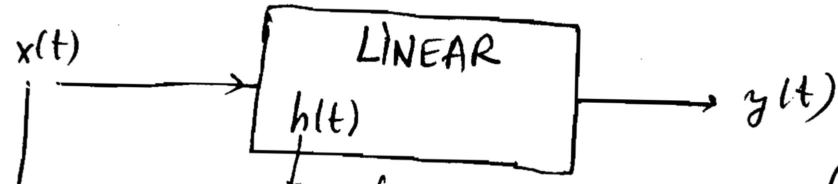
Fourier Series & Linear Systems :



impulse response (identifies the system)

What can we say about b_k's ??

periodic signal identified by a_k's & w_0 (by its F.S. representation)

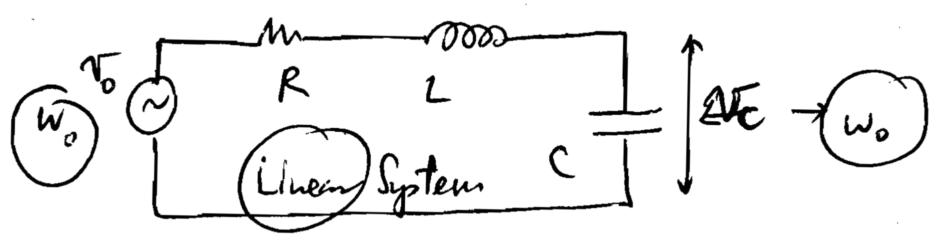
If we know the F.S. coefficients for $x(t)$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}$$

what are the F.S. coefficients for the output $y(t)$?

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jkw_0 t}$$

Linear System



1) Instead of inputing $x(t)$, I can input $a_k e^{jkw_0 t}$ $\forall k$ then sum over all k (there is a 1-1 correspondence b/w a periodic signal and its Fourier Series)

$$e^{jk\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow e^{jk\omega_0 t} * h(t)$$

$$e^{jk\omega_0 t} * h(t) = \int_{-\infty}^{\infty} d\tau \; h(\tau)\, e^{jk\omega_0 (t-\tau)}$$ (by definition of convolution)

$$= e^{jk\omega_0 t} \underbrace{\int_{-\infty}^{\infty} d\tau \; h(\tau)\, e^{-jk\omega_0 \tau}}_{\substack{\text{Fourier Transform of } h(t) \\ \hat{H}(jk\omega_0)}}$$

So

$$a_k e^{jk\omega_0 t} \longrightarrow \boxed{\begin{array}{c} h(t) \\ \text{Linear} \end{array}} \longrightarrow a_k e^{jk\omega_0 t} \hat{H}(jk\omega_0)$$

Sum over k:

$$\underset{x(t)}{\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}} \longrightarrow \boxed{\begin{array}{c} h(t) \\ \text{Linear} \end{array}} \longrightarrow \sum_{k=-\infty}^{\infty} a_k \hat{H}(jk\omega_0) e^{jk\omega_0 t}$$

$$= y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$$

$$\Rightarrow \boxed{b_k = a_k \hat{H}(jk\omega_0)}$$

$\Rightarrow$ <u>Conclusion</u>: for a linear system, the F.S. coeff. of the output are those of the input multiplied by the Fourier Transform of the system's impulse response. (for continuous-time signals)

For discrete-time signals:

$$x[n] \longrightarrow \boxed{\begin{array}{c} h[n] \\ \text{linear system} \end{array}} \longrightarrow y[n]$$

$x[n] \downarrow$ periodic

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\frac{2\pi}{N}n}$$

N period of $x[n]$

$$y[n] = \underbrace{\sum_{k=\langle N \rangle} b_k e^{jk\frac{2\pi}{N}n}}_{\text{same period as } x[n] \text{ b/c of the linear system.}}$$

I can apply $x[n]$ or $a_k e^{jk\frac{2\pi}{N}n}$ $\forall k$ and sum over k (N terms)

$$e^{jk\frac{2\pi}{N}n} \longrightarrow \boxed{h[n]} \longrightarrow e^{jk\frac{2\pi}{N}n} * h[n]$$

$$= \sum_{m=-\infty}^{\infty} h[m]\, e^{jk\frac{2\pi}{N}(n-m)}$$

$$= e^{jk\frac{2\pi}{N}n} \underbrace{\sum_{m=-\infty}^{\infty} h[m] e^{-jk\frac{2\pi}{N}m}}_{\text{Discrete-time Fourier Transform of } h[n]}$$

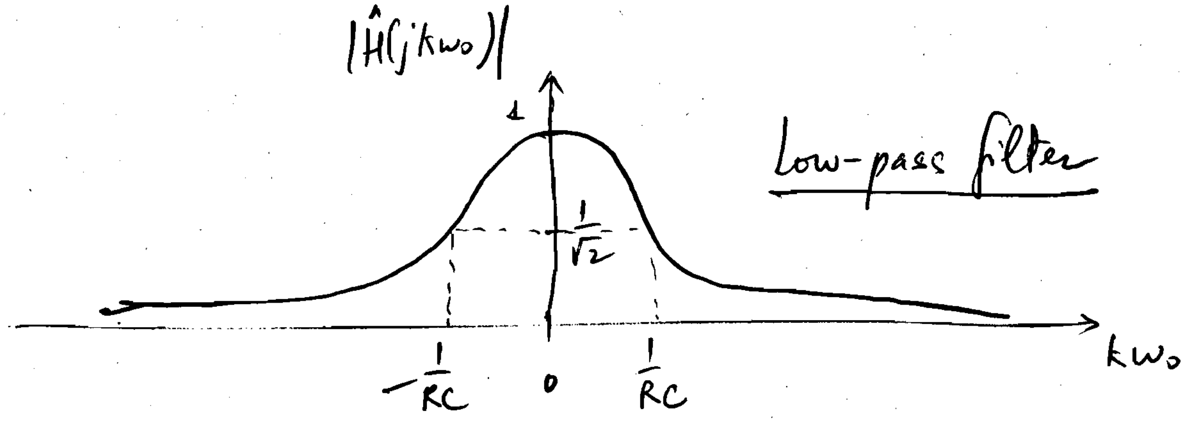
$\rightarrow \hat{H}(e^{jw})$; $w = \frac{2\pi}{N}$

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\frac{2\pi}{N}n} \longrightarrow \boxed{\begin{array}{c} h[n] \\ \text{linear} \end{array}} \longrightarrow \sum_{k=\langle N \rangle} a_k \hat{H}(e^{jw}) e^{jk\frac{2\pi}{N}n}$$

$$= y[n] = \sum_{k=\langle N \rangle} b_k e^{jk\frac{2\pi}{N}n}$$

$$\longrightarrow \boxed{b_k = a_k \hat{H}(e^{jw})}$$

$$= \left. \frac{e^{-(jk\omega_0 + \frac{1}{RC})t}}{-(1+jk\omega_0 RC)} \right]_0^\infty = \frac{0-1}{-(1+jk\omega_0 RC)} = \frac{1}{jk\omega_0 RC+1}$$



<u>Low-pass filter</u>

<u>Conclusion:</u> in general a system governed by $\frac{dy}{dt} + ay(t) = bx(t)$ is a low-pass filter.

Discrete-time: a discrete-time low-pass filter would be represented by the discrete-time version of $\frac{dy}{dt} + ay = bx$:

Use finite-difference approximation for time derivative:

$$\frac{dy}{dt} = \lim_{\Delta \to 0} \frac{y(t) - y(t-\Delta)}{\Delta} \simeq \frac{y(t) - y(t-\Delta)}{\Delta}$$

$$\text{or} \quad \frac{y[n] - y[n-1]}{\Delta}$$

low-pass filter:

$$\frac{dy}{dt} + ay = bx \longrightarrow \frac{y[n] - y[n-1]}{\Delta} + ay[n] = bx[n]$$

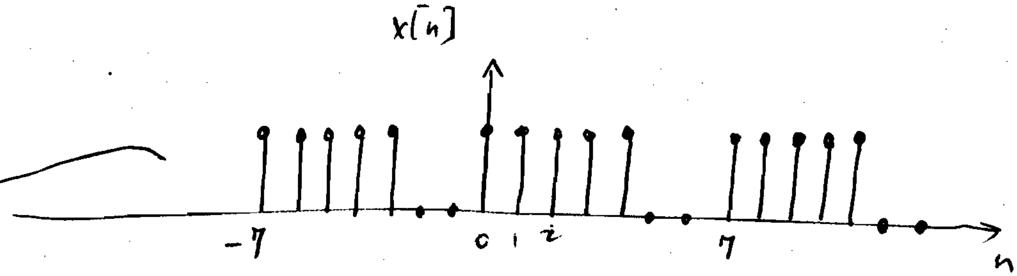
$$\rightarrow \left(\frac{1}{\Delta} + a\right) y[n] - \frac{1}{\Delta} y[n-1] = bx[n]$$

$$\text{or} \quad \underbrace{\frac{1}{b}\left(\frac{1}{\Delta} + a\right)}_{\beta} y[n] - \underbrace{\frac{1}{b\Delta}}_{\alpha} y[n-1] = x[n]$$

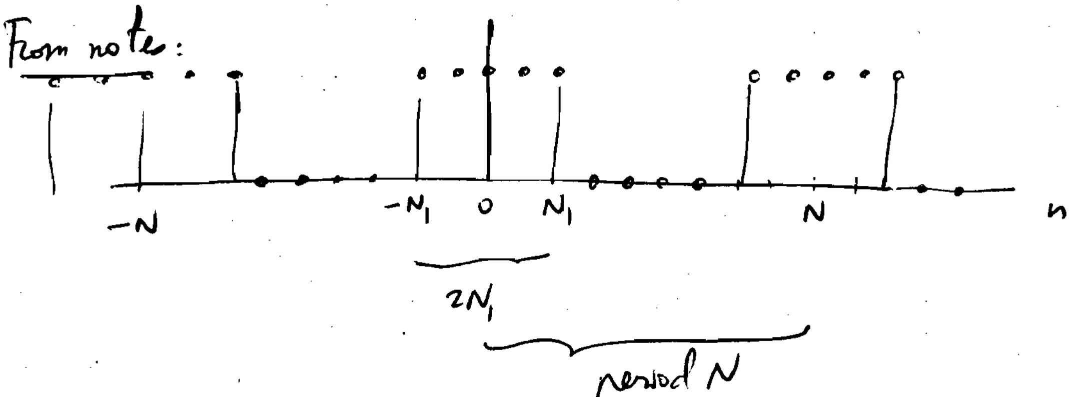
Discrete-time: $\boxed{\beta y[n] + \alpha y[n-1] = x[n]}$ This represents a discrete-time low-pass filter.

3.28 Find a_k

a)



From note:



$$a_k = \begin{cases} \dfrac{1}{N} \dfrac{\sin \frac{k2\pi}{N}\left(N_1 + \frac{1}{2}\right)}{\sin \frac{k2\pi}{N}} & k \neq 0 \text{ or } \neq \text{ multiple of } 2N \\ \dfrac{1}{N}(2N_1 + 1) & (k = 0 \text{ or } = \text{ multiple of } 2N) \end{cases}$$

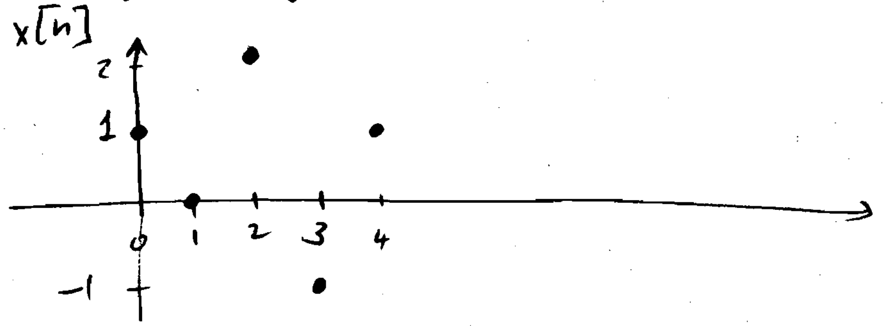
There is $\underbrace{\text{time shift of } 2}_{\text{Table 3.2 } e^{-jk\frac{2\pi}{N}2}}$ and $N_1 = 2$; $N = 7$

$$a_k = \begin{cases} \dfrac{1}{7} \dfrac{\sin \frac{k2\pi}{7}\left(2 + \frac{1}{2}\right)}{\sin\left(\frac{k2\pi}{7}\right)} e^{-jk\frac{4\pi}{N}} & k \neq 0 \text{ or } \neq \text{ multiple of } 14 \\ \dfrac{1}{7}(4+1) = \dfrac{5}{7} e^{-jk\frac{4\pi}{7}} \text{ if } k = 0 \text{ or } = \text{ multiple of } 14. \end{cases}$$

$$\rightarrow a_k = \begin{cases} \dfrac{1}{7} \dfrac{\sin\left(\frac{k5\pi}{7}\right)}{\sin\left(k\frac{2\pi}{7}\right)} e^{-jk\frac{4\pi}{N}} \\ \dfrac{5}{7} e^{-j\frac{k4\pi}{7}} \end{cases}$$

3.32 $$x[n] = \sum_{k=0}^{3} a_k e^{jk\left(\frac{2\pi}{4}\right)n}$$

a) Find a_k by solving system of 4 equations with 4 unknowns



(Fig. P3.32)

		$k=0$	$k=1$	$k=2$	$k=3$

$n=0$ $\quad x[0] = 1 = a_0 + a_1 + a_2 + a_3$

$n=1$ $\quad x[1] = 0 = a_0 + a_1 e^{j\frac{\pi}{2}} + a_2 e^{j\pi} + a_3 e^{j\frac{3\pi}{2}} = a_0 + ja_1 - a_2 - ja_3$

$n=2$ $\quad x[2] = 2 = a_0 + a_1 e^{j\pi} + a_2 e^{j2\pi} + a_3 e^{j3\pi} = a_0 - a_1 + a_2 - a_3$

$n=3$ $\quad x[3] = -1 = a_0 + a_1 e^{j\frac{3\pi}{2}} + a_2 e^{j3\pi} + a_3 e^{j\frac{9\pi}{2}} = a_0 - ja_1 - a_2 + ja_3$

$$e^{j\alpha} = \cos\alpha + j\sin\alpha \rightarrow e^{j\frac{\pi}{2}} = j;\ e^{j\pi} = -1;\ e^{j\frac{3\pi}{2}} = -j;\ e^{j2\pi} = 1;\ e^{j3\pi} = -1;$$
$$e^{j\frac{9\pi}{2}} = j$$

<u>In summary:</u>

1) $1 = a_0 + a_1 + a_2 + a_3$

2) $0 = a_0 + ja_1 - a_2 - ja_3$

3) $2 = a_0 - a_1 + a_2 - a_3$

4) $-1 = a_0 - ja_1 - a_2 + ja_3$

F.S coefficients a_0, a_1, a_2, a_3 are the solutions to this system

2) + 4) $\rightarrow$ $-1 = 2a_0 - 2a_2$

1) + 3) $\rightarrow$ $3 = 2a_0 + 2a_2$

$$2 = 4a_0 \rightarrow \boxed{a_0 = \frac{1}{2}} \rightarrow a_2 = \frac{3 - 2a_0}{2}$$

$$\rightarrow a_2 = \frac{3-1}{2} \Rightarrow \boxed{a_2 = 1}$$

2) $\quad 0 = \frac{1}{2} + ja_1 - 1 - ja_3 \quad$ or $\quad \frac{1}{2} = ja_1 - ja_3$

4) $\quad -1 = \frac{1}{2} - ja_1 - 1 + ja_3 \quad$ or $\quad -\frac{1}{2} = -ja_1 + ja_3$

1) $1 = \frac{1}{2} + a_1 + 1 + a_3$ or $-\frac{1}{2} = a_1 + a_3$

3) $2 = \frac{1}{2} - a_1 + 1 - a_3$ or $\frac{1}{2} = -a_1 - a_3$

2) or 4) $\rightarrow$ $\frac{1}{2}j = -a_1 + a_3$ $\quad (j^2 = -1)$

1) or 3) $\frac{1}{2} = -a_1 - a_3$

$\frac{1}{2}(1+j) = -2a_1 \rightarrow \boxed{a_1 = -\frac{1}{4}(1+j)}$

$a_3 = a_1 + \frac{1}{2}j \rightarrow \boxed{a_3 = -\frac{1}{4}(1-j)}$

5) Find the same F.S. coefficients a_0, a_1, a_2, a_3 using the formula for F.S coefficients:

$$a_k = \frac{1}{4}\sum_{n=0}^{3} x[n]\, e^{-jk\left(\frac{2\pi}{4}\right)n}$$

$$k=0: \quad a_0 = \frac{1}{4}\left\{ \overset{n=0}{1} + \overset{n=1}{0} + \overset{n=2}{2} \overset{n=3}{-1} \right\} = \frac{1}{2} \checkmark$$

$$k=1: \quad a_1 = \frac{1}{4}\left\{ 1 + 0 + 2\underbrace{e^{-j\pi}}_{-1} - 1\underbrace{e^{-j\frac{3\pi}{2}}}_{j} \right\} = \frac{-1-j}{4} \checkmark$$

$$k=2 \quad a_2 = \frac{1}{4}\left\{ 1 + 0 + 2\underbrace{e^{-j2\pi}}_{1} - 1\underbrace{e^{-j3\pi}}_{-1} \right\} = 1 \checkmark$$

$$k=3 \quad a_3 = \frac{1}{4}\left\{ 1 + 0 + 2\underbrace{e^{-j3\pi}}_{-1} - 1\underbrace{e^{-j\frac{9\pi}{2}}}_{-j} \right\} = \frac{-1+j}{4} \checkmark$$

$$e^{-j3\pi} = \left(e^{+j3\pi}\right)^{-1} = \left(e^{j(2\pi+\pi)}\right)^{-1} = \left(e^{j2\pi}\cdot e^{j\pi}\right)^{-1} = \left[(\overset{1}{\cos 2\pi} + j\overset{0}{\sin 2\pi})\cdot(\underset{-1}{\cos\pi} + j\underset{0}{\sin\pi})\right]^{-1}$$

$$= (-1)^{-1} = -1$$

$$(e^a)^b = e^{ab}$$

$$e^{j\frac{3\pi}{2}} = \underbrace{\cos\frac{3\pi}{2}}_{0} + j\underbrace{\sin\frac{3\pi}{2}}_{1} = j$$

$$e^{-j\frac{3\pi}{2}} = \left(e^{j\frac{3\pi}{2}}\right)^{-1} = (j)^{-1} = -j$$

In conclusion
$$\begin{cases} e^{-j3\pi} = e^{j3\pi} = -1 \\ e^{-j\frac{3\pi}{2}} = -e^{j\frac{3\pi}{2}} = j \end{cases}$$

3.46

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad ; \quad y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$$

(same period $T_0 = \frac{2\pi}{\omega_0}$)

a) $$z(t) = x(t)\,y(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

<u>Show that</u> $$c_k = \sum_{n=-\infty}^{\infty} a_n b_{k-n}$$ (= convolution of a_k's with b_k's)

$$x(t)\,y(t) = \underbrace{\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}_{\text{sum}} \underbrace{\sum_{l=-\infty}^{\infty} b_l e^{jl\omega_0 t}}_{\text{sum}}$$

(use different dummy indices for not missing cross terms in the product)

$$= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_k b_l \underbrace{e^{j(k+l)\omega_0 t}}_{\cancel{\text{use orthogonality of complex exponentials}}}$$

$\boxed{\text{Rename} : k \to k-l}$

$$= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_{k-l} b_l e^{j(k-l+l)\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} \left(\sum_{l=-\infty}^{\infty} a_{k-l} b_l \right) e^{jk\omega_0 t}$$

compare this with the series for $z(t) \rightarrow \boxed{c_k = \sum_{l=-\infty}^{\infty} a_{k-l} b_l}$

c_k's conv of b_k & a_k

a_k's: $\underbrace{\text{F.S. coefficients}}$ of $\cos 20\pi t$

$\hookrightarrow x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$; $a_k = \frac{1}{T}\int_0^T dt\, e^{-jk\omega_0 t} x(t)$

use this formula or:

$$\cos 20\pi t = \frac{e^{j20\pi t} + e^{-j20\pi t}}{2} \quad \left(\cos\alpha = \frac{e^{j\alpha} + e^{-j\alpha}}{2} \right)$$

What is ω_0? $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{3}$ (since T_0 for the pulse was 3 and the problem said both x & y have the same fundamental period)

$$e^{jk\omega_0 t} = e^{jk\frac{2\pi}{3}t}$$

$$\boxed{\cos 20\pi t = \frac{1}{2}\delta(k-30)e^{jk\frac{2\pi}{3}t} + \frac{1}{2}\delta(k+30)e^{+jk\frac{2\pi}{3}t}}$$

$e^{j\alpha} = \cos\alpha + j\sin\alpha$

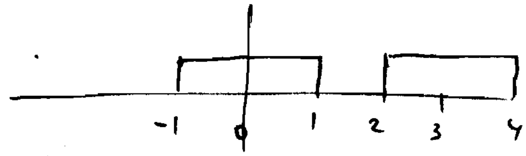
$e^{-j\alpha} = \cos\alpha - j\sin\alpha$

$e^{j\alpha} + e^{-j\alpha} = 2\cos\alpha \rightarrow \cos\alpha = \frac{1}{2}(e^{j\alpha} + e^{-j\alpha})$

$$\cos 20\pi t = \left[\frac{1}{2}\delta(k-30) + \frac{1}{2}\delta(k+30)\right] e^{jk\frac{2\pi}{3}t}$$

F.S. expansion for $\cos 20\pi t$ has only one term with $\boxed{a_k = \frac{1}{2}\delta(k-30) + \frac{1}{2}\delta(k+30)}$

b_k's : F.S. coefficients of



$T_0 = 3$ (period)

$T_1 = 1$ (half width)

↳ Note pg 47: $\quad b_k = \dfrac{2 \sin k\omega_0 T_1}{k\omega_0 T_0} = \dfrac{2 \sin k\frac{2\pi}{3}1}{k\frac{2\pi}{3}3}$

$$b_k = \frac{2 \sin k\frac{2\pi}{3}}{k\,2\pi}$$

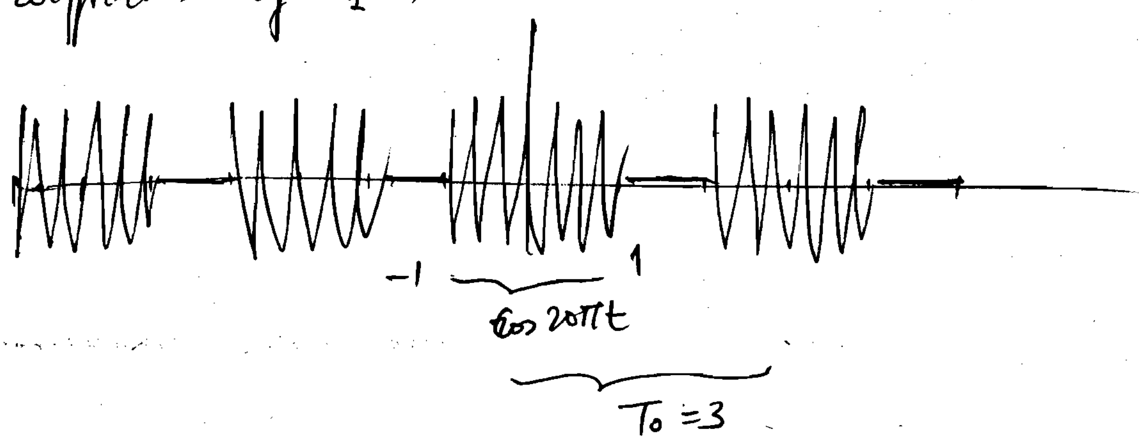
$$c_k = a_k * b_k = \left[\frac{1}{2}\delta(k-30) + \frac{1}{2}\delta(k+30)\right] * \frac{2\sin k\frac{2\pi}{3}}{k\,2\pi}$$

Remember a property of the convolution (notes pg 26)

$$f[n] * \delta[n-j] = f[n-j]$$

$$\boxed{c_k = \frac{\sin (k-30)\frac{2\pi}{3}}{(k-30)\,2\pi} + \frac{\sin (k+30)\frac{2\pi}{3}}{(k+30)\,2\pi}}$$

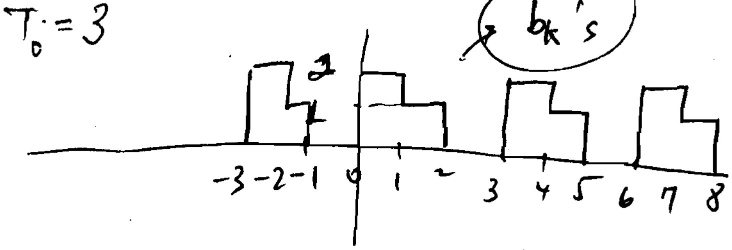
F.S. coefficients of $x_1(t)$



Now do it for $x_2(t)$; and $x_3(t)$

$x_2(t) = \cos 20\pi t$.

$T_0 = 3$



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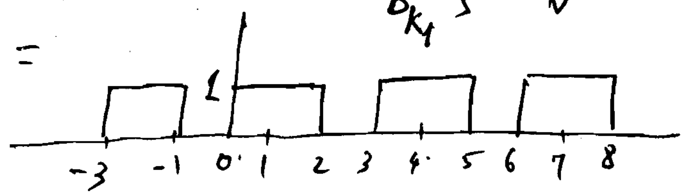
$T_0 = 3$



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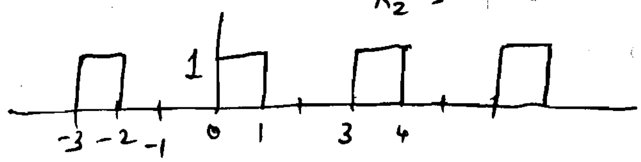
$T_0 = 3$



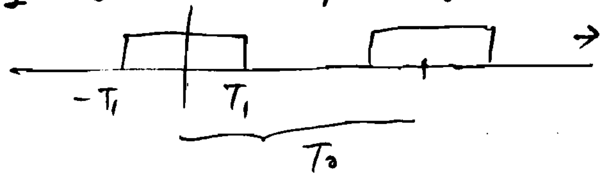
$$w_0 = \frac{2\pi}{T_0} = \frac{2\pi}{3}$$

$$\cos(20\pi t) = \underbrace{\left[\frac{1}{2}\delta(k-30) + \frac{1}{2}\delta(k+30)\right]}_{a_k} e^{jk\frac{2\pi}{3}t}$$

b_k's: If b_{k_1}'s & b_{k_2}'s are known $\Rightarrow b_k = b_{k_1} + b_{k_2}$

$b_{k_1} = \begin{cases} T_0 = 3 \\ T_1 = 1 \\ \text{shift of 1 in } t \end{cases}$ $\longrightarrow$ FS coeff multiplied by $e^{-jkw_0 t_0}$ $\rightarrow$ time shift (Notes pg 45)

Notes pg 47:



$\rightarrow b_k = \dfrac{2\sin k w_0 T_1}{k w_0 T_0}$

Centered at 0

$$b_{k_1} = \frac{2\sin k\frac{2\pi}{3}1}{k\frac{2\pi}{3}3} e^{-jk\frac{2\pi}{3}} = \frac{2\sin k\frac{2\pi}{3}}{k2\pi} e^{-jk\frac{2\pi}{3}}$$

due to a time-shift of 1

$$b_{k_2}\begin{cases} T_0 = 3 \\ T_1 = \frac{1}{2} \\ \text{shift } t_0 = \frac{1}{2} \end{cases} \rightarrow \text{F.S. coeff multiplied by } e^{-jk w_0 \frac{1}{2}}$$

$$b_{k_2} = \frac{2 \sin k \frac{2\pi}{3} \frac{1}{2}}{k \frac{2\pi}{3} 3} e^{-jk\frac{2\pi}{3}\cdot\frac{1}{2}}$$

$$b_{k_2} = \frac{2 \sin (k \frac{\pi}{3})}{k\, 2\pi} e^{-jk\frac{\pi}{3}}$$

$$\rightarrow c_k = a_k * b_k = a_k * (b_{k_1} + b_{k_2})$$

$$= \left[\frac{1}{2}\delta(k-30) + \frac{1}{2}\delta(k+30)\right] * \left\{ \frac{2 \sin k\frac{2\pi}{3}}{k 2\pi} e^{-jk\frac{2\pi}{3}} + \frac{2 \sin \frac{k\pi}{3}}{k\, 2\pi} e^{-jk\frac{\pi}{3}} \right\}$$

$$= \frac{\sin (k-30)\frac{2\pi}{3}}{(k-30)\, 2\pi} e^{-j(k-30)\frac{2\pi}{3}} + \frac{\sin (k+30)\frac{2\pi}{3}}{(k+30)\, 2\pi} e^{-j(k+30)\frac{2\pi}{3}}$$

Shift property when convoluting a δ

$$+ \frac{\sin (k-30)\frac{\pi}{3}}{(k-30)\, 2\pi} e^{-j(k-30)\frac{\pi}{3}} + \frac{\sin (k+30)\frac{\pi}{3}}{(k+30)\, 2\pi} e^{-j(k+30)\frac{\pi}{3}}$$

3.48

$$\underbrace{x[n]}_{\text{period is } N} = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n}$$ F.S. expansion for $x[n]$

a) Find b_k when x is shifted in time: $x[n-n_0]$

$$x[n-n_0] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}(n-n_0)}$$

$$= \sum_{k=\langle N\rangle} \underbrace{a_k e^{-jk\frac{2\pi}{N}n_0}}_{b_k} e^{jk\frac{2\pi}{N}n}$$

b) Find b_k for $x[n] - x[n-1]$

$$b_k = a_k - a_k e^{-jk\frac{2\pi}{N}} = \left(1 - e^{-jk\frac{2\pi}{N}}\right) a_k$$

Finish c) $\rightarrow$ h)

3.22 a)

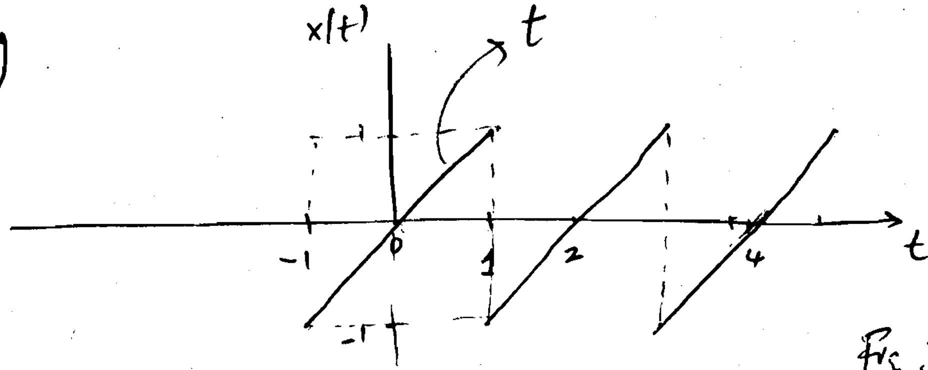


Fig 3.22 a)

F.S. representation $\begin{cases} a_k\text{'s} \\ T=2 \Rightarrow \omega_0 = \frac{2\pi}{2} = \pi \end{cases}$

$$a_k = \frac{1}{T}\int_T dt\, e^{-jk\omega_0 t} x(t) = \underbrace{\frac{1}{2}\int_{-1}^{1} dt\, e^{-jk\omega_0 t}\, t}_{\text{integration by parts, table of integrals / or}}$$

$$a_k = \frac{1}{2}\frac{1}{-jk}\frac{d}{d\omega_0}\underbrace{\left[\int_{-1}^{1} dt\, e^{-jk\omega_0 t}\right]}_{\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_{-1}^{1}} = \frac{1}{2(-jk)}\frac{d}{d\omega_0}\left[\underbrace{\frac{e^{-jk\omega_0} - e^{jk\omega_0}}{-jk\omega_0}}_{\frac{2\sin k\omega_0}{k\omega_0}}\right]$$

$$\sin\alpha = \frac{e^{j\alpha} - e^{-j\alpha}}{2j}$$

$$\to a_k = \frac{1}{-jk}\frac{d}{d\omega_0}\left[\frac{\sin k\omega_0}{k\omega_0}\right] = \frac{1}{-jk}\left[\frac{k\cos k\omega_0}{k\omega_0} - \frac{\sin k\omega_0}{k\omega_0^2}\right]$$

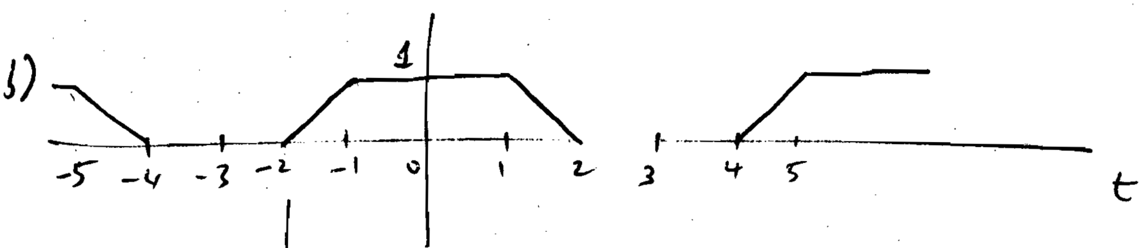
$$= \frac{1}{-jk\omega_0}\left[\cos k\omega_0 - \frac{\sin k\omega_0}{k\omega_0}\right] = \frac{1}{-jk\pi}\left[\cos k\pi - \frac{\sin k\pi}{k\pi}\right]$$

$$k \neq 0 \quad a_k = \frac{1}{-jk\pi}\left[(-1)^k\right] = \frac{j(-1)^k}{k\pi} \qquad k \neq 0$$

$$k = 0 \quad a_0 = \frac{1}{2}\int_{-1}^{1} dt\ t = \frac{1}{2}\left[\frac{t^2}{2}\right]_{-1}^{1} = 0$$

$\Big\}$ a_k's

b)



$$x(t) = \begin{cases} t+2 & -2 < t < -1 \\ 1 & -1 < t < 1 \\ -t+2 & 1 < t < 2 \end{cases}$$

3.52 $x[n]$ real with period N ; F.S. coefficients a_k's

$a_k = b_k + jc_k$ (b_k & c_k are real numbers)

a) Show $\boxed{a_{-k} = a_k^*}$ (complex conjugate of a_k) $= b_k - jc_k$

change sign of j

$$a_k = \frac{1}{N}\sum_n x[n]\, e^{-jk\frac{2\pi}{N}n}$$ (F.S. coeff. for a discrete-time signal)

$$\rightarrow a_{-k} = \frac{1}{N}\sum_n x[n]\, e^{jk\frac{2\pi}{N}n}$$

$$\rightarrow a_k^* = \frac{1}{N}\left(\sum_n x[n] e^{-jk\frac{2\pi}{N}n}\right)^* = \frac{1}{N}\sum_n\left(x[n]\, e^{-jk\frac{2\pi}{N}n}\right)^*$$

$$= \frac{1}{N}\sum_n x^*[n]\, e^{jk\frac{2\pi}{N}n} = \frac{1}{N}\sum_n x[n]\, e^{jk\frac{2\pi}{N}n} = a_{-k}$$

$x[n]$ real

Then: $\quad a_{-k} = a_k^* = b_k - j c_k$

$b_{-k} + j c_{-k}$

$\left.\begin{array}{l} b_{-k} = b_k \\ c_{-k} = -c_k \end{array}\right.$

b) Assume N is even $\rightarrow$ prove $a_{\frac{N}{2}}$ is real

$$a_{\frac{N}{2}} = \frac{1}{N}\sum_n x[n]\, e^{-j\frac{N}{2}\frac{2\pi}{N}n} = \frac{1}{N}\sum_n x[n]\, e^{-j\pi n}$$

$$e^{-j\pi n} = \cos \pi n - j \underbrace{\sin \pi n}_{0} = (-1)^n$$

$$a_{\frac{N}{2}} = \frac{1}{N}\sum_n \underbrace{(-1)^n x[n]}_{\text{real}} \rightarrow a_{\frac{N}{2}} \text{ is real!}$$