Ch3: Fourier Series Representation of Periodic Signals

HW3: 3.5; 3.12; 3.14; 3.22; 3.25; 3.28; 3.32; 3.46; 3.48; 3.52 a) b) c) d)

Periodic signals: (continuous time)

Examples:

\[ e^{jw_0 t} = \Re \left[ e^{j \omega t} \right] \]

\[ 2 \cos(\omega t) + \sin(\omega t) \rightarrow T = \pi \quad (1.10) \]

This periodic signal is a combination of 2 sinusoids.

Is there any situation in which a periodic signal is \textbf{not} a linear combination of sinusoids?

\[ \rightarrow \text{Yes} \rightarrow \text{a possible periodic signal that is \textbf{not} a linear combination of sinusoids is:} \]

\[ \rightarrow \text{No: any periodic signal can be expressed as an infinite sum of } e^{jkw_0 t} = e^{jkw_0 t} \quad \text{for integer } k \text{ and fundamental freq.} \]

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t} \]

- \( a_k \): amplitude, complex number, does not depend on time.

\( a_k \)'s and \( w_0 \)
Ch 3 Fourier series representation of periodic signals (cont.)

Is there any continuous-time periodic signal that is not a combination of sinusoids? → Yes

Counter example:

\[ e^{-at} \]

Can't be expressed as a combination of sinusoids

Train of rectangular pulses (digital electronics)

→ ANY continuous-time periodic signal can be expressed as a sum (finite or infinite) of \( \cos(k\omega_0 t) \) or \( e^{jkw_0 t} \)

Interger fundamental frequency

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t} \] \hspace{1cm} (1)

Fourier series representation of the periodic signal \( x(t) \)

Yes, it is possible to get a square wave by adding a sufficient number of odd harmonics (see attached MATLAB plots)

(\( w_0 \), \( w_0 (3w_0 t) \), \( w_0 (5w_0 t) \), etc.)

(In the MATLAB example 201 odd harmonics still give a slope for the side of the square pulses, they are not perfectly vertical yet)
This code shows we can build a square wave (train of rectangular pulses) from a linear combination of sinusoids.

Engin 321, October 16, 2007

clear all
close all
nf=1; %15; %45; %200; %this specifies the number of odd harmonics to sum
f0=15;
sr=1000;
deltat=1/sr;
ns=5000;
nt=ns*deltat/5;
sq=zeros(1,ns);
sqo=sq;
sql=zeros(1,ns);
n=deltat:deltat:nt;
for i=1:1:nf;
sq=sq+(4/(2*i+1)*pi)*sin(2*pi*(2*i+1)*f0*n/nf);
end
subplot(313), plot(nr, sq(1:length(nr))), title(strcat(num2str(nf+1), ' odd harmonics'))
Important consequence: we can identify a periodic signal $x(t)$ by giving the amplitude $a_k$'s and the fundamental freq $w_0$. How can I find $a_k$ given a signal $x(t)$?

Math result: orthogonality of complex exponentials (plane wave)

$$\int_0^T e^{jkw_0 t} e^{-jkw_0 t} \, dt = \begin{cases} T & \text{if } k=n \\ 0 & \text{if } k \neq n \end{cases}$$

Product of 2 complex exponentials of frequencies $k w_0 \pm n w_0$

Apply $\int_0^T e^{-jkw_0 t} \, dt$ to both sides of eq (1)

$$\int_0^T e^{-jkw_0 t} x(t) = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)w_0 t} \, dt$$

$$= T a_n$$

$$a_n = \frac{1}{T} \int_0^T e^{-jkw_0 t} x(t) \, dt$$

$$a_k = \frac{1}{T} \int_0^T e^{jkw_0 t} x(t) \, dt$$

$T = \frac{2\pi}{w_0}$ is the fundamental period.
Properties of Fourier Series Representation: (Table 3.1 p. 206)

**Linearity**: \( x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t} \) \( y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jkw_0 t} \)

\( x(t) \) and \( y(t) \) are both periodic with the same fundamental frequencies.

\[ y(t) \neq x(t) \]

- What are the amplitudes of the periodic signal \( Ax(t) + By(t) \)?

\[ Ax(t) + By(t) = \sum_{k=-\infty}^{\infty} A a_k e^{jkw_0 t} + \sum_{k=-\infty}^{\infty} B b_k e^{jkw_0 t} \]

- Constants

\[ = \sum_{k=-\infty}^{\infty} (A a_k + B b_k) e^{jkw_0 t} \]

- Amplitude of \( Ax + By \)

**Time-shift**:

- Fourier series amplitudes for \( x(t) \) are \( a_k \)’s.

- What are the amplitudes for \( x(t-t_0) \)?

We can find out by applying the time shift to the Fourier series representation:

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t} \]

\[ x(t-t_0) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 (t-t_0)} = \sum_{k=-\infty}^{\infty} a_k e^{-jkw_0 t_0} e^{jkw_0 t} \]

- They don’t depend on time!

- This is the time-shifted amplitude!
3.5

\[ x_1(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_1 t} \]  
\( (x_1(t) \text{ is specified by its fundamental freq. } w_1 \text{ and } a_k) \)

\[ x_2(t) = x_1(1-t) + x_1(t-1) \]  
\( (x_2(t) \text{ is specified by its fundamental freq. and } b_k) \)

**Problem:** find \( b_k \)'s in terms of \( a_k \)'s or

\[ x_2(t) = \sum_{k=-\infty}^{\infty} b_k e^{jkw_1 t} \]

\[ x_2(t) = x_1(1-t) + x_1(t-1) = \sum_{k=-\infty}^{\infty} a_k e^{jk(1-t)w_1} + \sum_{k=-\infty}^{\infty} a_k e^{jk(t-1)w_1} \]

\[ = \sum_{k=-\infty}^{\infty} \left( a_k e^{jkw_1} e^{-jkw_1 t} + a_k e^{-jkw_1} e^{jkw_1 t} \right) \]

\[ = \sum_{k=-\infty}^{\infty} \left( a_k e^{jkw_1} e^{j(k-1)w_1 t} + a_k e^{-jkw_1} e^{jkw_1 t} \right) \]

\[ = \sum_{k=-\infty}^{\infty} a_k e^{jkw_1} \left( e^{j(k-1)w_1 t} + e^{jkw_1 t} \right) \]
\[ x_2 = x_1(-\lambda) + x_1(\lambda) \quad \text{F.S. representation for a continuous-time periodic signal} \]

**Example 3.5:** \( a_k \) s for \( x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < \frac{T}{2} \end{cases} \)

\( x(t) \) is a train of rectangular pulses where \( T_1 \) is half width of the pulse; \( T \) is the period of the signal.

\[ a_k = \frac{1}{T} \int_{-T}^{T} e^{-jkw_0t} x(t) dt = \frac{1}{T} \int_{-T_1}^{T_1} e^{-jkw_0t} x(t) dt \]

\[ = \frac{1}{T} \int_{-T_1}^{T_1} dt e^{-jkw_0t} = \frac{1}{T} \left[ \frac{e^{-jkw_0t}}{-jkw_0} \right]_{-T_1}^{T_1} = \frac{1}{-jkw_0T} \left( -2j \sin(kw_0T_1) \right) = \frac{2 \sin(kw_0T_1)}{kw_0T} \]

\[ a_k = \frac{2 \sin(kw_0T_1)}{kw_0T} = \frac{\sin(kw_0T_1)}{kT} \quad k \neq 0 \]
We would like to write both terms into one sum. To get rid of the sign in the exponent of the first $e^{j(-k)\omega_1 t}$ we use a math trick of renaming the dummy index of summation: $k \rightarrow -k$

\[
\sum_{k=-\infty}^{\infty} \left( a_k e^{j k \omega_1} e^{j k \omega_1 t} \right) + \sum_{k=-\infty}^{\infty} \left( a_k e^{-j k \omega_1} e^{j k \omega_1 t} \right)
\]

\[
\uparrow \quad \text{same sum!} \quad \uparrow
\]

\[x_2(t) = \sum_{k=-\infty}^{\infty} (a_k + a_{-k}) e^{-j k \omega_1} e^{j k \omega_1 t}\]

Recall $x_2 = \sum_{k=-\infty}^{\infty} b_k e^{j k \omega_1 t}$ \rightarrow $b_k = (a_k + a_{-k}) e^{-j k \omega_1}$

Alternatively, let's find $b_k$ using the table 3.1 of properties of Fourier series:

- **Time shifting**: $x(t-t_0) \rightarrow a_k e^{-j k \omega_1 t}$
- **Time reversal**: $x(-t) \rightarrow a_{-k}$

\[x_2(t) = x_1(1-t) + x_1(t-1)\]

- **Time reversal** and **time shift of +1**

\[b_k = (a_k + a_{-k}) e^{-j k \omega_1}\]

or:

\[x_2(t) = x_1(1-t) + x_1(t-1) \rightarrow \text{define } \lambda = t-1 \downarrow \text{ time shift of +1}\]

\[x_2 = x_1(-\lambda) + x_1(\lambda)\]

\[\downarrow \text{ time reversal} \quad \downarrow \text{ same sum!} \quad \downarrow \]

\[\begin{align*}
&\begin{array}{c}
\text{Recall } x_2 = \sum_{k=-\infty}^{\infty} b_k e^{j k \omega_1 t} \\
&\text{Recall } x_2 = \sum_{k=-\infty}^{\infty} b_k e^{j k \omega_1 t}
\end{array}
\end{align*}\]
\[ a_0 = \frac{1}{T} \int_{-T}^{T} dt \ e^{j \omega_0 t} \cdot 1 = \frac{2T}{T} \quad (k=0) \]

**Fourier Series representation for a discrete-time periodic signal:**

\[ x[n] \quad \text{periodic} \quad x[n+N] = x[n] \]

**Example:**

\[ \cos [\omega_0 n] = \cos [\omega_0 (n+N)] = \cos [\omega_0 n] \]

\[ \frac{\omega_0 N}{2\pi} = 2\pi \]

\[ \omega_0 = \frac{2\pi}{N} \]

Real part of \( e^{j \omega_0 n} = e^{j \frac{2\pi}{N} n} \)

\[ e^{j \omega_0 t} \quad (\text{continuous time}) \]

\[ \downarrow \]

\[ \text{different for different } k's \]

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j \omega_0 t} \]

\[ x[n] = \sum_{k=-N}^{N} a_k e^{j \frac{2\pi}{N} n} \quad \text{always} \]

Implication for Fourier series representation:

\[ x[n+N] = x[n] \quad \text{for the same signal} \]

\[ e^{j (k+N) \frac{2\pi}{N} n} = e^{j k \frac{2\pi}{N} n} \]

Only \( N \) terms are needed, where \( N \) is period of the discrete-time signal.
F.S. Representation of a discrete-time periodic signal:

\[ x[n] = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kn/N} \]

where each pulse is \( 2\pi n \) in period. For a discrete-time rectangular pulse:

\[ a_k = \frac{1}{N} \sum_{n=-N}^{N-1} x[n] e^{-j2\pi kn/N} \]

Example 3.12:

Math. result:

\[ a_k = \frac{1}{N} \sum_{n=-N}^{N-1} x[n] e^{-j2\pi kn/N} \]

\[ a_k = \frac{1}{N} \left( \sum_{n=-N}^{N-1} x[n] e^{-j2\pi kn/N} \right) \]

For a discrete-time rectangular pulse:

\[ a_k = \frac{1}{N} \sum_{n=-N}^{N-1} x[n] e^{-j2\pi kn/N} \]

Change dummy variable:

\[ a_k = \frac{1}{N} \sum_{m=0}^{2N} x[n] e^{-j2\pi km/N} \]

For a discrete-time periodic signal:

\[ a_k = \frac{1}{N} \sum_{n=-N}^{N-1} x[n] e^{-j2\pi kn/N} \]

\[ a_k = \frac{1}{N} \sum_{n=-N}^{N-1} x[n] e^{-j2\pi n} \]

\[ a_k = \frac{1}{N} \sum_{n=-N}^{N-1} x[n] e^{-j2\pi kn/N} \]
To write this $a_k$ in terms of sine functions, use this trick:

$$1 - e^{-j\beta} = e^{-j\beta/2} \left( e^{j\beta/2} - e^{-j\beta/2} \right)$$

$$a_k = \frac{1}{N} e^{j\frac{k2\pi}{N} \left(N_1 + \frac{1}{2}\right)} \left( e^{j\frac{k\pi}{2N}(N_1 + \frac{1}{2})} - e^{j\frac{k\pi}{2N}(N_1 + \frac{1}{2})} \right)$$

When $\sin \frac{k\pi}{2N} = 0 \rightarrow$ this is not valid - when $k = 0$ or a multiple of $2N$. We need another expression for $a_k$ in these situations:

When $k = 0$ or a multiple of $2N$:

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} x[n] e^{-j\frac{k2\pi}{N}n} \rightarrow \begin{cases} k = 0 \rightarrow e^{j\frac{k2\pi}{N}n} = 1 \\ k = 2N \rightarrow e^{-j\frac{k2\pi}{N}n} = 1 \end{cases}$$

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} 1 = \frac{1}{N} (2N + 1)$$

$$\rightarrow a_k \begin{cases} \frac{1}{N} \sin \frac{k\pi}{2N} \left(N_1 + \frac{1}{2}\right) & k \neq 0 \text{ or multiple of } 2N \\ \frac{1}{N} (2N + 1) & k = 0 \text{ or } \text{multiple of } 2N \end{cases}$$
Sec. 3.6 Fourier Series Representation of Discrete-Time Periodic Signals

Figure 3.17 Fourier series coefficients for the periodic square wave of Example 3.12, plots of $a_n$ for $2N_1 + 1 = 5$ and (a) $N = 10$, (b) $N = 20$, and (c) $N = 40$.

Sec. 3.4 Convergence of the Fourier Series

Figure 3.7 Plots of the scaled Fourier series coefficients $T_n$ for the periodic square wave with $T_1$ fixed and for several values of $T$: (a) $T = 4T_1$, (b) $T = 8T_1$, (c) $T = 16T_1$. The coefficients are regularly spaced samples of the envelope $2 \sin \omega T_1 \nu_0$, where the spacing between samples, $2\pi/T$, decreases as $T$ increases.
$x_1[n]$, period $N=4$; $a_k : a_0 = a_3 = 1; a_1 = a_2 = 2$

$x_2[n]$, period $N=4$; $b_k : b_0 = b_1 = b_2 = b_3 = 1$

Table 3.1: Multiplication property:

- **Continuous-time** $x(t)y(t) \rightarrow c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$
- **Discrete-time** $x_1[n]x_2[n] \rightarrow c_k = \sum_{l=-N}^{\infty} a_l b_{k-l}$

Assuming the period of $x_1[n]x_2[n]$ is also $N$.

From MATLAB figure, not necessarily!

$$c_k = a_0 b_k + a_1 b_{k-1} + a_2 b_{k-2} + a_3 b_{k-3}$$

We have only $N$ terms, the rest are 0.

$$c_k = b_k + 2(b_{k-1} + b_{k-2}) + b_{k-3}$$

Since $x_2[n]$ is discrete time: $b_k = b_{k+N}$

- $b_1 = b_{1+4} = b_5 = 1$
- $b_2 = b_2 = 1$
- $b_3 = b_1 = 1$

\[
\begin{align*}
    c_0 &= b_0 + 2(b_{1} + b_{2}) + b_3 \\
     &= 1 + 2(1+1) + 1 \\
     &= 6
\end{align*}
\]

\[
\begin{align*}
    c_1 &= b_1 + 2(b_0 + b_{-1}) + b_{-2} \\
     &= 1 + 2(1+1) + 1 \\
     &= 6
\end{align*}
\]

\[
\begin{align*}
    c_2 &= 0 \\
    c_3 &= -1 \\
    c_4 &= -1
\end{align*}
\]