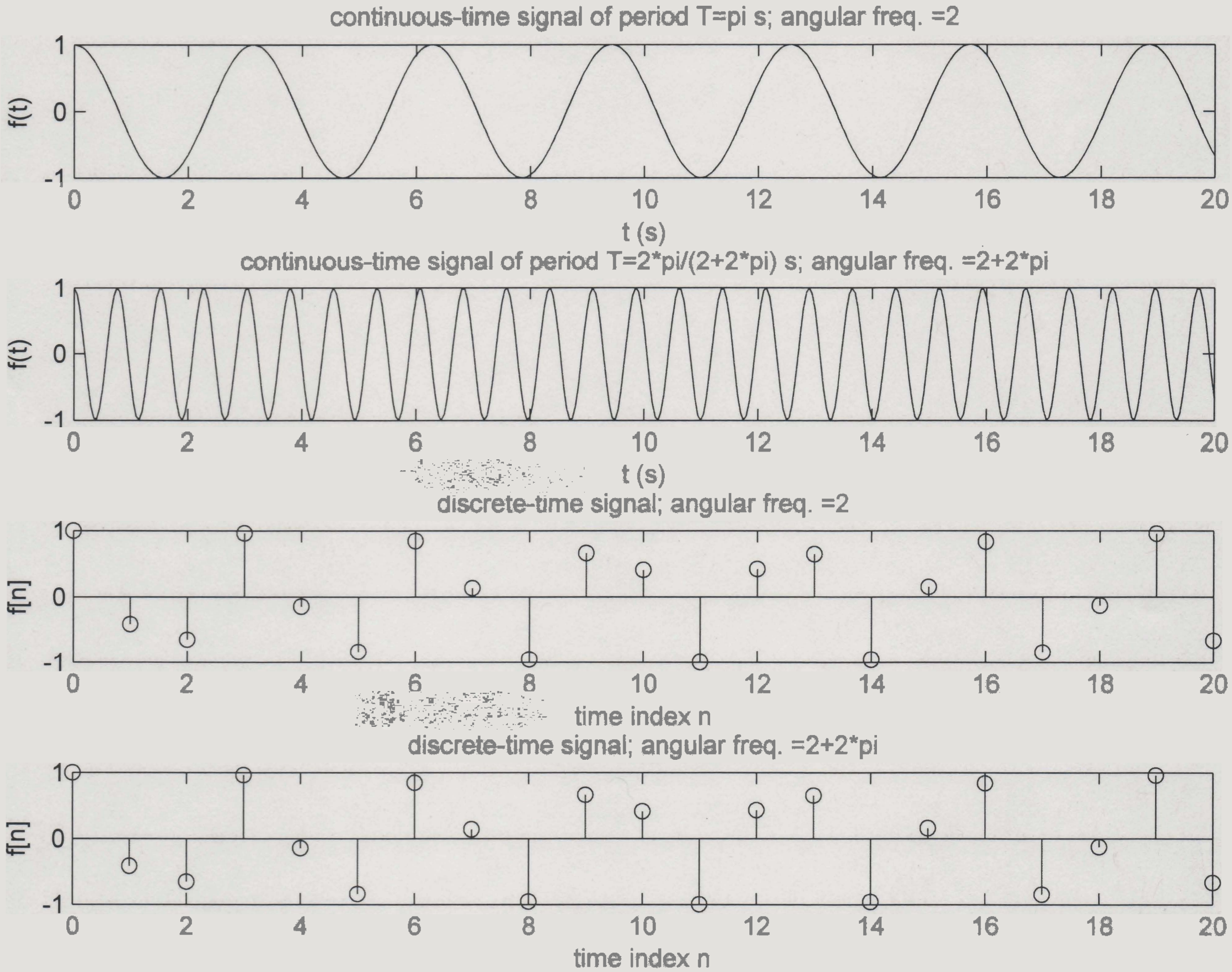
```
%Engin 321/TM
%Sept 9, 2008
%To visualize a second difference between continuous-time and discrete-↙
time
%signals: for example, exp(j2t) is not the same as exp(j(2+2pi)t), while
%exp[j2n] is the same as exp[j(2+2*pi)t]

%Continuous-time signal with angular freq. omega1=2
omega1=2;
tc=0:.05:20;
figure(2), subplot(4,1,1), plot(tc,cos(omega1*tc))
title('continuous-time signal of period T=pi s; angular freq. =2')
xlabel('t (s)')
ylabel('f(t)')

%Continuous-time signal with angular freq. omega2=2+2*pi
omega2=2+2*pi;
subplot(4,1,2), plot(tc,cos(omega2*tc))
title('continuous-time signal of period T=2*pi/(2+2*pi) s; angular freq.↙
=2+2*pi')
xlabel('t (s)')
ylabel('f(t)')

%Discrete-time signal with angular freq. omega1=2
td=0:1:20;
subplot(4,1,3), stem(td,cos(omega1*td))
title('discrete-time signal; angular freq. =2')
xlabel('time index n')
ylabel('f[n]')

%Discrete-time signal with angular freq. omega2=2+2*pi
subplot(4,1,4), stem(td,cos(omega2*td))
title('discrete-time signal; angular freq. =2+2*pi')
xlabel('time index n')
ylabel('f[n]')
```


continuous-time signal of period T=pi s; angular freq. =2
f(t)
t (s)
continuous-time signal of period T=2*pi/(2+2*pi) s; angular freq. =2+2*pi
f(t)
t (s)
discrete-time signal; angular freq. =2
f[n]
time index n
discrete-time signal; angular freq. =2+2*pi
f[n]
time index n

1.10 Determine fundamental period of $x(t) = \underbrace{2\cos(10t+1)}_{\substack{\text{periodic \#1} \\ x_1(t)}} - \underbrace{\sin(4t-1)}_{\substack{\text{periodic \#2} \\ x_2(t)}}$

Periodic signal #1: $x_1(t+T_1) = x_1(t) \Rightarrow 10T_1 = 2\pi n_1$ ($n_1 = 1, 2, 3$, etc)
period T_1

$$\rightarrow T_1 = \frac{2\pi}{10} n_1 \rightarrow T_1 = \left\{ \overset{n_1=1}{\frac{2\pi}{10}}, \overset{n_1=2}{\frac{4\pi}{10}}, \overset{n_1=3}{\frac{6\pi}{10}}, \overset{n_1=4}{\frac{8\pi}{10}}, \boxed{\pi}, \ldots \right\}$$

Periodic signal #2: $x_2(t+T_2) = x_2(t) \Rightarrow 4T_2 = 2\pi n_2$ ($n_2 = 1, 2, 3$, etc.)
period T_2

$$\rightarrow T_2 = \frac{2\pi}{4} n_2 \rightarrow T_2 = \left\{ \frac{\pi}{2}, \boxed{\pi}, \frac{3\pi}{2}, 2\pi, \text{etc.} \right\}$$

→ What is the period T of $x(t) = 2x_1(t) - x_2(t)$:
Both x_1 & x_2 repeat themselves(*) @ $T_1 = T_2 = \pi$ → this will be the fundamental(*) period of $x(t)$ → $\boxed{T = \pi}$

(*) For the 1st time.

1.11 Determine the fundamental period for $x[n] = 1 + \underbrace{e^{j\frac{4\pi n}{7}}}_{\substack{x_1[n] \\ \downarrow \\ \text{periodic}}} - \underbrace{e^{j\frac{2\pi n}{5}}}_{\substack{x_2[n] \\ \downarrow \\ \text{periodic}}}$

Periodic signal #1: $x_1[n+N_1] = x_1[n] \rightarrow e^{j\frac{4\pi}{7}N_1} = 1 \rightarrow \frac{4\pi}{7}N_1 = 2\pi n_1$, ($n_1 = 1, 2, 3, 4, \ldots$)
period N_1

$$N_1 = \frac{7}{2} n_1 \rightarrow \underset{\substack{\downarrow \\ \text{integers}}}{N_1} = \{7, 14, 21, 28, 35, 42, \ldots\}$$

Periodic signal #2: $x_2[n+N_2] = x_2[n] \rightarrow e^{j\frac{2\pi}{5}N_2} = 1 \rightarrow \frac{2\pi}{5}N_2 = 2\pi n_2$ ($n_2 = 1, 2, 3, 4, \ldots$)
period N_2

$$N_2 = 5n_2 \rightarrow N_2 = \{5, 10, 15, 20, 25, 30, 35, 40, \ldots\}$$

The first time both $x_1[n]$ & $x_2[n]$ repeat themselves is when $N_1 = N_2 = 35$ → This is the fundamental period of

$x[n] = 1 + x_1[n] - x_2[n]$ → $\boxed{N = 35}$.

1.36 $x(t) = e^{j\omega_0 t}$: periodic with fundemental freq. ω_0

→ fundamental period $\boxed{T_0 = \frac{2\pi}{\omega_0}}$. Look at the discrete-time signal with $t = \boxed{n}\boxed{T}$ (where T is the time increment) → $x[n] = \underbrace{e^{j\omega_0 nT}}_{e^{j(\omega_0 T)n}}$ is a discrete-time signal

(a) Show that $x[n]$ is periodic if & only if $\frac{T}{T_0}$ is a rational number (ratio of two integers) =

A discrete-time sinusoid is periodic when the angular frequency carries a factor of π :

$$\omega_0 T = k\pi \Rightarrow \frac{2\pi}{T_0} T = k\pi \quad (k \text{ integer})$$

$$\rightarrow \frac{T}{T_0} = \frac{k}{2} \quad (k \text{ integer}) \Rightarrow \frac{T}{T_0} = \text{a rational number}$$

(b) Suppose $x[n]$ is periodic $\overset{(a)}{\rightarrow}$ $\frac{T}{T_0} = \frac{p}{q}$ (p & q are integers)

→ what is the fundamental period N_0 for $x[n]$?

if $x[n]$ is periodic with fundemental period N_0:

$$x[n+N_0] = x[n] \Rightarrow \underbrace{e^{j\omega_0 T(n+N_0)}}_{\cancel{e^{j\omega_0 Tn}} \cdot e^{j\omega_0 T N_0}} = \cancel{e^{j\omega_0 Tn}}$$

$\downarrow$
$x[n] = e^{j\omega_0 Tn}$

$$\Rightarrow e^{j w_0 T N_0} = 1 \rightarrow w_0 T N_0 = 2\pi m \quad (m: \text{integer})$$

since $w_0 = \frac{2\pi}{T_0} \longrightarrow \cancel{2\pi} N_0 \underbrace{\frac{T}{T_0}}_{\frac{p}{q}} = \cancel{2\pi} m \quad (m \text{ integer})$

$$\rightarrow \boxed{N_0 = m \frac{q}{p}}$$

For example: 1) $q = 7$ & $p = 3 \rightarrow N_0 = m \frac{7}{3}$ since N_0 has to be an integer $\rightarrow N_0 = 7 \; (m = 3)$

2) $q = 28; \; p = 12 \rightarrow N_0 = m \frac{28}{12} = m \frac{7}{3} \rightarrow N_0 = 7$

$\rightarrow N_0 = q$ if $\frac{q}{p}$ is irreducible $\rightarrow \boxed{N_0 = \frac{q}{\gcd(q,p)}}$

gcd: "greatest common divisor" $\rightarrow$ check: $q = 28; \; p = 12$

$$\rightarrow \frac{q}{\gcd(q,p)} = \frac{28}{\gcd(28,12)} = \frac{28}{4} = 7$$

Fundamental period is $N_0 \rightarrow$ fundamental freq is $w_{0,D} = \frac{2\pi}{N_0}$

$$w_{0,D} = \frac{2\pi}{\frac{q}{\gcd(q,p)}} = 2\pi \frac{\gcd(q,p)}{q}$$

c) Assume $\frac{T}{T_0} = \frac{p}{q}$ (p & q are integers) or $x[n]$ is periodic

$\rightarrow$ how many periods T_0 of $x(t)$ are needed to obtain samples that form one period N_0 of $x[n]$

One period of $x[n]$ in time index is $N_0 \rightarrow$ in time is $\underline{N_0} \, \underline{T}$

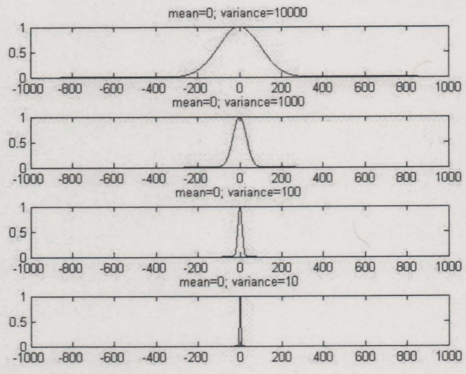
→ This would contain $\frac{N_0 T}{T_0}$ periods T_0 of $x(t)$

$$\frac{N_0 T}{T_0} = \frac{q}{\gcd(q,p)} \underbrace{\frac{T}{T_0}}_{\frac{p}{q}} = \frac{\cancel{q}\ \boxed{p}}{\cancel{q}\ \boxed{\gcd(q,p)}}$$

This is how many periods of the original $x(t)$ are contained in one period of $x[n]$

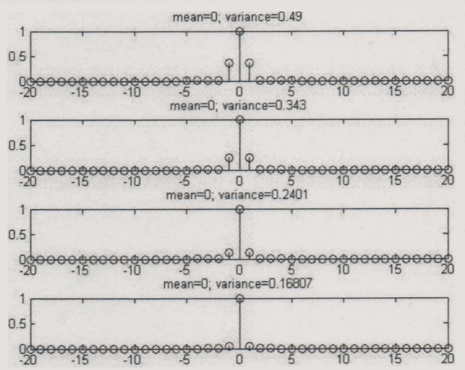
More on signals:

Unit impulse: δ(t)



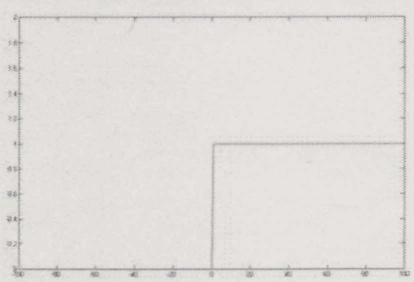

```
%Gaussian distribution
clear all
close all
for i=1:4
var=10^(5-i);%1000;%variance
sigma=sqrt(var);%standard deviation
xbar=0;%%mean
x=-1000:1000;
fx=1/(sqrt(2*pi)*sigma)*exp(-(x-
xbar).^2/(2*var));
fx=fx/max(fx);
figure(1), subplot(4,1,i),plot(x,fx),
title(strcat('mean=',num2str(xbar),';
variance=',num2str(var)))
end
```

A unit impulse is the limit of gaussian function when the width (variance; standard deviation) tends to zero. "Unit" when the maximum value is 1. The discrete-time version of it is δ[n], as shown in the figure below.

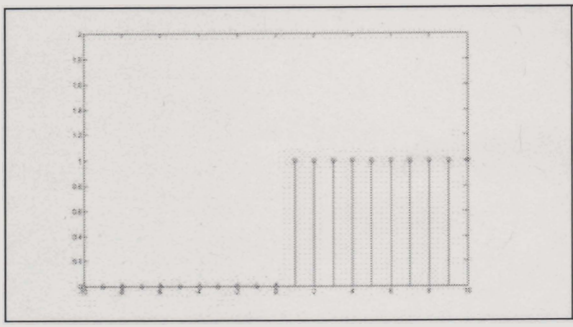



```
%Gaussian distribution: discrete-time
clear all
%close all
for i=1:4
var=.7^(i+1);%1000;%variance
sigma=sqrt(var);%standard deviation
xbar=0;%%mean
x=-20:20;
fx=1/(sqrt(2*pi)*sigma)*exp(-(x-
xbar).^2/(2*var));
fx=fx/max(fx);
figure(2), subplot(4,1,i),stem(x,fx),
title(strcat('mean=',num2str(xbar),';
variance=',num2str(var)))
end
```

Unit step function: u(t) or u[n]



```
close all
clear all
xstep=-100:1:100;
step=[zeros(1, 101) ones(1,100)];
figure(1),plot(xstep,step)
```



```
xstepdt=-10:1:10;
stepdt=[zeros(1, 11) ones(1,10)];
figure(2), stem(xstepdt, stepdt)
```

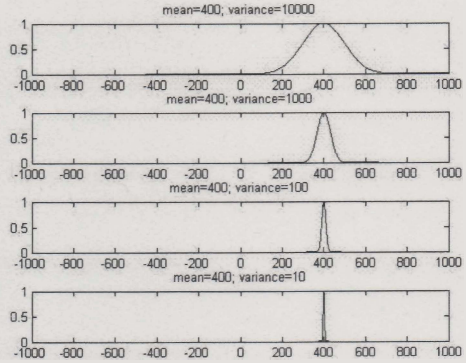
Relationship:

$$\delta(t) = \frac{du}{dt}$$

Integrals involving impulse functions δ(t):

$$\int_{-\infty}^{+\infty} dt \quad f(t)\delta(t) \; = \; f(0)$$

Below is a impulse centered at 400



$$\int_{-\infty}^{+\infty} dt \quad f(t)\delta(t-400) \; = \; f(400) \quad \text{(impulse centered at 400)}$$

$$\int_{-\infty}^{+\infty} dt \quad f(t)\delta(t+400) \; = \; f(-400) \quad \text{(impulse centered at -400)}$$

$$\int_{-3}^{+3} dt \quad f(t)\delta(t) \; = \; f(0)$$

$$\int_{-3}^{+3} dt \quad f(t)\delta(t-400) \; = \; 0$$

1.39 From the relationship between impulse and the step functions:

$$u_\Delta(t) = \int_{-\infty}^{t} d\tau\, \delta_\Delta(\tau); \quad u(t) = \lim_{\Delta \to 0} u_\Delta(t)$$

u_Δ is a step function with a finite slope during Δ at the transition from 0 to 1; when $\Delta \to 0$ the transition has an infinite slope, we get the actual step function u(t). Similarly δ_Δ is an impulse with a finite width Δ starting from 0, when $\Delta \to 0$ we get back the actual impulse $\delta(t)$.

The problem asks for this proof:

$$\lim_{\Delta \to 0}\left[u_\Delta(t)\delta(t)\right] = 0$$

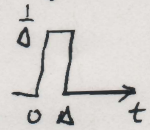
This is obvious if we replace the definition of u_Δ:

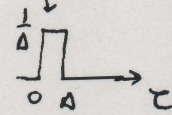
$$\lim_{\Delta \to 0}\left[u_\Delta(t)\delta(t)\right] = \delta(t)\lim_{\Delta \to 0}\int_{-\infty}^{t} d\tau\, \delta_\Delta(\tau);$$ since when t=0 the lim is 0; same when t<0; when t>0 the limit is nonzero however $\delta(t)=0$, so the result is still 0.

The problem also asks for this proof:

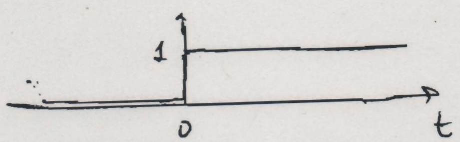
$$\lim_{\Delta \to 0}\left[u_\Delta(t)\delta_\Delta(t)\right] = \frac{1}{2}\delta(t)$$

$$\lim_{\Delta \to 0} \left[u_\Delta(t)\, \delta_\Delta(t) \right] = \lim_{\Delta \to 0} \left[\quad \times \int_{-\infty}^{t} \delta_\Delta(\tau)\, d\tau \right]$$

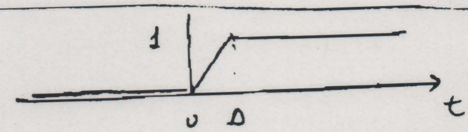




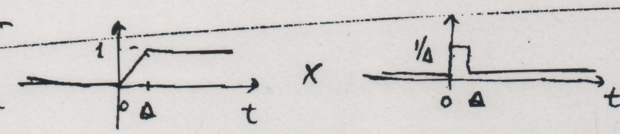
$u(t)$

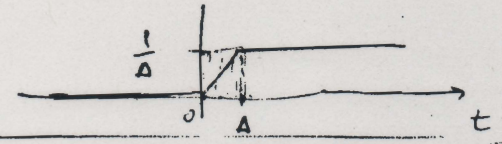


$u_\Delta(t)$



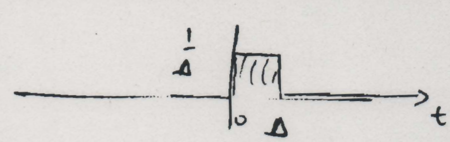
$$\lim_{\Delta \to 0} \left[u_\Delta(t)\, \delta_\Delta(t) \right] = \lim_{\Delta \to 0} \left[\quad \times \quad \right]$$





→ Recall:

$$\lim_{\Delta \to 0} \quad = \delta(t)$$



So → $\frac{1}{2}\delta(t)$.

$\frac{1}{2}$ → half area.