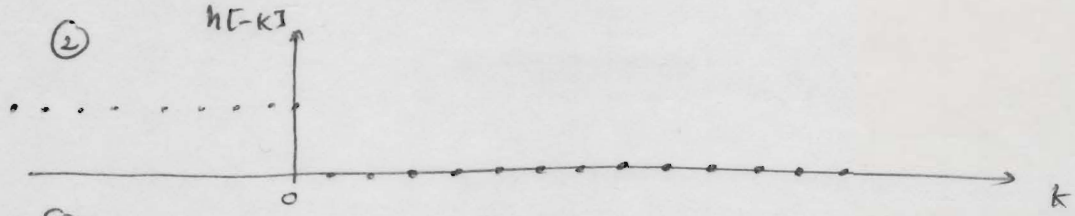
Let's repeat the graphical solution for the case $n=0$:



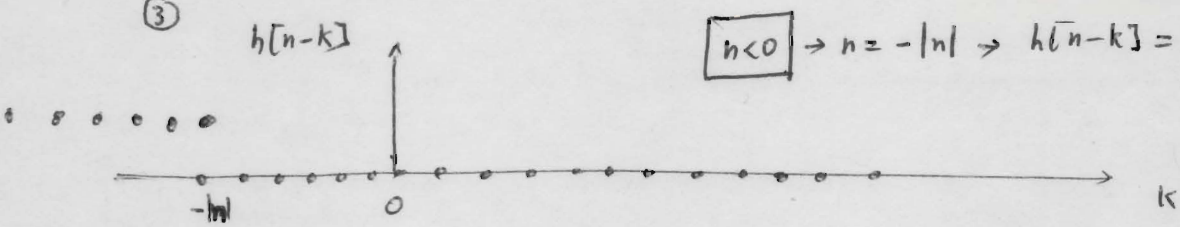
$\rightarrow y[n] = 1$ $\boxed{n=0}$

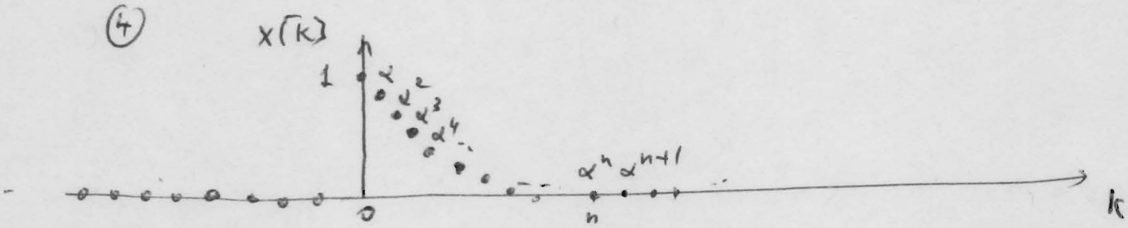
$n=0: \quad y[n] = \sum_{k=0}^{n} \delta[k] = 1$

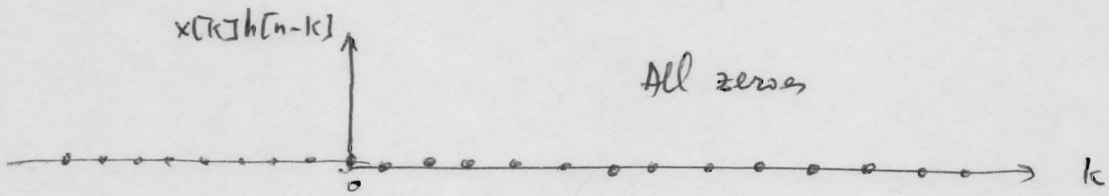
Let's repeat the graphical solution for $n<0$:



$\boxed{n<0} \rightarrow n = -|n| \rightarrow h[n-k] = h\,\underbrace{[-|n|-k]}_{\text{shift to the left of origin.}}$







$n<0: \quad y[n] = \sum_{n=0}^{n} 0 = 0$

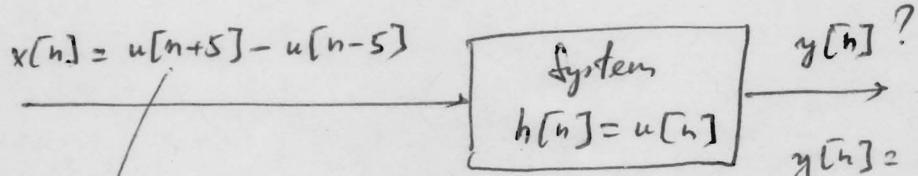
Summary:

$x[n] = \alpha^n u[n] \longrightarrow$ System $h[n] = u[n]$ $\longrightarrow y[n] = \begin{cases} 0; & n<0 \\ 1; & n=0 \\ \dfrac{1-\alpha^{n+1}}{1-\alpha}; & n>0 \end{cases}$

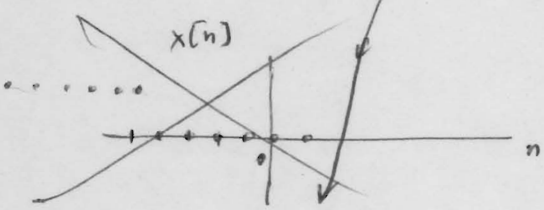
$= \dfrac{1-\alpha^{n+1}}{1-\alpha}\, u[n]$

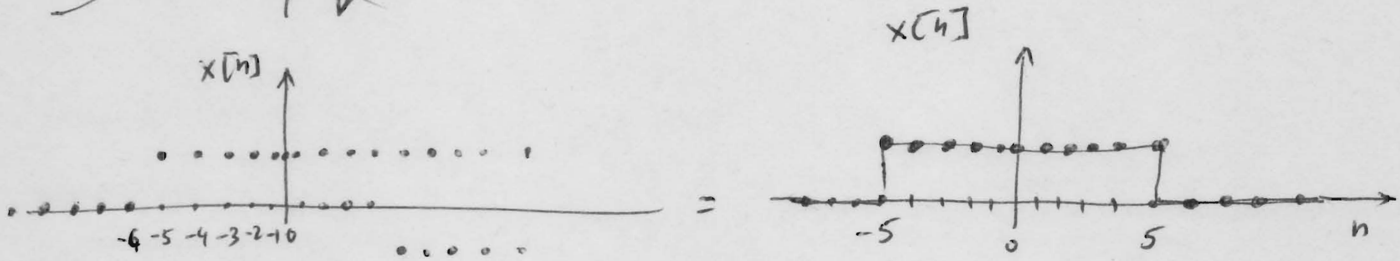
Observation: to calculate a convolution we may need to repeat the calculation for different intervals of n (in the previous example: $n<0$; $n=0$; $n>0$)

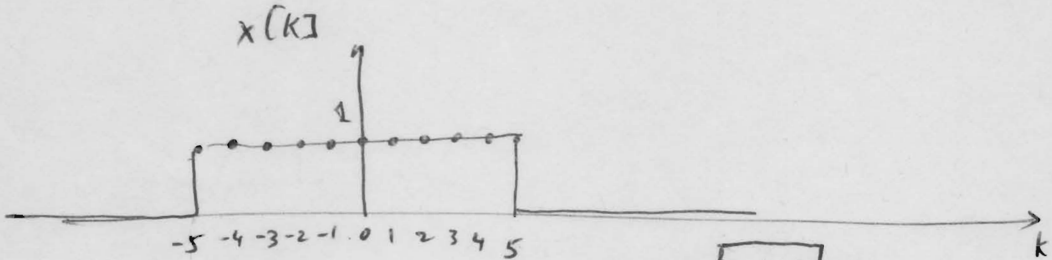
Another example of the convolution:



$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, \underbrace{h[n-k]}$$

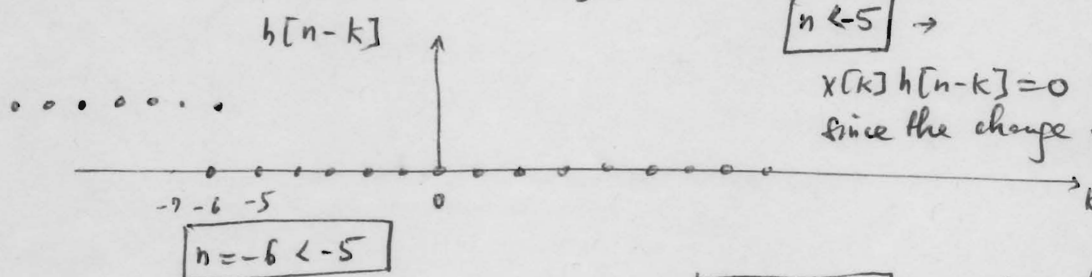




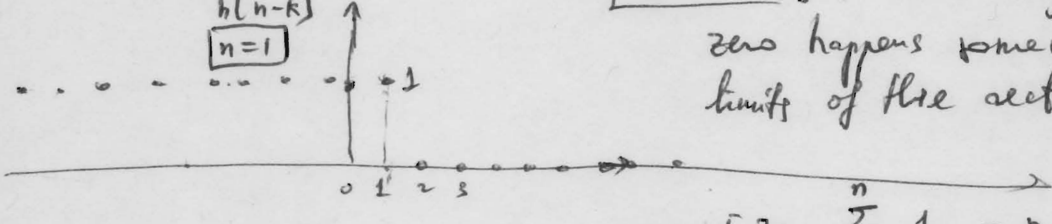


$\boxed{n < -5} \rightarrow$

$x[k]\,h[n-k] = 0 \;\; \forall k \rightarrow y[n] = 0 \;\; (n < -5)$ since the change of level 1 to 0 happens outside the limits of the rectangular pulse



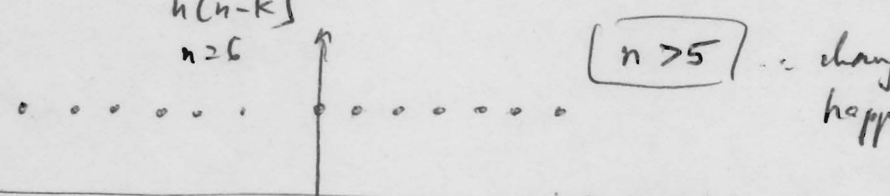
$\boxed{-5 \le n \le 5} \rightarrow$ the change of level from 1 to zero happens somewhere within the limits of the rectangular pulse



$$y[n] = \sum_{k=-5}^{n} 1 = n - (-5) + 1 = n + 6$$

$(-5 \le n < 5)$

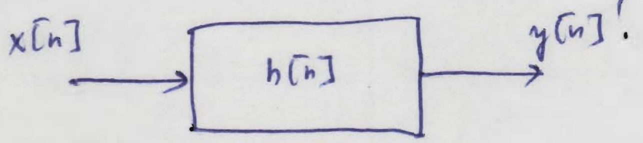
$\boxed{n > 5}$ .. change of level from 1 to zero happens to the right of the rectangular pulse → overlap b/w x & h is the whole rectangular pulse.



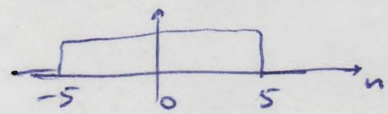
$$y[n] = \sum_{k=-5}^{5} 1 = 5 - (-5) + 1 = 11 \quad (n > 5)$$

$$y[n] = \begin{cases} 0 & n < -5 \\ n+6 & -5 \le n \le 5 \\ 11 & n > 5 \end{cases}$$

<u>Third example of convolution:</u> Let's convolute two rectangular pulses together.



$x[n] = u[n+5] - u[n-5]$



$h[n] = u[n+5] - u[n-5]$

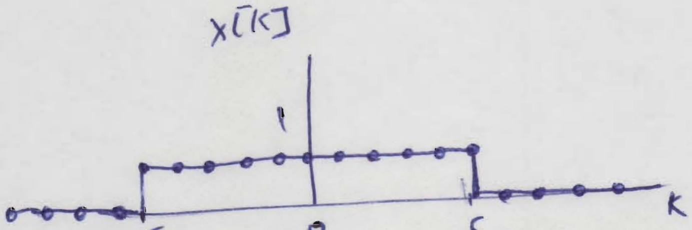
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{K} x[k] \cdot \underbrace{h[n-k]}_{\text{time reversal + shift by } n}$$

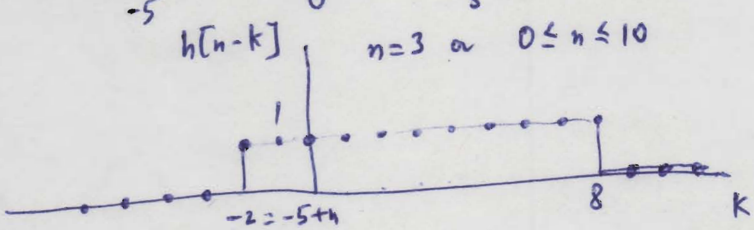
time-reversal + shift by n

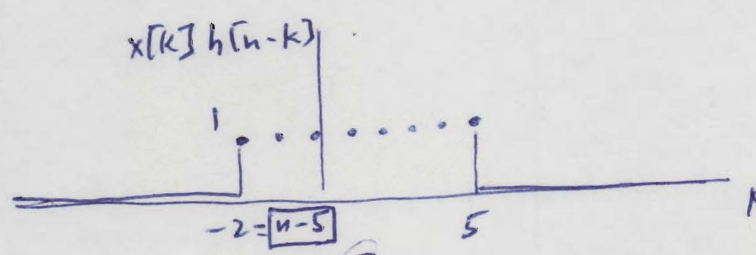
reflection wrt. the origin

↓

same for an ~~centered~~ even signal centered @ the origin

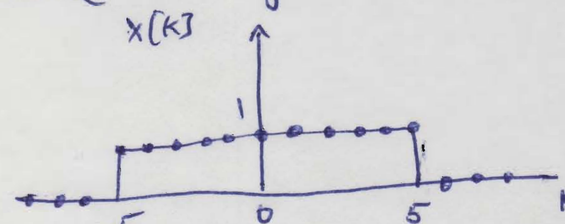


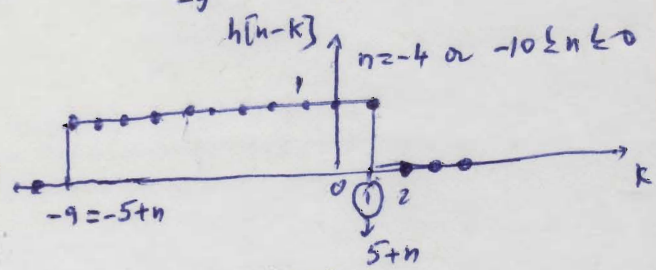


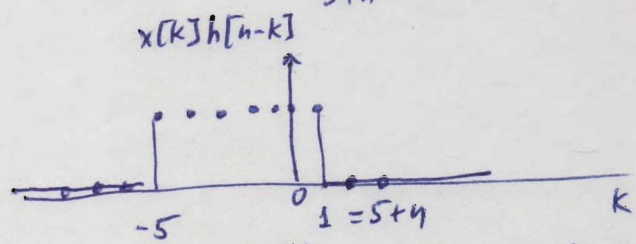


$$\rightarrow y[n] = \sum_{k=n-5}^{5} 1 = 5-(n-5)+1 = 11-n$$

$y[n] = 0$ if $n > 10$ (no overlap b/w $x[k]$ & $h[n-k]$)

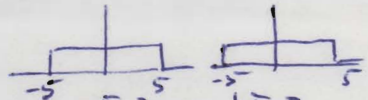




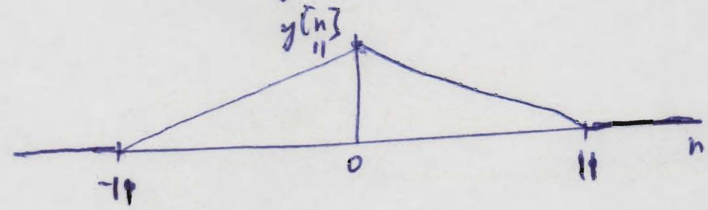


$$y[n] = \sum_{k=-5}^{5+n} 1 = (5+n)-(-5)+1 = 11+n$$

$y[n] = 0$ if $n < -10$



$$x[n] * h[n] = y[n] \begin{cases} n < -10 & : 0 \\ -10 \le n \le 0 & : 11+n \text{ (slope } +1) \\ 0 \le n \le 10 & : 11-n \text{ (slope } -1) \\ n > 10 & : 0 \end{cases}$$



Convolution for continuous-time signal:

$$x(t) \longrightarrow \boxed{\text{System } h(t)} \longrightarrow y(t) = x(t) * h(t)$$

impulse response : it the output to the $\delta(t)$ input:

$$\delta(t) \longrightarrow \boxed{\text{System}} \longrightarrow h(t)$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} d\lambda \; x(\lambda)\, h(t-\lambda) \underset{\substack{\text{change of} \\ \text{variable}}}{=} \int_{-\infty}^{\infty} d\lambda \; x(t-\lambda)\, h(\lambda)$$

$$= h(t) * x(t)$$

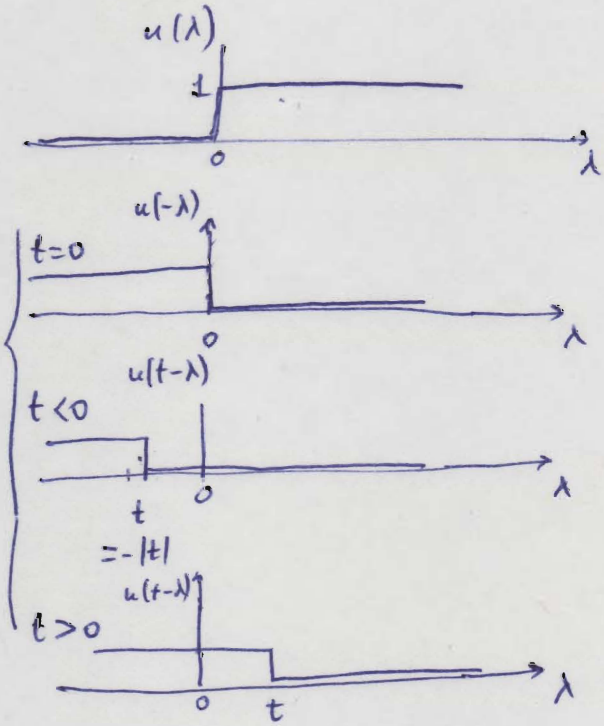
- The integral's change of variable leads to the commutative property of the convolution : $x * h = h * x$
  (we can apply time-reversal and shift on either h or x getting the same result)
- λ is the continuous-time counterpart of the discrete-time index k ; same with t & n

Let's find $y = x * h$ where $\begin{cases} x(t) = e^{-at}\, u(t) \\ h(t) = u(t) \end{cases}$

$$y(t) = \int_{-\infty}^{\infty} d\lambda \; x(\lambda)\, h(t-\lambda) = \cdots \;?$$

$$= \int_{-\infty}^{\infty} d\lambda \; e^{-a\lambda}\, \underbrace{u(\lambda) \cdot u(t-\lambda)}$$

$u(t-\lambda) =$



Conclusion: $u(\lambda)\, u(t-\lambda)$ is nonzero only when $0 \le t \rightarrow$

$$y(t) = \begin{cases} 0 & t<0 \\ \int_0^t d\lambda \; e^{-a\lambda} \cdot 1 = \left[\dfrac{e^{-a\lambda}}{-a} \right]_0^t = \dfrac{1-e^{-at}}{a} \end{cases}$$

In summary: $\boxed{y(t) = x(t) * h(t) = \dfrac{1-e^{-at}}{a}\, u(t)}$

Do HW#2 bring in questions 9/30

2.8 Find $\underline{x(t) * h(t)}$ $\quad \begin{cases} x(t) = \begin{cases} t+1 & 0 \le t \le 1 \\ 2-t & 1 \le t \le 2 \\ 0 & \text{elsewhere} \end{cases} \\ h(t) = \delta(t+2) + 2\delta(t+1) \end{cases}$

$$\int_{-\infty}^{\infty} d\lambda \; x(\lambda)\, h(t-\lambda) = \int_{-\infty}^{\infty} d\lambda \; x(\lambda) \left[\delta(t-\lambda+2) + 2\delta(t-\lambda+1) \right]$$

$$= \int_{-\infty}^{\infty} d\lambda \; x(\lambda) \underbrace{\delta(t+2-\lambda)}_{\text{centered @ } t+2} + 2\int_{-\infty}^{\infty} d\lambda \; x(\lambda) \underbrace{\delta(t+1-\lambda)}_{\text{centered @ } t+1}$$

$$= x(t+2) + 2\, x(t+1) \longrightarrow \text{plot;}$$