**Introduction to Systems:**

A system is represented with a box whose content we don't know. For us, a system is identified by the equation that relates the output $y(t)$ to the input $x(t)$:

$$y(t) = 2x(t) - x^2(t+1)$$

This is the "input-output" equation.

**Properties of Systems:**

1. **Memory:** A system has memory when current output depends on previous or past inputs.

2. **Invertibility:** A system $S$ is invertible if we can find the inverse system $S^{-1}$ such that:

$$x(t) \xrightarrow{S} y(t) \xrightarrow{S^{-1}} x(t)$$

3. **Causality:** A system $S$ is causal if its output only depends on current or past inputs.

4. **Stability:** A system $S$ is stable if it will not produce extremely large output for normal input.

5. **Time-invariance:** A system $S$ is time-invariant when it produces same output for an input regardless of when it is applied.

6. **Linearity:** A system $S$ is linear if:
\[ y_1(t) = a_1 x(t) \]

\[ y_2(t) = a_2 x(t) \]

\[ y_1(t) + y_2(t) \]

\[ \sqrt{y_1(t) y_2(t)} \]

\[ \sin(ax(t)) = ay(t) \]

\[ y_1[n] = 2x_1[n] + 4x_1[n-1] \]

\[ y_2[n] = x_1[n-1] + \frac{1}{2} x_2[n-3] \]
a) Input-Output equation for $S$ = how to get $y_2[n]$ from $x_1[n]$.

Use: $y_1[n] = x_2[n]$ as output to $S_2$, is input to $S_2$ in a serial connection:

$$y_2[n] = y_1[n-2] + \frac{1}{2} y_1[n-3]$$

$$= 2x_1[n-2] + 4x_1[n-3] + \frac{1}{2} x_1[n-3] + \frac{1}{4} x_1[n-4]$$

$$\therefore$$

$$y_2[n] = 2x_1[n-2] + 5x_1[n-3] + 2x_1[n-4]$$

b) Input-output equation for system $S$, which is the serial combination of $S_2$ & $S_1$.

$$S = x_1 \xrightarrow{S_1} y_1, y_2, S_2 \xrightarrow{S_1} y_2(n) = 2x_1(n-2) + 5x_1(n-3) + 2x_1(n-4)$$

4) Use $y_2[n] = x_1[n]$.

$$y_1[n] = 2y_2[n] + 4y_2[n-1]$$

$$\bar{y} = \begin{array}{c}
2x_2[n-2] + 2 \frac{1}{2} x_2[n-3] + 4 x_2[n-3] + 4 \frac{1}{2} x_2[n-4] \\
= 2x_2[n-2] + 5x_2[n-3] + 2x_2[n-4]
\end{array}$$

→ Some system if order of connections is switched!

→ Some input-output equation:

$$y(n) = 2x[n-2] + 5x[n-3] + 2x[n-4]$$
\[ y[n] = x[n] x[n-2] \]

a) Is this system memoryless? → No, it has memory since current output depends on past input. 

b) What is \( y[n] \) if \( x[n] = A \delta[n] \)? A any real or complex number.

\[ y[n] = A \delta[n] A \delta[n-2] = A^2 \delta[n] \delta[n-2] = 0 \]

Since \( \delta[n] \) is centered at \( n=0 \) → it is \( 1 \) at \( n=0 \) \& 0 otherwise.
\( \delta[n-2] \) is centered at \( n=2 \) → it is \( 1 \) at \( n=2 \) \& 0 otherwise.

c) Is this system invertible?
→ If it is → we should be able to find \( S^{-1} \) such that:

\[ x[n] \xrightarrow{S} y[n] \xrightarrow{S^{-1}} x[n] \]

Only way:
\[ y[n] = x[n] \frac{1}{x[n-2]} \]

→ Division: this will blow up with any zero in the input → Not possible.

→ This system is not invertible.
(1.27) \[ y(t) = x(t-2) + x(2-t) \]

1) Memory? Yes
2) Time invariant? \( t \to t+a \) \[
\begin{align*}
  x(t-2) &\to x(t+a-2) \\
  x(2-t) &\to x(2-t-a)
\end{align*}
\]

\[ y(t+a) = x(t+a-2) + x(2-t-a) \to \text{Yes!} \]

(\( x(t) \to x(t+a) \) then \( y(t) \to y(t+a) \))

3) Linear? Yes
4) Causal? (If current output depends on previous inputs only).

\[
\begin{align*}
  y(0) &= x(-2) + x(2) \quad \to \text{not causal} \\
  y(1) &= x(-1) + x(1) \quad \to \text{causal} \\
  y(2) &= x(0) + x(0) \quad \to \text{causal}
\end{align*}
\]

\( \to \) system is causal if \( t \geq 1 \)

5) Stable? Yes (for normal input system won't produce \( \infty \) output)

(b) \[ y(t) = (\cos 3t) x(t) \]
\[ f(t) \text{ any function} \rightarrow \begin{cases} \text{Even} = \text{Ev}[f(t)] = \frac{f(t) + f(-t)}{2} \\ \text{Odd} = \text{Odd}[f(t)] = \frac{f(t) - f(-t)}{2} \end{cases} \]

- a function \( g(t) \) is even when \( g(-t) = g(t) \) or \( g(t) \) is symmetric w.r.t. the origin.
- a function \( g(t) \) is odd when \( g(-t) = -g(t) \) or \( g(t) \) is antisymmetric w.r.t. the origin.

\[ \rightarrow \begin{cases} \text{Ev}[f(t)] \text{ is even or symmetric w.r.t. origin} \\ \text{Odd}[f(t)] \text{ is odd or antisymmetric w.r.t. origin} \end{cases} \]

1.78 d):
\[ y[n] = \text{Ev}[x[n-1]] = \frac{1}{2}(x[n-1] + x[1-n]) \]
- Memory: \( \checkmark \)
- Time invariance: \( n \rightarrow n+k \) \( \checkmark \)
- Causality: \( n \geq 0 \) \( y[0] = \frac{1}{2}(x[-1] + x[1]) \) Not causal \( n \geq 1 \rightarrow \) causal.
- Linearity: \( \checkmark \)
- Stability: \( \checkmark \)

2) \[ y[n] = \begin{cases} x[n] & n \geq 1 \\ 0 & n = 0 \\ x[n+1] & n \leq -1 \end{cases} \]
- Memory: \( \text{No} \)
- Time inv: \( \checkmark \)
- Causality: \( \text{No} \)
- Linearity: \( \checkmark \)
- Stability: \( \checkmark \)
1.46

\[
x[n] \rightarrow e[n] \rightarrow y[n] = x[n-1] \rightarrow y[n]
\]

\[\text{Assume } y[-1] = 0 \quad \forall n < 0\]

a) Sketch \(y[n]\) when \(x[n] = \delta[n]\)  
b) when \(x[n] = u[n]\)

Find \(y[n]\) in terms of \(x[n]\) or the input-output equation.

\[e[n] = x[n] + y[n]\]

\[y[n] = x[n-1] + y[n-1]\]

\[y[n] = 0 \quad \forall n < 0\]

\[\begin{align*}
y[0] &= x[-1] + y[-1] = x[-1] \\
y[1] &= x[0] + y[0] \\
\end{align*}\]

\[
\begin{array}{c|c|c|c}
\text{delta} & \text{step} & x[n] = \delta[n] & x[n] = u[n-1] \\
\hline
0 & 0 & \delta[-1] = 0 & u[-1] = 0 \\
1 & 1 & 1+0 = 1 & 1+0 = 1 \\
2 & 1 & 0+1 = 1 & 1+1 = 2 \\
3 & 1 & 0+1 = 1 & 1+2 = 3 \\
4 & 1 & 0+1 = 1 & 1+3 = 4 \\
\end{array}
\]

\[
x[n] = \delta[n]
\]

\[
x[n] = u[n-1]
\]
\[ y[n] \quad x[n] = h[n] \]
\[ x_3(t) = x_1(t+1) + x_1(t) \]

\[ y_3(t) \rightarrow \text{LTI} \rightarrow y_1(t+1) + y_1(t) \]

\[ y_2(t) \rightarrow \text{neg. slope} + \text{pos. slope} = 0 \text{ slope} \] (horizontal)
**Ch2: Linear Time-Invariant Systems**

HW 2: 2.7; 2.8; 2.11; 2.17; 2.22; 2.31; 2.40; 2.47; 2.6

**Convolution:** "\( \ast \)" is a mathematical operation that allows us to calculate \( y[n] \) from \( x[n] \) using impulse response \( h[n] \).

\[
y[n] = x[n] \ast h[n]
\]

Impulse response \( h[n] \) is the output of a \( d[n] \) input.

\( h[n] \) can be used to characterize the system.

\[
y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]
\]

Simple multiplication.

**Example:** Find \( y[n] \) from input \( x[n] = \delta[n] \) step function

and impulse response \( h[n] = u[n] \) a constant

Switch: when you turn the switch @ \( t = 0 \), i.e. the \( d[n] \) input, the light turns on and stays on; i.e. flat step function.