4.19 ; 4.34 → Note

Ch5: Discrete-time Fourier Transform

HW5: 5.8; 5.13, 5.19; 5.22 (a → e); 5.26; 5.27; 5.33, 5.36 c) ; 5.41 b) ; 5.51

Continuous-time

Direct: $\hat{X}(j\omega) = \int_{-\infty}^{\infty} dt\, x(t)\, e^{-j\omega t}$

Inverse: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \hat{X}(j\omega) e^{j\omega t}$

$\hat{X}(j(\omega+2\pi)) \neq \hat{X}(j\omega)$

↳ The continuous-time Fourier Transform is NOT periodic in frequency

Discrete-time

$$\hat{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} d\omega\, \hat{X}(e^{j\omega}) e^{j\omega n}$$

i) $e^{j(\omega+2\pi)n} = e^{j\omega n} \cdot \underbrace{e^{j2\pi n}}_{1}$

↳ Only integrate ω over an interval of 2π for the inverse discrete-time Fourier Transform

ii) $\hat{X}(e^{j(\omega+2\pi)}) = \hat{X}(e^{j\omega})$

↳ The discrete-time Fourier Transform is always periodic in frequency.

Properties: see Tables 5.1 (linearity, time-shift, convolution, etc.) & 5.2 (basic pairs of DT-FT)

5.8 Use Tables 5.1 & 5.2 to find the inverse DTFT for

$$\hat{X}(e^{j\omega}) = \frac{1}{1-e^{-j\omega}} \frac{\sin \frac{3}{2}\omega}{\sin \frac{\omega}{2}} + 5\pi\,\delta(\omega)$$

$-\pi < \omega < \pi$ (DTFT is periodic in frequency)

In Table 5.1 : "Accumulation Property 5.3 5":

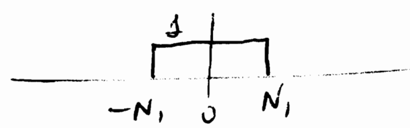
$$\sum_{k=-\infty}^{n} x[k] \xrightarrow{\text{DTFT}} \frac{1}{1-e^{-j\omega}} \hat{X}(e^{j\omega}) + \pi \hat{X}(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

If $\hat{X}(e^{j\omega}) = \boxed{\frac{\sin \frac{3}{2}\omega}{\sin \frac{1}{2}\omega}} \longrightarrow x_1[n]$?

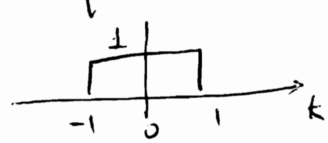
In Table 5.2 : Basic Pairs of DTFT :

Row # 9: $x[n] = \begin{cases} 1 & |n| \le N_1 \\ 0 & |n| > N_1 \end{cases} \xrightarrow{\text{DTFT}} \frac{\sin(N_1 + \frac{1}{2})\omega}{\sin \frac{1}{2}\omega}$

$\boxed{N_1 = 1}$!



$$\rightarrow x_1[k] = \begin{cases} 1 & |k| \le 1 \\ 0 & |k| > 1 \end{cases}$$



Now going back to the "Accumulation Property"

$$\sum_{k=-\infty}^{n} x_1[k] \xrightarrow{\text{DTFT}} \frac{1}{1-e^{-j\omega}} \underbrace{\frac{\sin \frac{3}{2}\omega}{\sin \frac{1}{2}\omega}}_{\hat{X}(e^{j\omega})} + \underbrace{\pi \left[\frac{\sin \frac{3}{2}\omega}{\sin \frac{1}{2}\omega}\right]_{\omega \to 0} \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)}$$

$\delta(\omega)$

since

$-\pi < \omega < \pi$

$\rightarrow k = 0$

L'Hôpital rule

$$\left[\frac{\sin\frac{3}{2}\omega}{\sin\frac{1}{2}\omega}\right]_{\omega\to 0} \overset{\uparrow}{=} \left[\frac{\frac{3}{2}\cos\frac{3}{2}\omega}{\frac{1}{2}\cos\frac{1}{2}\omega}\right]_{\omega\to 0} = \frac{\frac{3}{2}}{\frac{1}{2}} = 3$$

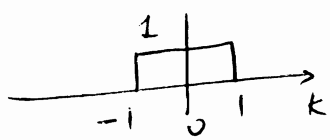
$$\Rightarrow \sum_{k=-\infty}^{n} x_1[k] \xrightarrow{DTFT} \frac{1}{1-e^{-j\omega}}\,\frac{\sin\frac{3}{2}\omega}{\sin\frac{1}{2}\omega} + 3\pi\delta(\omega)$$

$+\boxed{2\pi\delta(\omega)}$ To get $\hat{X}(e^{j\omega})$

$$1 \xrightarrow{DTFT} 2\pi\delta(\omega) = 2\pi\sum_{l=-\infty}^{\infty}\delta(\omega-2\pi l)$$

since $-\pi<\omega<\pi \Leftrightarrow l=0$

$$1+\sum_{k=-\infty}^{n}\boxed{x_1[k]} \xrightarrow{DTFT} \hat{X}(e^{j\omega}) = \frac{1}{1-e^{-j\omega}}\,\frac{\sin\frac{3}{2}\omega}{\sin\frac{1}{2}\omega} + 5\pi\delta(\omega)$$

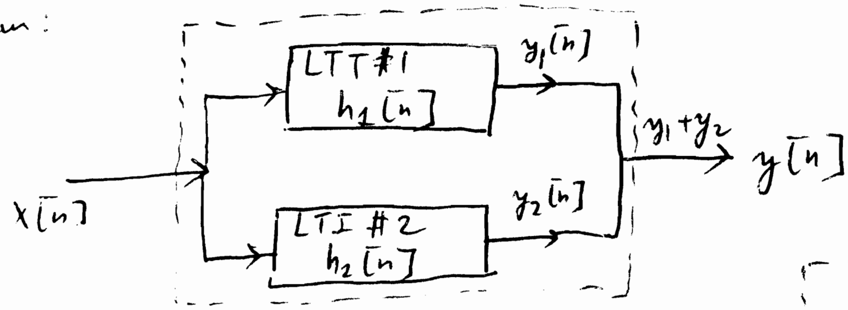


or $x_1[k] = \begin{cases} 1 & |k|\le 1 \\ 0 & |k|>1 \end{cases}$

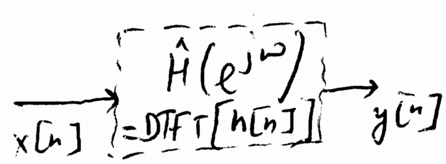
5.13 LTI system #1 $\left(h_1[n] = \left(\frac{1}{3}\right)^n u[n]\right)$ is in parallel with LTI system #2 ($h_2[n]$). The resulting parallel connection has: $\hat{H}(e^{j\omega}) = \dfrac{-12+5e^{-j\omega}}{12-7e^{-j\omega}+e^{-j2\omega}}$. $\boxed{\text{Find } h_2[n]}$

Diagram:

* Parallel connection: both systems receive a same input
* Series connection output of system #1 is the input of system #2



↳ Combined system :



$h[n]$: impulse response of the combined system:

In frequency = applying DTFT on

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$\hat{X}(e^{j\omega}) \longrightarrow \boxed{\hat{H}(e^{j\omega})} \longrightarrow \hat{Y}(e^{j\omega})$$

where $\boxed{\hat{Y} = \hat{X} \cdot \hat{H}}$ ①

Find $h_2[n]$:

$$y = y_1 + y_2 = x * h_1 + x * h_2 = x * [h_1 + h_2]$$

$\hookrightarrow$ $y = x * [h_1 + h_2]$ Now applying DTFT on both sides:

$$\boxed{\hat{Y}(e^{j\omega}) = \hat{X}(e^{j\omega}) \cdot \left[\hat{H}_1(e^{j\omega}) + \hat{H}_2(e^{j\omega})\right]}$$ ② (convolution property of DTFT)

Comparing ① with ②: $\underset{\text{given}}{\hat{H}} = \underset{\text{DTFT}[h_1[n]]}{\hat{H}_1} + \hat{H}_2$

i) $\hat{H}_2 = \hat{H} - \mathrm{DTFT}[h_1[n]]$

ii) Then $h_2[n] = \mathrm{DTFT}^{-1}[\hat{H}_2]$ (inverse)

---

i) $\hat{H}_2 = \underset{\text{given}}{\hat{H}} - \underbrace{\mathrm{DTFT}[h_1[n]]}_{\text{Need to find}}$

$$h_1[n] = \left(\frac{1}{3}\right)^n u[n] \xrightarrow[\text{Table 5-2 row \#8}]{\text{DTFT}} \frac{1}{1 - \frac{1}{3} e^{-j\omega}} = \frac{3}{3 - e^{-j\omega}}$$

$$a^n u[n] \xrightarrow{\text{DTFT}} \frac{1}{1 - a e^{-j\omega}}$$

$|a| < 1$

$$\hat{H}_2 = \frac{-12+5e^{-j\omega}}{12-7e^{-j\omega}+e^{-j2\omega}} - \frac{3}{3-e^{-j\omega}}$$

Complex algebra to simplify $\hat{H}_2$:

$$\hat{H}_2 = \frac{(-12+5e^{-j\omega})(3-e^{-j\omega}) - 3(12-7e^{-j\omega}+e^{-j2\omega})}{(12-7e^{-j\omega}+e^{-j2\omega})(3-e^{-j\omega})}$$

$$= \frac{-72+48e^{-j\omega}-8e^{-j2\omega}}{36-33e^{-j\omega}+10e^{-j2\omega}-e^{-j3\omega}}$$

$$= -2\frac{36-24e^{-j\omega}+4e^{-j2\omega}}{36-33e^{-j\omega}+10e^{-j2\omega}-e^{-j3\omega}}$$

Division

$$\begin{array}{rrrr|l} 36 & -33e^{-j\omega} & +10e^{-j2\omega} & -e^{-j3\omega} & 36-24e^{-j\omega}+4e^{-j2\omega} \\ + & 4e^{-j\omega} & -6e^{-j2\omega} & +e^{-j3\omega} & \frac{1}{4}e^{-j\omega}\boxed{-1} \\ \hline 36 & -24e^{-j\omega} & 4e^{-j2\omega} & +0 & \end{array}$$

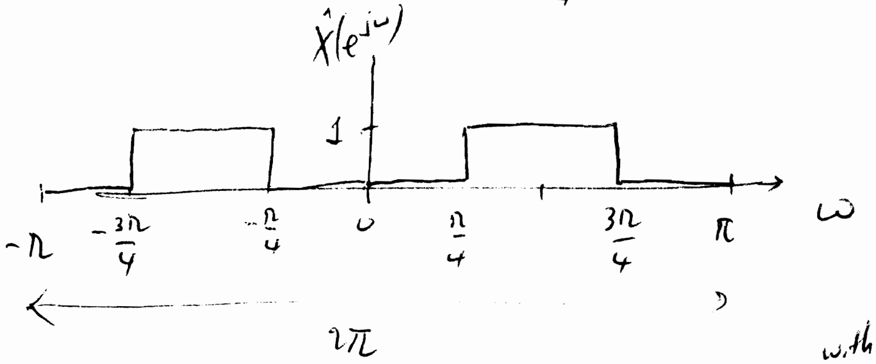
$$\to 36-33e^{-j\omega}+10e^{-j2\omega}-e^{-j3\omega} = (36-24e^{-j\omega}+4e^{-j2\omega})\left(\tfrac{1}{4}e^{-j\omega}-1\right)$$

$$\to \hat{H}_2 = -2\frac{1}{\frac{1}{4}e^{-j\omega}-1} = 2\frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$\to \text{DTFT}^{-1} \longrightarrow +2\left(\tfrac{1}{4}\right)^n u[n]$$

again using Table 5.2 row #8

5-22 a) $\hat{X}(e^{j\omega}) = \begin{cases} 1 & \frac{\pi}{4} \le |\omega| \le \frac{3\pi}{4} \\ 0 & \frac{3\pi}{4} \le |\omega| \le \pi \;;\; 0 < |\omega| < \frac{\pi}{4} \end{cases}$



$\hat{X}(e^{j\omega})$ is periodic in frequency (as DTFT with any) $\rightarrow$ Find $x[n]$

$$x[n] = DTFT^{-1}\left[\hat{X}(e^{j\omega})\right] = \frac{1}{2\pi}\int_{2\pi} d\omega\, \hat{X}(e^{j\omega})\, e^{j\omega n}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega\, \hat{X}(e^{j\omega})\, e^{j\omega n} = \frac{1}{2\pi}\int_{-\frac{3\pi}{4}}^{-\frac{\pi}{4}} d\omega\, 1\, e^{j\omega n} + \frac{1}{2\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\omega\, 1\, e^{j\omega n}$$

$$= \frac{1}{2\pi}\,\frac{e^{-j\frac{\pi}{4}n} - e^{-j\frac{3\pi}{4}n}}{jn} + \frac{1}{2\pi}\,\frac{e^{j\frac{3\pi}{4}n} - e^{j\frac{\pi}{4}n}}{jn}$$

$$= \frac{e^{-jn\frac{2\pi}{4}}}{2\pi jn}\underbrace{\left(e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}\right)}_{2j\sin\frac{\pi}{4}n} + \frac{e^{jn\frac{2\pi}{4}}}{2\pi jn}\underbrace{\left(e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}\right)}_{2j\sin\frac{\pi}{4}n}$$

$$\boxed{x[n] = \frac{2}{\pi n}\cos\frac{2\pi}{4}n\,\sin\frac{\pi}{4}n}$$

5.19

$$x[n] \longrightarrow \boxed{\text{Linear System}} \longrightarrow y[n]$$

input/output difference equation given as

$$y[n] - \frac{1}{6}y[n-1] - \frac{1}{6}y[n-2] = x[n]$$

a) Use DTFT to find $\hat{H}(e^{j\omega}) \equiv \dfrac{\hat{Y}(e^{j\omega})}{\hat{X}(e^{j\omega})}$

DTFT $\rightarrow$ $\hat{Y}(e^{j\omega}) - \frac{1}{6}e^{-j\omega}\hat{Y}(e^{j\omega}) - \frac{1}{6}e^{-2j\omega}\hat{Y}(e^{j\omega}) = \hat{X}(e^{j\omega})$

$$\hookrightarrow \hat{H}(e^{j\omega}) = \frac{\hat{Y}(e^{j\omega})}{\hat{X}(e^{j\omega})} = \frac{1}{1 - \frac{1}{6}e^{-j\omega} - \frac{1}{6}e^{-2j\omega}}$$

b) Find $h[n]$ using $DTFT^{-1}$.

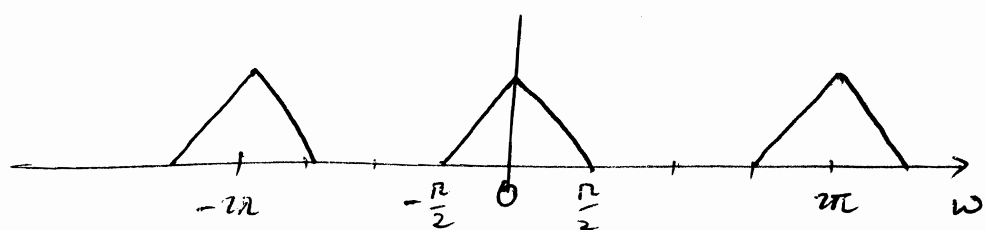
$$\hat{H}(e^{j\omega}) = \frac{1}{(1-\frac{1}{2}e^{-j\omega})(1+\frac{1}{3}e^{-j\omega})} = \frac{\frac{3}{5}}{1-\frac{1}{2}e^{-j\omega}} + \frac{\frac{2}{5}}{1+\frac{1}{3}e^{-j\omega}}$$

Partial Fraction Expansion

$DTFT^{-1}$

$$h[n] = \frac{3}{5}\left(\frac{1}{2}\right)^n u[n] + \frac{2}{5}\left(-\frac{1}{3}\right)^n u[n].$$

5-27 a) Given $x[n]$ whose DTFT is $\hat{X}(e^{j\omega})$:



(i) Sketch $\hat{W}(e^{j\omega})$ where $w[n] = x[n]p[n]$ being

$$p[n] = \cos \pi n$$

Strategy: to obtain $\hat{W}(e^{j\omega})$: 1) Find $\hat{P}(e^{j\omega})$ (DTFT of $p[n]$)

$$\hat{P}(e^{j\omega}) = \pi[\delta(\omega-\pi)+\delta(\omega+\pi)]$$

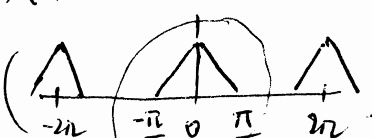
(used Table 5.2: $\cos\omega_0 n \xrightarrow{DTFT} \pi\sum_{\ell}[\delta(\omega-\omega_0-2\pi\ell)+\delta(\omega+\omega_0-2\pi\ell)]$

& in our case DTFT is periodic → one interval is needed. $(-\pi,\pi) \rightarrow \ell=0$)

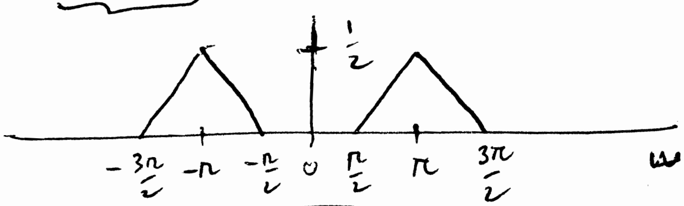
2) Use multiplication property: $x[n]y[n] \xrightarrow{DTFT} \underbrace{\frac{1}{2\pi}\int_{2\pi} d\theta\, \hat{X}(e^{j\theta})\hat{Y}(e^{j(\omega-\theta)})}_{\frac{1}{2\pi}\hat{X} * \hat{Y}}$

$\downarrow$ convolution.

$$\hat{W} = \frac{1}{2\pi}\hat{X} * \hat{P}$$

$= \frac{1}{2\pi}\Big($  $\Big) * \pi[\delta(\omega-\pi)+\delta(\omega+\pi)]$

=



Sketch of DTFT of $x[n] \cdot p[n]$

> Convolution with a delta function:
>
> $$Z(\omega) * \delta(\omega-\omega_0) = Z(\omega-\omega_0)$$

iv) Sketch $\hat{W}$ when $w[n] = x[n]p[n]$ being $p[n] = \sum_{k=-\infty}^{\infty} \delta[n-2k]$

Similar strategy: 1) Find $\hat{P}(e^{j\omega})$

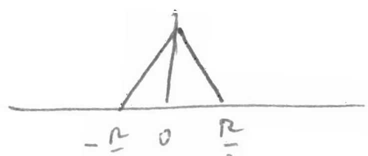
Table 5.2:

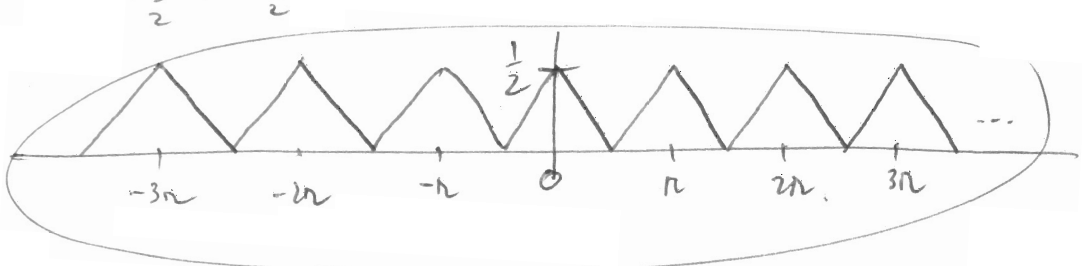
$$\sum_{k=-\infty}^{\infty} \delta[n-kN] \xrightarrow{DTFT} \frac{2\pi}{N}\sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$$

$N=2$ $\Rightarrow$ $p[n] \xrightarrow{DTFT} \pi \sum_{k=-\infty}^{\infty} \delta(\omega - \pi k) = \hat{P}(e^{j\omega})$

2) Use multiplication property for DTFT:

$$w[n] = x[n]p[n] \xrightarrow{DTFT} \hat{W} = \frac{1}{2\pi}\hat{X} * \hat{P}$$

$\hat{W} = \frac{1}{2\cancel{\pi}}$  $* \ \cancel{\pi} \sum_{k=-\infty}^{\infty} \delta(\omega - \pi k)$

= 

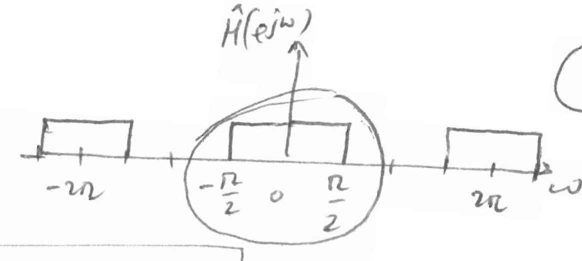
b)

$w[n] \longrightarrow$ $\boxed{\text{LTI system } h[n]}$ $\longrightarrow y[n]$

$$h[n] = \frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n}$$

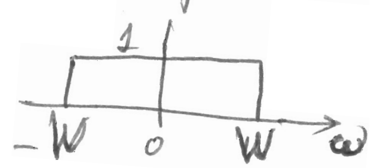
Determine output $y[n]$ when $w[n] = x[n]p[n]$

iv) Being $p[n] = \sum_{k=-\infty}^{\infty} \delta[n-2k]$ (in a) we found a sketch for $\hat{W}$: strategy: 1) $\hat{Y} = \hat{W} \cdot \hat{H}$ 2) $y[n] = DTFT^{-1}[\hat{Y}]$

$\hat{H} = \text{DTFT}\left\{\frac{\sin\frac{\pi}{2}n}{\pi n}\right\} \Rightarrow \hat{H} =$ 

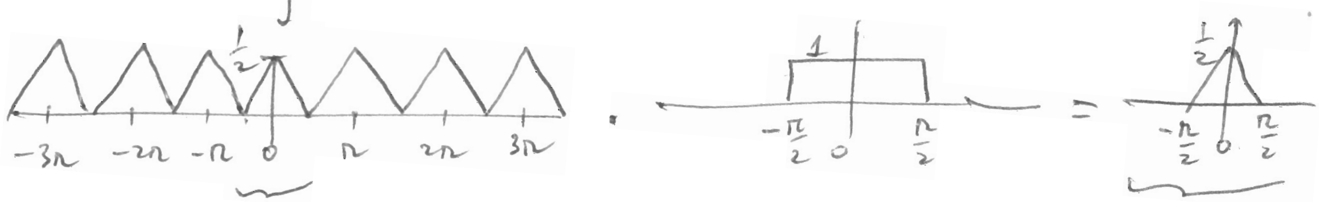
$W = \frac{\pi}{2}$

Table 5-2: $\boxed{\frac{\sin Wn}{\pi n} = \frac{W}{\pi}\,\text{sinc}\left(\frac{Wn}{\pi}\right)} \xrightarrow{\text{DTFT}}$ Rectangular pulse of halfwidth W

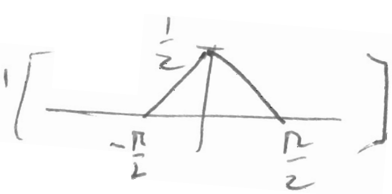


$$\underbrace{\begin{cases} 1 & 0 \le |\omega| \le W \\ 0 & W < |\omega| \le \pi \end{cases}}_{\text{Periodic of period } 2\pi}$$

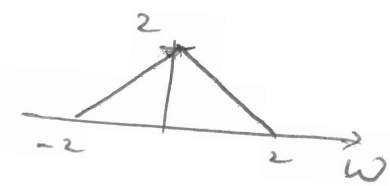
2) $\hat{Y} = \hat{W} \cdot \hat{H}$



$\hat{Y}$

$\rightarrow y[n] = \text{DTFT}^{-1}\Big[$  $\Big]$

Recall: 4.10:

$\left(\frac{\sin t}{\pi t}\right)^2 \xrightarrow{FT} \frac{1}{2\pi}$ 

$$= \frac{\pi}{2}\left(\frac{\sin\frac{\pi}{4}n}{\pi n}\right)^2$$

Ch 6: Time & Frequency Characterization of Signals & Systems

HW 6: 6.5; 6.18; 6.19; 6.23; 6.27; 6.30; 6.31; 6.33;
6.40; 6.51

6.5 System is a continuous-time band-pass filter

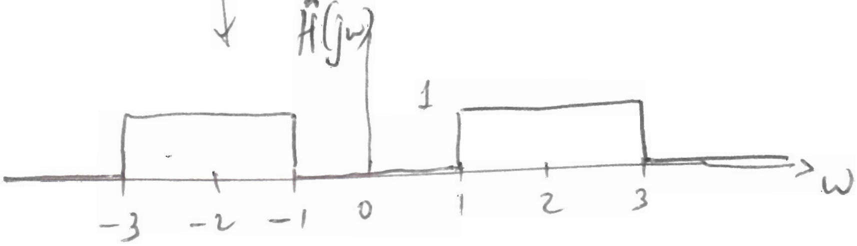
$$x(t) \to \boxed{h(t)} \to y(t)$$

$$\hat{H}(j\omega) = \begin{cases} 1 & \omega_c \le |\omega| \le 3\omega_c \\ 0 & \text{outside} \end{cases}$$

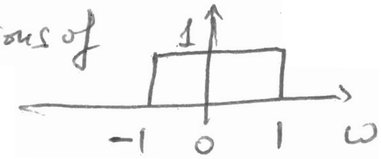
$\omega_c \to$ cut-off frequency

a) Determine $g(t)$ such that $h(t) = g(t)\,\dfrac{\sin \omega_c t}{\pi t}$

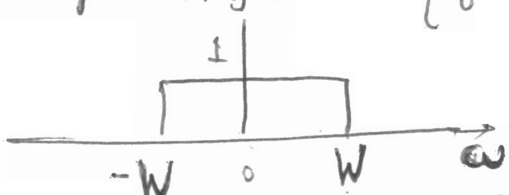
$$\to g(t) = \frac{h(t)}{\dfrac{\sin \omega_c t}{\pi t}} = \frac{F^{-1}[\hat{H}(j\omega)]}{\dfrac{\sin \omega_c t}{\pi t}}$$



Observation: this has 2 translations of



Let's define $\hat{X}(j\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| > W \end{cases}$ $\overset{F^{-1}}{\longleftrightarrow}$ $\dfrac{\sin Wt}{\pi t}$



In our problem $W = 1$ & $\hat{H}(j\omega) = \hat{X}(j(\omega - 2\omega_c)) + \hat{X}(j(\omega + 2\omega_c))$

$\omega_c = W = 1$

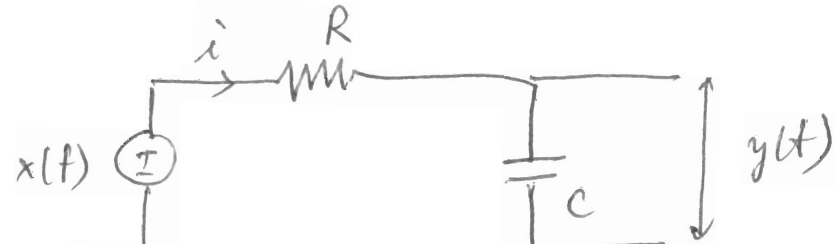
$$h(t) = F^{-1}\left[\hat{H}(j\omega)\right] = e^{j2\omega_c t}\,\frac{\sin \omega_c t}{\pi t} + e^{-j2\omega_c t}\,\frac{\sin \omega_c t}{\pi t}$$

$$= 2\cos 2\omega_c t \;\frac{\sin \omega_c t}{\pi t}$$

$$g(t) = \frac{h(t)}{\dfrac{\sin \omega_c t}{\pi t}} = 2\cos 2\omega_c t$$

b) if $\omega_c \uparrow$ is $h(t)$ more or less concentrated around the origin? (Plot $h(t) = \dfrac{2\cos 2\omega_c t \sin \omega_c t}{\pi t}$ for increasing values of ω_c, observe results).

6.18:



(Can a step response of the RC circuit be oscillatory?
(if we turn on $x(t)$, can $y(t)$ be oscillatory?)

$u(t) \longrightarrow$ $\boxed{\text{an RC circuit}}$ $\longrightarrow y(t)$ oscillatory?

Let's look at $y(t) = F^{-1}[\hat{Y} = \hat{X}\cdot\hat{H}]$

$\hat{H}$ for an RC circuit: $x(t) = Ri + \underbrace{\frac{1}{C}\int i\,dt}_{y(t)}$ $\rightarrow i(t) = C\dfrac{dy}{dt}$

$$\boxed{x(t) = RC\frac{dy}{dt} + y(t)}$$

F $\hat{X} = RCj\omega\hat{Y} + \hat{Y}$

$$\rightarrow \hat{H} = \frac{\hat{Y}}{\hat{X}} = \frac{1}{RCj\omega + 1}$$

$$y(t) = F^{-1}[\hat{H} \cdot \hat{X}] = \boxed{F^{-1}\left[\frac{1}{RCj\omega+1} \cdot \left(\frac{1}{j\omega} + \pi\delta(\omega)\right)\right]}$$

$x(t) = u(t) \xrightarrow[F]{} \frac{1}{j\omega} + \pi\delta(\omega)$

→ No oscillation, only exponential behavior in time.

Need $\delta(\omega \pm \omega_0)$ where $\omega_0 \neq 0$.