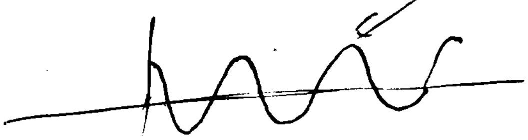
Ch3: Fourier Series Representation of Periodic Signals

HW3: 3.5; 3.12; 3.14; 3.22; 3.25; 3.28; 3.32; 3.46; 3.48; 3.52 a) b) c) d)

Periodic signals: (continuous-time)

↳ examples: $\cos(\omega t) = \mathrm{Re}\left[e^{j\omega t}\right]$

$\underbrace{2\cos(10t) + \sin(4t)}$ → $T = \pi$ (1.10)



this periodic signal is a combination of 2 sinusoids.

Is there any situation in which a periodic signal is NOT a linear combination of sinusoids?

→ Yes → a possible periodic signal that is not a linear comb. of sinusoids is:

→ No: → any periodic signal can be expressed as an ∞ sum of $\cos(k\omega_0 t)$ or $e^{jk\omega_0 t}$

k → integer; ω_0 → fundamental freq.

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}$$

a_k: amplitude, complex number, doesn't depend on time.

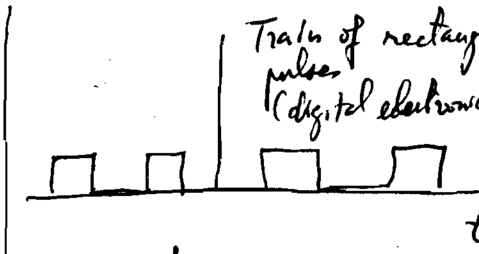
$$\boxed{x(t) \longleftrightarrow a_k\text{'s and } \omega_0}$$

Ch 3 Fourier Series Representation of Periodic Signals (cont.)

Is there any continuous-time periodic signal that is not a combination of sinusoids? $\longrightarrow$ { Yes / No }

Counter examples:

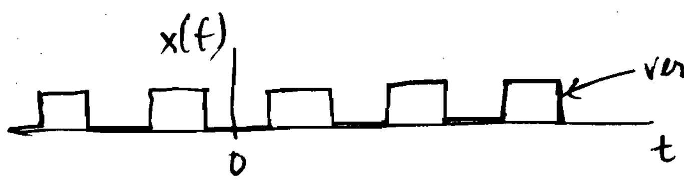
Can't be expressed as a combination of sinusoids. But this is not a periodic signal.



Yes, it is possible to get a square wave by adding a sufficient number of odd harmonics (see attached Matlab plots).

$\rightarrow$ ANY continuous & periodic signal can be expressed as a sum (finite or infinite) of $\cos(k w_0 t)$ or e^{jkw_0t}

k: integer; w_0: fundamental frequency

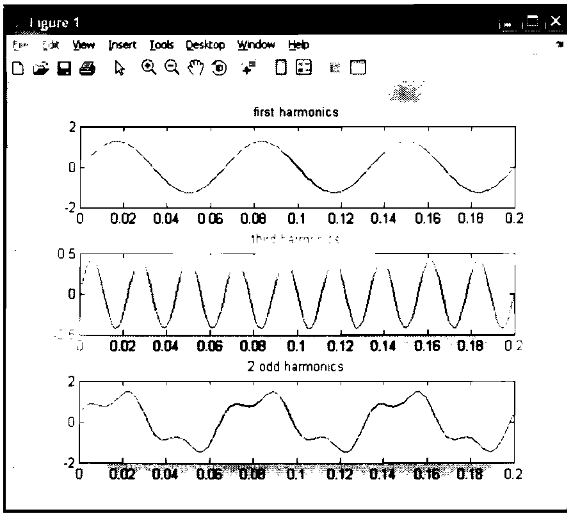


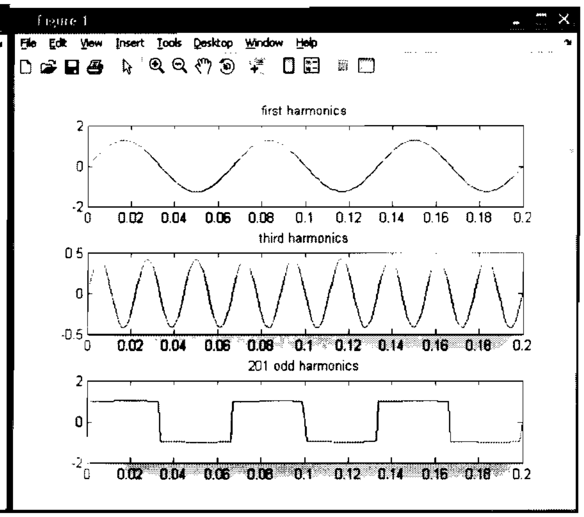
is a ∞ sum of odd harmonics: $\cos(w_0t)$, $\cos(3w_0t)$, $\cos(5w_0t)$ etc.

(In the Matlab example 201 odd harmonics still gives a slope for the sides of the rect. pulses, they are not perfectly vertical yet).

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0t}} \quad (1)$$

Fourier series representation of the periodic signal $x(t)$






```
%This code shows we can build a square wave (train of rectangular
pulses)
%from a linear combination of sinusoids
%Engin 321, October 16, 2007


clear all
close all
nf=1;%15;%45;%200;%this specifies the number of odd harmonics to sum
f0=15;
sr=1000;
deltat=1/sr;
ns=5000;
nt=ns*deltat/5;
sq=zeros(1,ns);
sq1=zeros(1,ns);
n=deltat:deltat:nt;
nr=deltat:deltat:nt/5;
clear sq;
sq=(4/pi)*sin(2*pi*f0*n);
sq0=sq;
subplot(311), plot(nr, sq(1:length(nr))), title('first harmonics')
sq1=(4/(3*pi))*sin(2*pi*3*f0*n);
subplot(312), plot(nr, sq1(1:length(nr))), title('third harmonics')
for i=1:1:nf;
    sq=sq+(4/((2*i+1)*pi))*sin(2*pi*(2*i+1)*f0*n+1/nf);
end
subplot(313), plot(nr, sq(1:length(nr))), title(strcat(num2str(nf+1),'
odd harmonics'))
```

Important consequence: we can identify a periodic signal $x(t)$ by giving the amplitude a_k's and the fundamental freq w_0

How can I find a_k given a signal $x(t)$?

→ Math result: orthogonality of complex exponentials (plane waves)

$$\underbrace{\int_0^T e^{jkw_0 t} e^{-jnw_0 t}\, dt}_{\substack{\text{product of 2 complex} \\ \text{exponentials of frequencies } kw_0 \,\&\, nw_0}} = \begin{cases} T & \text{if } k=n \\ 0 & \text{if } k \neq n \end{cases}$$

Apply $\int_0^T dt\, e^{-jnw_0 t}$ to both sides of eq (1)

$$\int_0^T dt\, e^{-jnw_0 t} x(t) = \sum_{k=-\infty}^{\infty} a_k \underbrace{\int_0^T dt\, e^{j(k-n)w_0 t}}_{T\delta[k-n]}$$

$$= T a_n$$

$$\rightarrow a_n = \frac{1}{T}\int_0^T dt\, e^{-jnw_0 t} x(t) \quad \text{or} \quad \boxed{a_k = \frac{1}{T}\int_0^T dt\, e^{-jkw_0 t} x(t)}$$

$T = \frac{2\pi}{w_0}$ is the fundamental period.

Properties of Fourier Series Representation: (Table 3.1 p. 206)

Linearity:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$$

$x(t)$ & $y(t)$ are both periodic with the same fundamental frequencies.

b_k for $y(t) \neq x(t)$

→ what are the amplitudes of the periodic signal $Ax(t) + By(t)$? (A, B constants)

$$Ax(t) + By(t) = \sum_{k=-\infty}^{\infty} A a_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{\infty} B b_k e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{(A a_k + B b_k)}_{\text{amplitudes of } Ax + By} e^{jk\omega_0 t}$$

Time-shift:

Fourier-series amplitudes for $x(t)$ are a_k's

→ What are the amplitudes for $x(t - t_0)$?

We can find out by applying the time-shift to the Fourier series representation:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x(t-t_0) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 (t-t_0)} = \sum_{k=-\infty}^{\infty} \underbrace{a_k e^{-jk\omega_0 t_0}}_{\text{they don't depend on time! this is the time-shifted amplitude!}} e^{jk\omega_0 t}$$

Other properties (see table 3.1):
Frequency - shift ; time - reversal ; time scaling ;
multiplication ; differentiation , integration ,

<u>Parseval 's relation</u> :
$$\boxed{\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2}$$
$$\underbrace{\frac{1}{T}\int_T |x(t)|^2 dt}_{\text{an operation in time domain}} \qquad \underbrace{\sum_{k=-\infty}^{\infty} |a_k|^2}_{\text{operation in frequency domain.}}$$

<u>3.5</u>
$$x_1(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_1 t}$$
($x_1(t)$ is specified by its fundamental freq. ω_1 and a_k's)

$$x_2(t) = x_1(1-t) + x_1(t-1)$$
($x_2(t)$ is specified by its fundamental freq. and b_k's)
freq. → ω_1; $\underbrace{b_k\text{'s}}_{\neq a_k\text{'s}}$

time - shift & time reversal

<u>Problem</u>: find b_k's in terms of a_k's or
$$x_2(t) = \sum_{k=-\infty}^{\infty} \underbrace{b_k} e^{jk\omega_1 t}$$

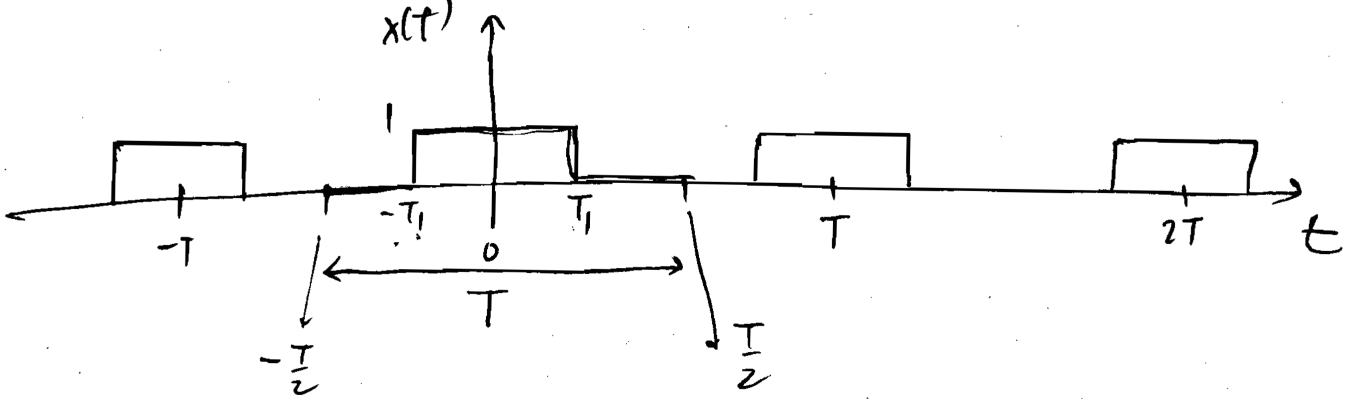
$$x_2(t) = x_1(1-t) + x_1(t-1) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_1(1-t)} + \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_1(t-1)}$$
$$= \sum_{k=-\infty}^{\infty} \left(a_k e^{jk\omega_1} e^{-jk\omega_1 t} + a_k e^{-jk\omega_1} e^{jk\omega_1 t} \right)$$
$$= \sum_{k=-\infty}^{\infty} \left(a_k e^{jk\omega_1} e^{j(-k)\omega_1 t} \right) + \sum_{k=-\infty}^{\infty} \left(a_k e^{-jk\omega_1} \underline{e^{jk\omega_1 t}} \right)$$

$$X_2 = \underbrace{x_1(-\lambda)}_{\substack{\downarrow \\ a_{-k}e^{-jk\omega_0} \\ \text{Time reversal}}} + \underbrace{x_1(\lambda)}_{a_k e^{-jk\omega_0}} \quad \rightarrow \quad b_k = (a_k + a_{-k})e^{-jk\omega_0}$$

F.S. representation for a continuous-time periodic signal

Example 3.5: a_k's for $x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < \frac{T}{2} \end{cases}$

$x(t)$ is a train of rectangular pulses where T_1 is half width of the pulse; T is the period of the signal.



one period of the signal $x(t)$

$$a_k = \frac{1}{T}\int_T dt\, e^{-jk\omega_0 t} x(t) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} dt\, e^{-jk\omega_0 t} x(t)$$

$$= \frac{1}{T}\int_{-T_1}^{T_1} dt\, e^{-jk\omega_0 t} = \frac{1}{T}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_{-T_1}^{T_1} = \frac{1}{-jk\omega_0 T}\left(-2j\sin k\omega_0 T_1\right)$$

$$e^{ja} - e^{-ja} = 2j\sin a$$

$$\boxed{a_k = \frac{2\sin k\omega_0 T_1}{k\omega_0 T} \underset{\omega_0 = \frac{2\pi}{T}}{=} \frac{\sin(k\omega_0 T_1)}{k\pi}} \quad k \neq 0$$

We would like to write both terms into one sum. To get rid of the sign in the exponent of the first $e^{j(-k)\omega_1 t}$ we use a math trick of renaming the dummy index of summation: $k \longrightarrow -k$

$$= \sum_{k=+\infty}^{-\infty} \left(a_{-k} e^{-jk\omega_1} e^{jk\omega_1 t} \right) + \sum_{k=-\infty}^{\infty} \left(a_k e^{-jk\omega_1} e^{jk\omega_1 t} \right)$$

same sum!

$$x_2(t) = \sum_{k=-\infty}^{\infty} (a_{-k} + a_k)\, e^{-jk\omega_1} e^{jk\omega_1 t}$$

Recall $x_2 = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_1 t} \rightarrow b_k = (a_{-k} + a_k)\, e^{-jk\omega_1}$

Alternatively, let's find b_k using the table 3.1 of properties of Fourier series:

<u>Time shifting</u>: $x(t-t_0) \longrightarrow a_k e^{-jk\omega_0 t_0}$

<u>Time reversal</u>: $x(-t) \longrightarrow a_{-k}$

$$x_2(t) = \underbrace{x_1(1-t)}_{\substack{\text{time reversal}\\ \text{and a time shift}\\ \text{of } (-1)\\ a_{-k} e^{jk\omega_1}}} + \underbrace{x_1(t-1)}_{\substack{\text{Time shift of } +1\\ a_k e^{-jk\omega_1}}} \quad \Bigg\} \; b_k = (a_{-k} + a_k)\, e^{-jk\omega_1}$$

or: $x_2(t) = x_1(1-t) + x_1(t-1) \rightarrow$ define $\lambda \equiv t-1$ ↳ time shift of +1

$$x_2 = \underbrace{x_1(-\lambda)}_{\substack{\text{time reversal}\\ a_{-k}}} + \underbrace{x_1(\lambda)}_{a_k}$$

$$\left[a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt\; e^{j0\omega_0 t}\cdot 1 = \frac{2T_1}{T}\right] \quad (k=0)$$

F.S. representation for a discrete-time periodic signal:

$x[n]$; periodic: $x[n+N] = x[n]$

Example: $\cos[\omega_0 n]$: $\cos[\omega_0(n+N)] = \cos[\omega_0 n]$

$$\omega_0 N = 2\pi$$

$$\rightarrow \boxed{\omega_0 = \frac{2\pi}{N}}$$

Real part of $e^{jk\omega_0 n} = e^{jk\frac{2\pi}{N}n}$

$e^{jk\omega_0 t}$ (continuous time)

$\downarrow$

different for different k's

$e^{jk\frac{2\pi}{N}n}$ (discrete-time)

$\downarrow$

$k \Rightarrow k+N$ we have the same signal!

$$e^{j(k+N)\frac{2\pi}{N}n} = e^{jk\frac{2\pi}{N}n}\underbrace{e^{j2\pi n}}_{\text{always } 1}$$

Implication for Fourier series representation.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n}$$

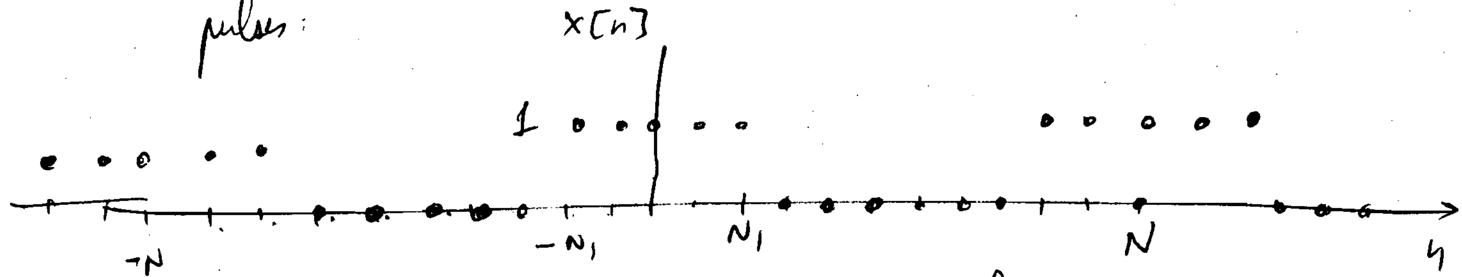
only N terms are needed, where N is period of the discrete-time signal

F.S. Representation of a discrete-time periodic signal:

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n} \quad \text{(only } N \text{ terms)}$$

$$\hookrightarrow a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jk\frac{2\pi}{N}n}$$

Example 3.12: a_k's for a discrete-time train of rectangular pulses:



Width of each pulse is $2N_1$; period is N

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\sum_{n=-N_1}^{N_1} \overset{1}{x[n]}\, e^{-jk\frac{2\pi}{N}n}$$

center the period around $n=0$

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1}\left(e^{-jk\frac{2\pi}{N}}\right)^n \quad \text{a geometric sum:}$$

$$\boxed{\text{Math result: } \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha} \quad \text{(geometric sum)}}$$

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} \alpha^n = \frac{1}{N}\sum_{m=0}^{2N_1} \alpha^{m-N_1} = \frac{1}{N}\alpha^{-N_1}\frac{1-\alpha^{2N_1+1}}{1-\alpha}$$

$\alpha \equiv e^{-jk\frac{2\pi}{N}}$

change dummy index $n \rightarrow m = n + N_1$

$$\rightarrow \boxed{a_k = \frac{1}{N}\, e^{jk\frac{2\pi N_1}{N}} \cdot \frac{1-e^{-jk\frac{2\pi}{N}(2N_1+1)}}{1-e^{-jk\frac{2\pi}{N}}}}$$

For a discrete-time train of rectangular pulse.

To write this a_k in term of sin functions, use this trick:

$$1 - e^{-j\beta} = e^{-j\frac{\beta}{2}} \underbrace{\left(e^{j\frac{\beta}{2}} - e^{-j\frac{\beta}{2}} \right)}_{2j\sin\frac{\beta}{2}}$$

$$a_k = \frac{1}{N} \; \frac{e^{jk\frac{2\pi N_1}{N}} \; e^{-j\frac{k2\pi}{N}(N_1+\frac{1}{2})} \left(e^{j\frac{k2\pi}{N}(N_1+\frac{1}{2})} - e^{-j\frac{k2\pi}{N}(N_1+\frac{1}{2})} \right)}{e^{-jk\frac{2\pi}{2N}} \left(e^{jk\frac{2\pi}{2N}} - e^{-jk\frac{2\pi}{2N}} \right)}$$

$$\text{"} 1$$

$$a_k = \frac{1}{N} \; \frac{\cancel{2j} \sin \frac{k2\pi}{N}(N_1+\frac{1}{2})}{\cancel{2j} \sin \frac{k2\pi}{2N}}$$

when $\sin \frac{k2\pi}{2N} = 0 \rightarrow$ this is not valid: when $k=0$ or a multiple of $2N$. We need another expression for a_k in these situations:

When $k=0$ or a multiple of $2N$:

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} \underbrace{x[n]}_{1} \; \underbrace{e^{-jk\frac{2\pi}{N}n}}$$

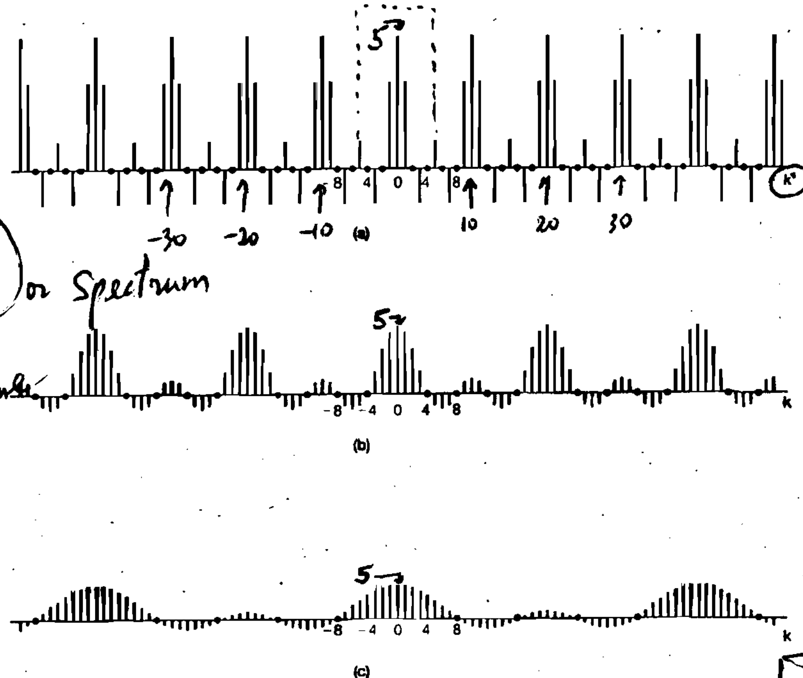
(train of rectangular pulses of amplitude 1)

$$\begin{cases} k=0 \rightarrow e^{-jk\frac{2\pi}{N}n} = 1 \\ k=2N \rightarrow e^{-j2\cancel{N}\frac{2\pi}{\cancel{N}}n} = e^{-j4\pi n} = 1 \end{cases}$$

$$a_k = \frac{1}{N} \underbrace{\sum_{n=-N_1}^{N_1} 1 \cdot 1} = \frac{1}{N}(2N_1+1)$$

$$\rightarrow a_k \begin{cases} \dfrac{1}{N} \dfrac{\sin \frac{k2\pi}{N}(N_1+\frac{1}{2})}{\sin \frac{k2\pi}{N}} & k \neq 0 \text{ or multiple of } 2N \\[2ex] \dfrac{1}{N}(2N_1+1) & k=0 \text{ or a multiple of } 2N \end{cases}$$

(53)

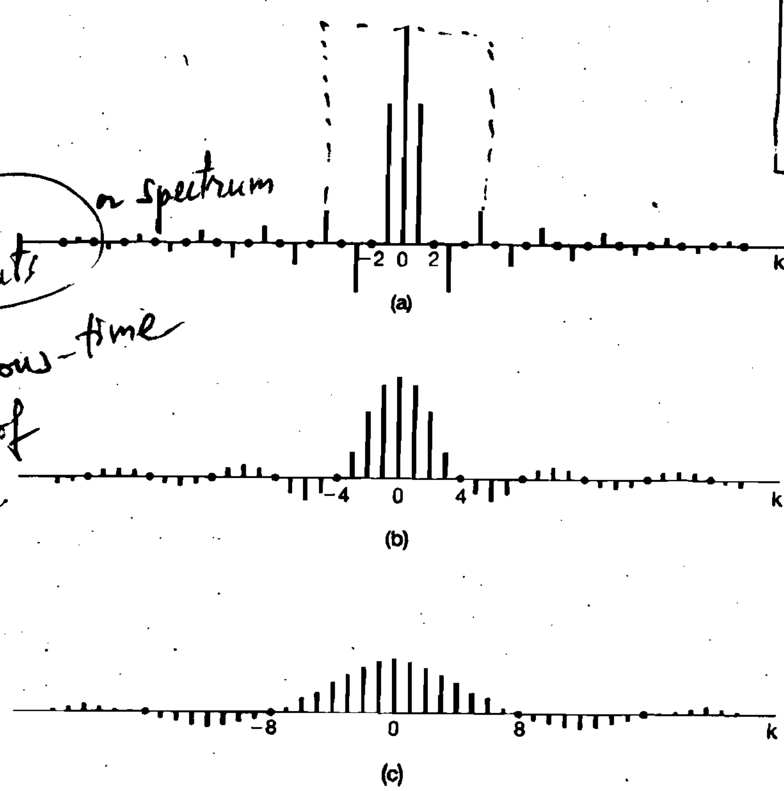


F.S. Coefficients or Spectrum of discrete-time train of pulse

Figure 3.17 Fourier series coefficients for the periodic square wave of Example 3.12; plots of Na_k for $2N_1 + 1 = 5$ and (a) $N = 10$; (b) $N = 20$; and (c) $N = 40$.

The effect of discretizing a signal is the repetition of the original Spectrum in frequency

Reconstruction of a discrete-time signal requires prior application of a low-pass filter



F.S. Coefficients or spectrum of continuous-time train of pulse

Figure 3.7 Plots of the scaled Fourier series coefficients Ta_k for the periodic square wave with T_1 fixed and for several values of T: (a) $T = 4T_1$; (b) $T = 8T_1$; (c) $T = 16T_1$. The coefficients are regularly spaced samples of the envelope $(2 \sin \omega T_1)/\omega$, where the spacing between samples, $2\pi/T$, decreases as T increases.

3.12

$x_1[n]$, period $N=4$; a_k : $a_0 = a_3 = 1$; $a_1 = a_2 = 2$

$x_2[n]$, period $N=4$; b_k : $b_0 = b_1 = b_2 = b_3 = 1$

Table 3.1: Multiplication property:

↳ continuous-time $x(t)y(t) \rightarrow c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$

↳ discrete-time $x_1[n]x_2[n] \rightarrow c_k = \sum_{l=\langle N \rangle} a_l b_{k-l}$

Assuming the period of $x_1[n]x_2[n]$ is <u>also</u> $N \rightarrow$
From MatLab figures, not necessarily!

$$c_k = \underset{l=0}{a_0 b_k} + \underset{l=1}{a_1 b_{k-1}} + \underset{l=2}{a_2 b_{k-2}} + \underset{l=3}{a_3 b_{k-3}} + \underset{l=4}{\cancel{a_4 b_{k-4}}} \ldots$$

We have only N terms, the rest are 0.

$$C_k = b_k + 2(b_{k-1} + b_{k-2}) + b_{k-3}$$

Since $x_2[n]$ is discrete time : $b_k = b_{k+N}$

$(N=4)$

$b_{-1} = b_{-1+4} = b_3 = 1$

$b_{-2} = b_2 = 1$

$b_{-3} = b_1 = 1$

$c_0 = b_0 + 2(b_{-1} + b_{-2}) + b_{-3}$

$= 1 + 2(1+1) + 1$

$= 6$

$c_1 = b_1 + 2(b_0 + b_{-1}) + b_{-2}$

$= 6$

$c_2 =$

$c_3 =$

$c_4 =$ —

$c_5 =$ — — — $c_6 =$ — —