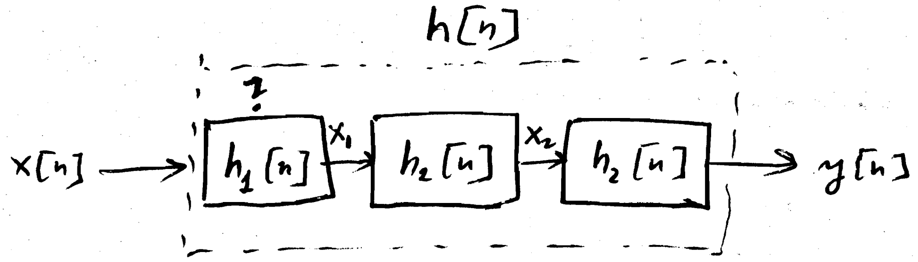
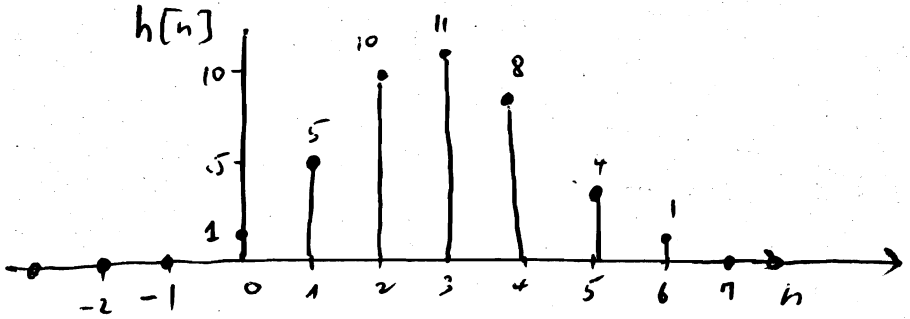
2.24)



$h_2[n] = u[n] - u[n-2] \longrightarrow h_1[n]\ ?$



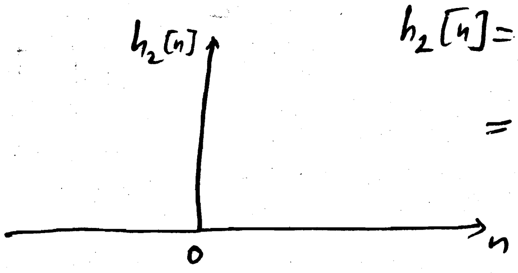
$y[n] = x_2[n] * h_2[n] = \left(x_1[n] * h_2[n]\right) * h_2[n]$

$= x[n] * \underbrace{\left(h_1[n] * h_2[n] * h_2[n]\right)}_{}$

$h[n]$: impulse response of combined system

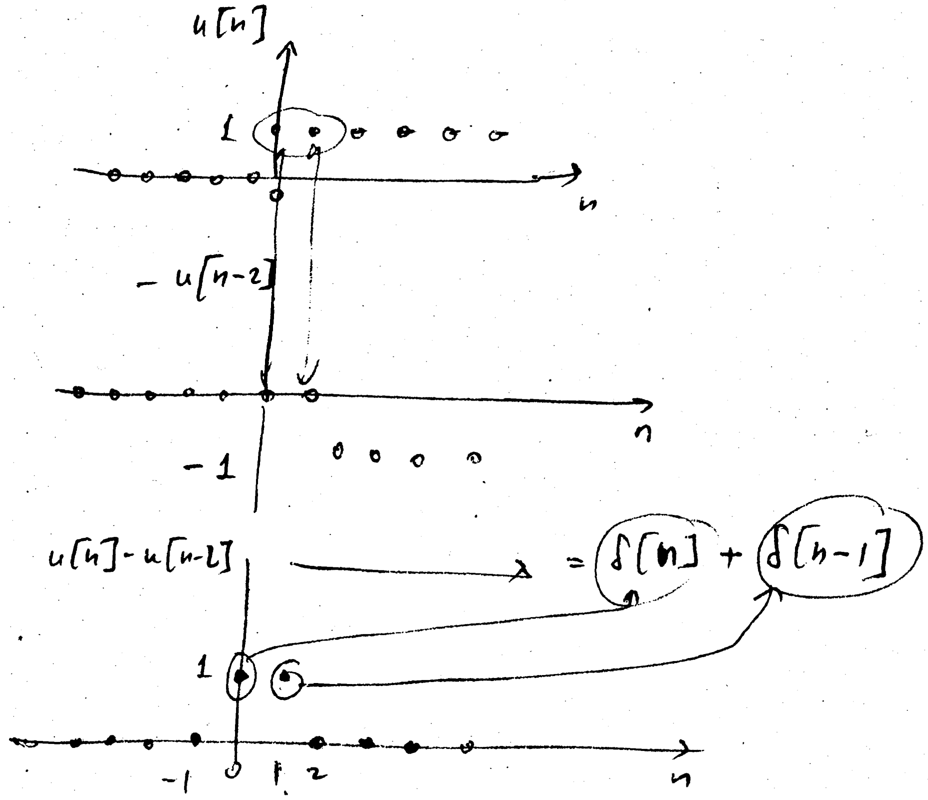
$h = h_1 * h_2 * h_2$

known ? known



$h_2[n] = \underbrace{u[n]} - u[n-2]$

$= \delta[n] + \delta[n-1]$



$$\Rightarrow \quad h_2[n] = u[n] - u[n-2] = \delta[n] + \delta[n-1]$$

$$h_2 * h_2 = \delta[n] * \delta[n] + \delta[n] * \delta[n-1] + \delta[n-1] * \delta[n]$$
$$+ \delta[n-1] * \delta[n-1] =$$

Property: $\delta[n] * \underbrace{\delta[n-j]}_{\text{centered at } j} = \delta[j]$ } Convolution with a δ centered at j (shifts the function by j)

$$= \delta[n] + \delta[n-1] + \delta[n-1] + \delta[n-2]$$

$$\boxed{h_2 * h_2 = \delta[n] + 2\delta[n-1] + \delta[n-2]}$$

We know: $h[n] = h_1 * h_2 * h_2$

$= h_1[n] * \left(\delta[n] + 2\delta[n-1] + \delta[n-2]\right)$

Convolution with a δ property:

$= \underbrace{h_1[n] * \delta[n]} + 2\underbrace{h_1[n] * \delta[n-1]} + h_1[n] * \delta[n$

$= h_1[n] + 2h_1[n-1] + h_1[n-2]$

$$\Rightarrow \boxed{h[n] = h_1[n] + 2h_1[n-1] + h_1[n-2]}$$

$h[-2] = 0$
$h[-1] = 0$
$h[0] = 1$
$h[1] = 5$
$h[2] = 10$
$h[3] = 11$
$h[4] = 8$
$h[5] = 4$
$h[6] = 1$
$h[7] = 0$
$\dots \quad 0$

Causal systems
impulse response starts at $n = 0$
$\hookrightarrow h_1[n] = 0$ for $n < 0$

(impulse is at 0)

$h[0] = h_1[0] + 2 \times \underset{h_1[-1]}{0} + \underset{h_1[-2]}{0} = 1$

$\hookrightarrow \boxed{h_1[0] = 1}$

$h[1] = h_1[1] + 2h_1[0] + \underset{h_1[-1]}{0}$

$\hookrightarrow \boxed{h_1[1] = 3}$

$h[2] = h_1[2] + 2h_1[1] + h_1[0]$

$\hookrightarrow \boxed{h_1[2] = 3}$

$h[3] = h_1[3] + 2h_1[2] + h_1[1]$

$\hookrightarrow \boxed{h_1[3] = 2}$

$h[4] = h_1[4] + 2h_1[3] + h_1[2]$

$\hookrightarrow \boxed{h_1[4] = 1}$

$h[5] = h_1[5] + 2h_1[4] + h_1[3]$

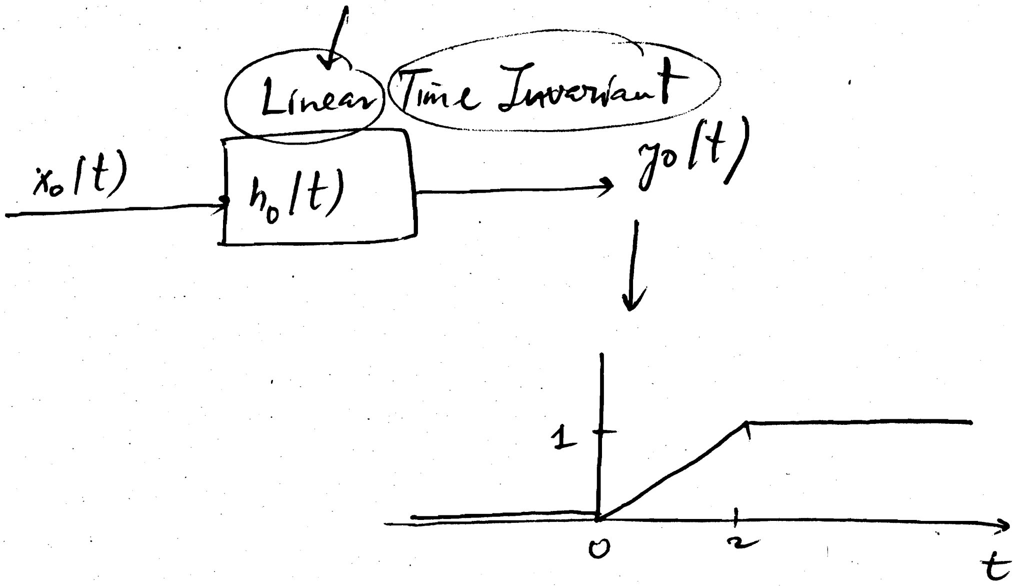
$\hookrightarrow \boxed{h_1[5] = 0}$

$h[6] = h_1[6] + 2h_1[5] + h_1[4]$

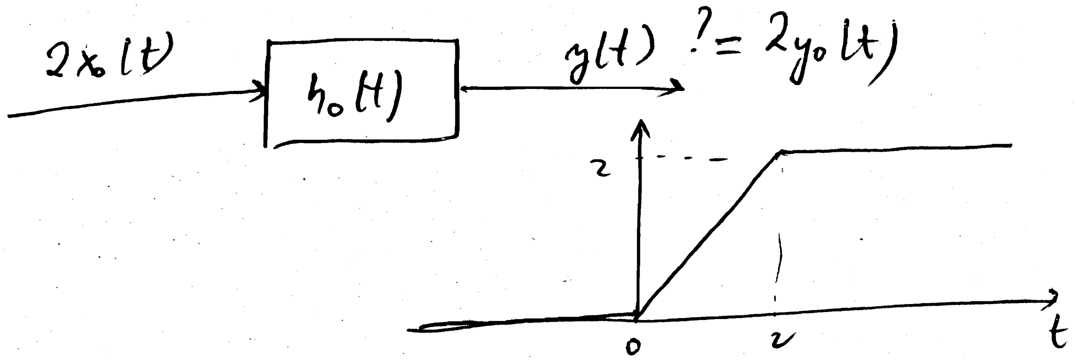
$\hookrightarrow \boxed{h_1[6] = 0}$. $h_1[7] = 0 \dots$

$$\boxed{h_1[n] = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ 3 & n = 1 \,\&\, 2 \\ 2 & n = 3 \\ 1 & n = 4 \\ 0 & n \geq 5 \end{cases}}$$

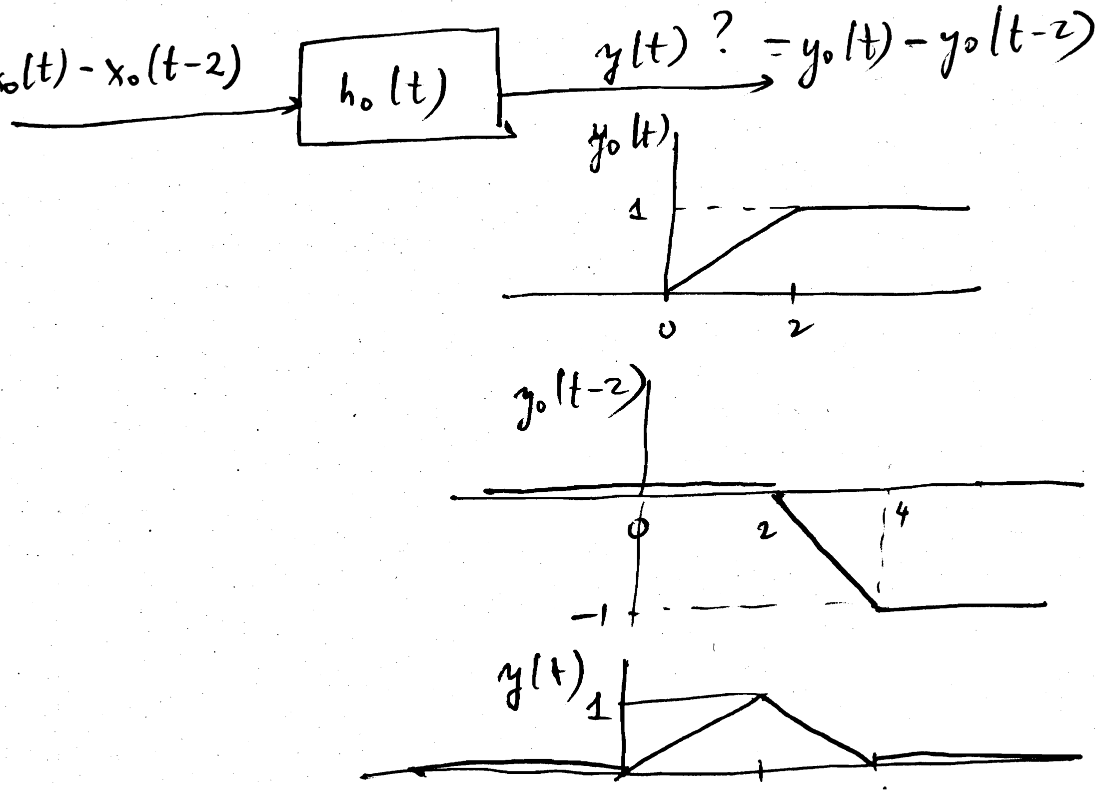
2.47


Linear Time Invariant
$x_0(t)$
$h_0(t)$
$y_0(t)$
1
0
2
t

(a)


$2x_0(t)$
$h_0(t)$
$y(t) \stackrel{?}{=} 2y_0(t)$
2
0
2
t

(b)


$x_0(t) - x_0(t-2)$
$h_0(t)$
$y(t) \stackrel{?}{=} y_0(t) - y_0(t-2)$
$y_0(t)$
1
0
2
$y_0(t-2)$
0
2
4
-1
$y(t)$
1

2.22

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

a) $x(t) = e^{-\alpha t} u(t)$; $h(t) = e^{-\beta t} u(t)$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} d\lambda \; x(\lambda)\, h(t-\lambda)$$

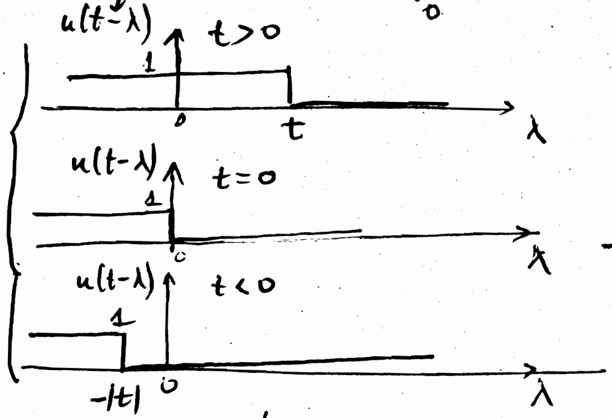
$$= \int_{-\infty}^{\infty} d\lambda \; e^{-\alpha\lambda} u(\lambda)\, e^{-\beta(t-\lambda)} u(t-\lambda)$$

Factors not involving λ can be brought outside the integral

$$= e^{-\beta t} \int_{-\infty}^{\infty} d\lambda \; e^{-(\alpha-\beta)\lambda} \underbrace{u(\lambda)}_{0 \;\lambda<0} u(t-\lambda)$$

$$= e^{-\beta t} \int_{0}^{\infty} d\lambda \; e^{-(\alpha-\beta)\lambda} u(t-\lambda)$$

$u(t-\lambda)$



$\longrightarrow$ ①

$\boxed{\alpha \neq \beta}$ $\longrightarrow$ ②

$\longrightarrow$ ③

$y(t) =$ ($\alpha \neq \beta$)

① ($t>0$): $e^{-\beta t} \int_0^t d\lambda \; e^{-(\alpha-\beta)\lambda} = e^{-\beta t} \dfrac{\left[e^{-(\alpha-\beta)t} - 1\right]}{-(\alpha-\beta)} = \dfrac{e^{-\beta t} - e^{-\alpha t}}{\alpha - \beta}$

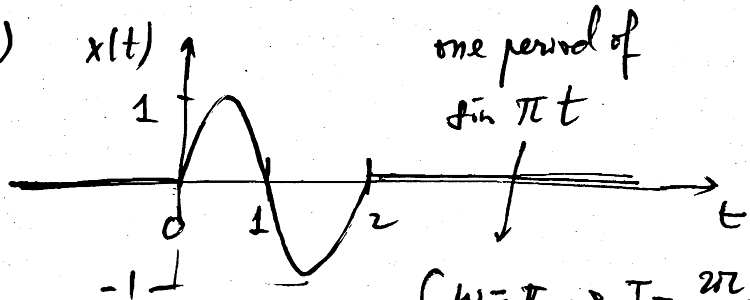
② ($t=0$): $e^{-\beta t} \int_0^0 d\lambda \; e^{-(\alpha-\beta)\lambda} = 0$

③ ($t<0$): 0

$$y(t) \underset{\alpha=\beta}{=} \begin{cases} ① \; t>0 : & e^{-\beta t}\int_0^t d\lambda \; 1 = t e^{-\beta t} \\ ② \; t=0 & \\ ③ \; t<0 & \end{cases} \Bigg\} \; 0$$

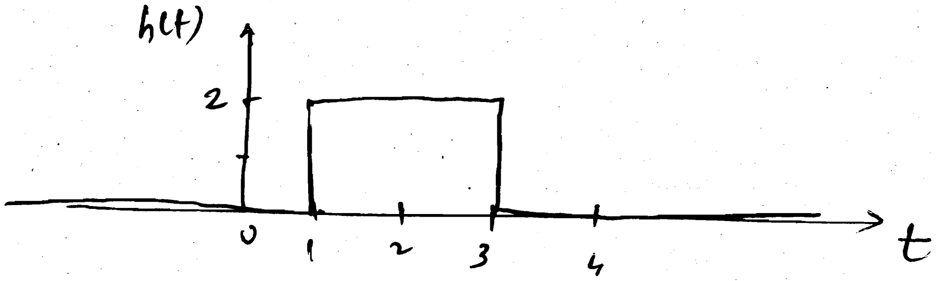
(b) Do it

(c)



one period of $\sin \pi t$

$(\omega = \pi \rightarrow T = \frac{2\pi}{\omega} = 2s)$



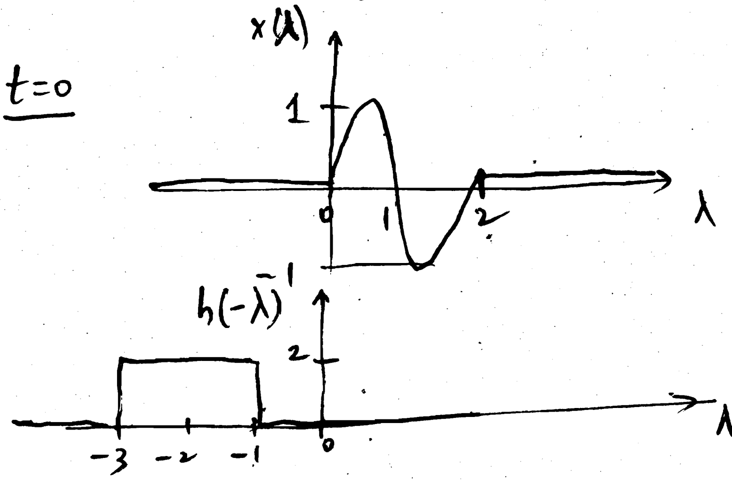
Two methods for calculating the convolution

- Graphics & integrals
- Use math & integrals

$$\begin{cases} x(t) = \sin \pi t \,[u(t) - u(t-2)] \\ h(t) = 2[u(t-1) - u(t-3)] \end{cases}$$

Convolution: Graphics & integrals : $y(t) = \int_{-\infty}^{\infty} d\lambda \; x(\lambda)\, h(t-\lambda)$ (by definition)

t=0



$$y(t) = \int_{-\infty}^{\infty} d\lambda \; 0 = 0$$

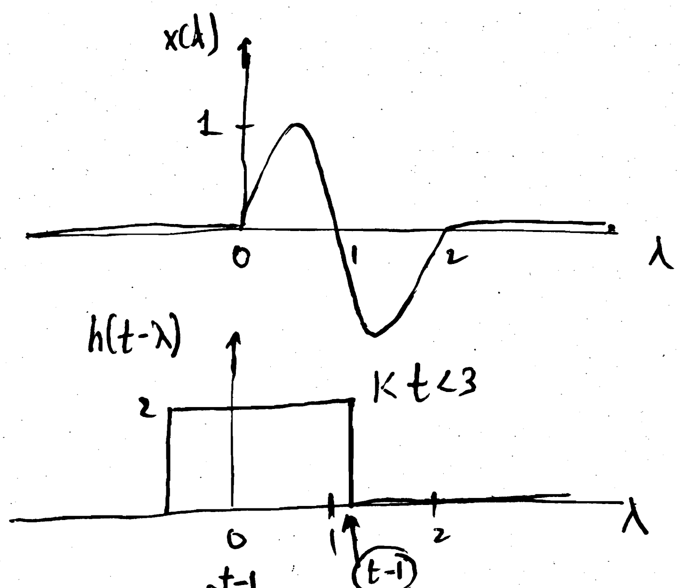
t<0 : $h(t-\lambda)$ is $h(-\lambda)$ translated further to the left! $\Big\} \rightarrow y(t) = 0$

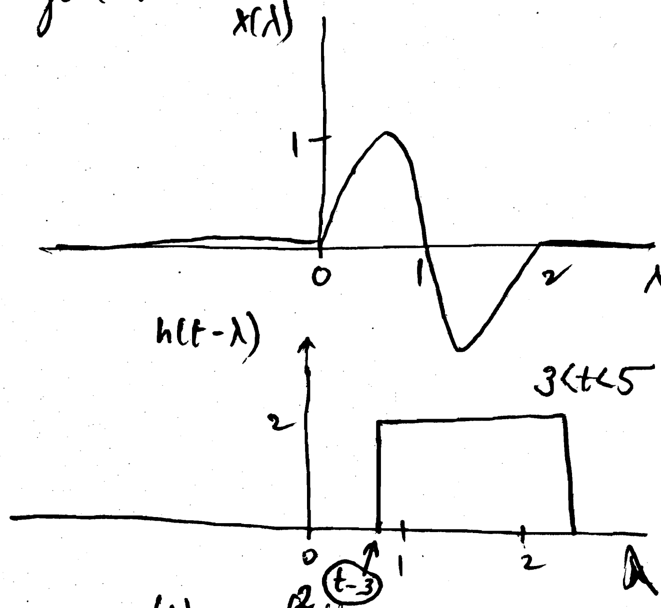
$t > 0$ we will have some overlap b/w $x(\lambda)$ and $h(t-\lambda)$, more specifically $t > 1$ and < 5

$$y(t) = \begin{cases} 0 & t \le 1 \;;\; t \ge 5 \\ \underbrace{\neq 0}_{?} & 1 < t < 5 \end{cases}$$

$$1<t<5: \quad y(t) = \int d\lambda \; 2 \sin \pi\lambda$$

What are the limits of the integral?





$$y(t) = 2\int_0^{t-1} d\lambda \; \sin \pi\lambda$$

$$= -2\left[\frac{\cos \pi\lambda}{\pi}\right]_0^{t-1}$$

$$= \frac{2}{\pi}\left(1 - \cos \pi(t-1)\right)$$

$$y(t) = 2\int_{t-3}^{2} d\lambda \; \sin \pi\lambda$$

$$= -2\left(\frac{\cos \pi\lambda}{\pi}\right)_{t-3}^{2}$$

$$= \frac{2}{\pi}\left(\cos \pi(t-3) - 1\right)$$

$$\rightarrow y(t) = \begin{cases} 0 & t \le 1 \text{ or } \ge 5 \\ \frac{2}{\pi}\left(1 - \cos \pi(t-1)\right) & 1 < t < 3 \\ \frac{2}{\pi}\left(\cos \pi(t-3) - 1\right) & 3 < t < 5 \end{cases}$$

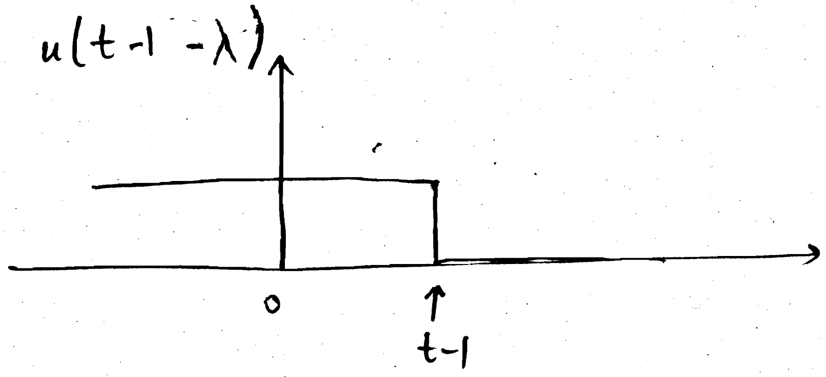
Convolution: using math & integrals:

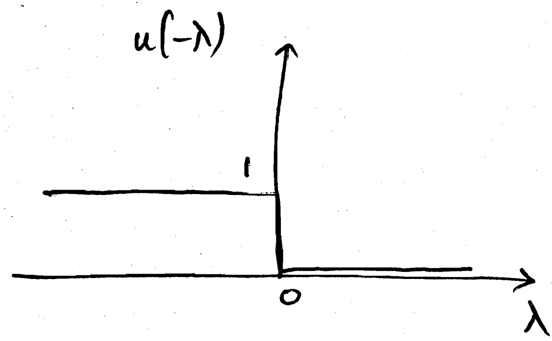
$$y(t) = 2\int_{-\infty}^{\infty} d\lambda \, \sin \pi\lambda \left[u(\lambda) - u(\lambda-2) \right] \cdot \left[u(t-\lambda-1) - u(t-\lambda-3) \right]$$

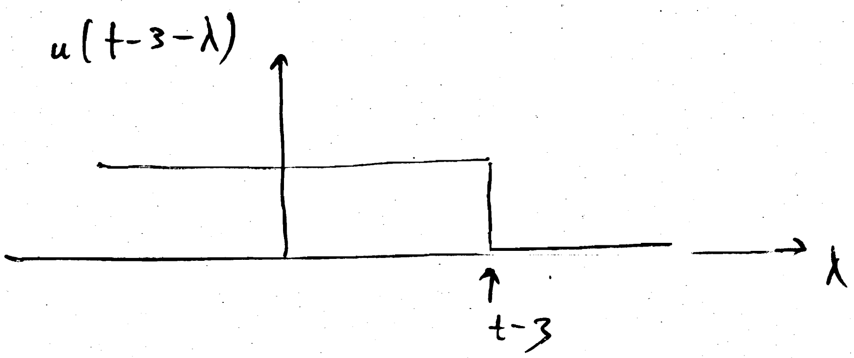
$$= 2\int_{-\infty}^{\infty} d\lambda \, \sin \pi\lambda \Big[u(\lambda)u(t-\lambda-1) - u(\lambda)u(t-\lambda-3) - u(\lambda-2)u(t-\lambda-1) + u(\lambda-2)u(t-\lambda-3) \Big]$$

$$= 2\int_{0}^{\infty} d\lambda \, \sin \pi\lambda \, u(t-\lambda-1) - 2\int_{0}^{\infty} d\lambda \, \sin \pi\lambda \, u(t-\lambda-3) - 2\int_{2}^{\infty} d\lambda \, \sin \pi\lambda \, u(t-\lambda-1) + 2\int_{2}^{\infty} d\lambda \, \sin \pi\lambda \, u(t-\lambda-3)$$

$$= 2\int_{0}^{t-1} d\lambda \, \sin \pi\lambda - 2\int_{0}^{t-3} d\lambda \, \sin \pi\lambda - 2\int_{2}^{t-1} d\lambda \, \sin \pi\lambda + 2\int_{2}^{t-3} d\lambda \, \sin \pi\lambda$$







$$y(t) = \underbrace{2\int_0^{t-1} d\lambda \sin\pi\lambda - 2\int_0^{t-3} d\lambda \sin\pi\lambda}\ \underline{-2\int_2^{t-1} d\lambda \sin\pi\lambda} + 2\int_2^{t-3} d\lambda \sin\pi\lambda .$$

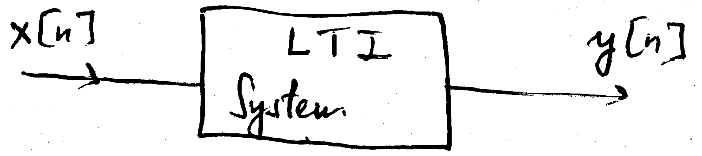
$$y(t) = \begin{cases} t<1 & 0 \\ t>1 & \begin{cases} 1<t<3 & = 2\int_0^{t-1} d\lambda \sin\pi\lambda - \cancel{\int_2^{t-1}} = -2\left[\dfrac{\cos\pi\lambda}{\pi}\right]_0^{t-1} = \dfrac{2}{\pi}\left(1-\cos\pi(t-1)\right) \\ 3<t<5 & = 2\int_{t-3}^{t-1} d\lambda \sin\pi\lambda - 2\int_2^{t-1} d\lambda \sin\pi\lambda . \end{cases} \end{cases}$$

$$= 2\int_{t-3}^{2} \sin\pi\lambda \, d\lambda = \frac{2}{\pi}\left(\cos\pi(t-3)-1\right)$$

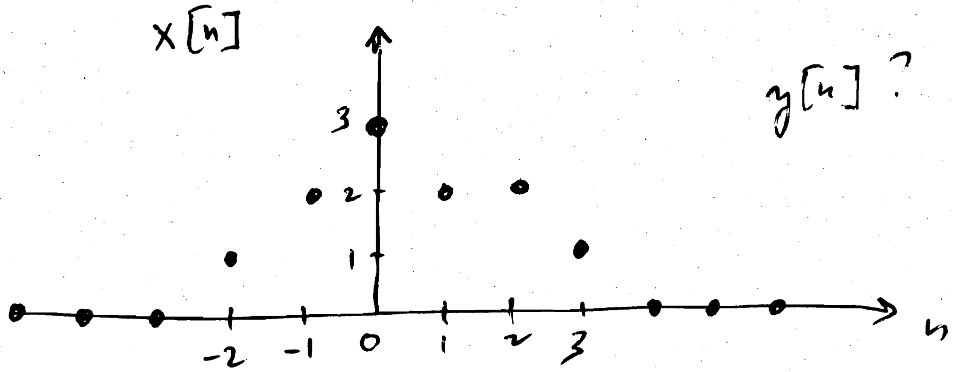
$3<t<5 \rightarrow \quad 2<t-1<4$

$0<t-3<2$

2.31



$$y[n] + 2y[n-1] = x[n] + 2x[n-2]$$



$y[n]$?

System initially at rest : $y[n] = 0$ when $x[n] = 0$

or $y[n] = 0 \quad n < -2$

$(y[-3] = y[-4] = \dots = 0)$

1) $y[-1] + 2\underbrace{y[-2]}_{?} = \underbrace{x[-1]}_{2} + 2\underbrace{x[-3]}_{0}$

2) $y[-2] + 2\underbrace{y[-3]}_{0} = \underbrace{x[-2]}_{1} + 2\underbrace{x[-4]}_{0} \rightarrow y[-2] = 1$

$y[-1] = 2 - 2x[1] = 0$

$y[-2]=1$
$y[-1]=0$
$y[0]=5$
$y[1]=-4$
$y[2]=16$
$y[3]=-27$
$y[4]=+58$
$y[5]=-114$
$y[6]=$

3) $y[0] + 2y[-1] = \underbrace{x[0]}_{3} + 2\underbrace{x[-2]}_{1} \rightarrow y[0] = 5$

4) $y[1] + 2y[0] = \underbrace{x[1]}_{2} + 2\underbrace{x[-1]}_{2} \rightarrow y[1] = -4$

5) $y[2] + 2y[1] = \underbrace{x[2]}_{2} + 2\underbrace{x[0]}_{3} \Rightarrow 16 = y[2]$

6) $y[3] + 2y[2] = \underbrace{x[3]} + 2\underbrace{x[1]} \rightarrow y[3] = -27$

7) $\quad y[4] + 2\underbrace{y[3]}_{-27} = \underbrace{x[4]}_{0} + 2\underbrace{x[2]}_{2} \rightarrow y[4] = 58$

8) $\quad y[5] + 2\underbrace{y[4]}_{58} = \underbrace{x[5]}_{0} + 2\underbrace{x[3]}_{1} \rightarrow y[5] = -114$

9) $\quad y[6] + 2y[5] = \underbrace{x[6]}_{0} + 2\underbrace{x[4]}_{0} \rightarrow y[6] = -2y[5]$

$$= -2(-114)$$

10) $\quad$ ____________________, $\quad y[7] = (-2)(-2)(-114)$

$$\boxed{y[n] = -114\,(-2)^{n-5} \quad (n \geq 5)}$$

$y[-2] = 1$; $y[-1] = 0$; $y[0] = 5$; $y[1] = -4$; $y[2] = 16$;
$y[3] = -27$; $y[4] = 58$.

2.40

a) $\quad x(t) \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ h(t)\,?\end{matrix}} \longrightarrow y(t) = \displaystyle\int_{-\infty}^{t} e^{-(t-\tau)}\, x(\tau-2)\, d\tau$

impulse response

In general: $\quad y(t) = x(t) * h(t) = \displaystyle\int_{-\infty}^{\infty} d\tau\; x(\tau)\, \underbrace{h(t-\tau)}$

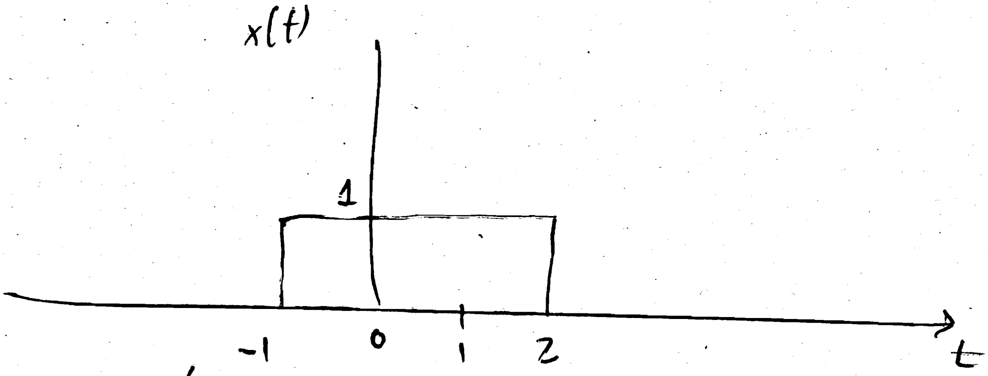
To identify $h(t-\tau)$ we need to have a change of variable
$x(\tau-2) \longrightarrow x(\tau')$: $\tau' \equiv \tau - 2$ or $d\tau' = d\tau$

$$\rightarrow y(t) = \int_{-\infty-2=-\infty}^{t-2} d\tau'\; e^{-(t-\tau'-2)}\, x(\tau') = \int_{-\infty}^{\infty} d\tau'\; \underbrace{u(t-2-\tau')\, e^{-(t-\tau'-2)}}_{h(t-\tau')}\, x(\tau')$$

$\longrightarrow h(t-\tau') = u(t-\tau'-2)\, e^{-(t-\tau'-2)} \rightarrow h(-\tau') = u(-\tau'-2)\, e^{-(-\tau'-2)}$

$h(\tau') = u(\tau'-2)\, e^{-(\tau'-2)} \longrightarrow \boxed{h(t) = u(t-2)\, e^{-(t-2)}}$

b) What is $y(t)$ if $x(t)$ is



$$x(t) = u(t+1) - u(t-2)$$

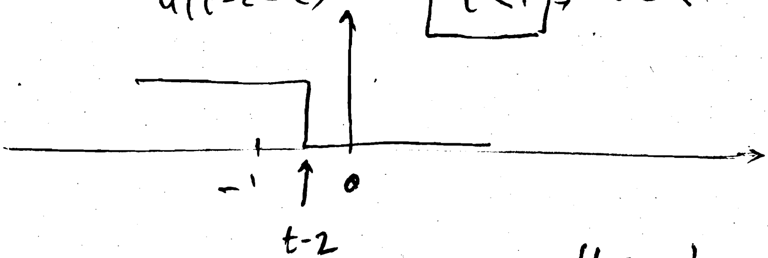
$$\rightarrow y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} d\tau \; x(\tau)\, h(t-\tau)$$

$$= \int_{-\infty}^{\infty} d\tau \; [u(\tau+1) - u(\tau-2)]\, u(t-\tau-2)\, e^{-(t-\tau-2)}$$

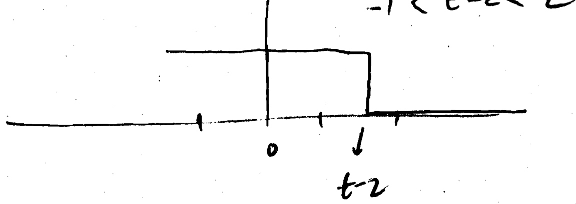
$$= \int_{-\infty}^{\infty} d\tau \; \underbrace{u(\tau+1)}\, u(t-2-\tau)\, e^{-(t-2-\tau)} - \int_{-\infty}^{\infty} d\tau \; u(\tau-2)\, u(t-2-\tau)\, e^{-(t-2-\tau)}$$

$$= \int_{-1}^{\infty} d\tau \; \underbrace{u(t-2-\tau)}\, e^{-(t-2-\tau)} - \int_{2}^{\infty} d\tau \; \underbrace{u(t-2-\tau)}\, e^{-(t-2-\tau)}$$

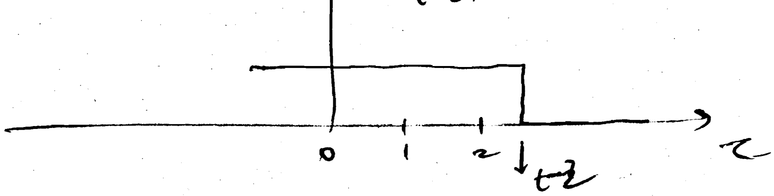
$u(t-2-\tau)$, $\boxed{t<1}$, $t-2<-1$



$u(t-2-\tau)$ $\boxed{1<t<4}$ $-1<t-2<2$



$u(t-2-\tau)$, $\boxed{t>4}$ $t-2>2$



$$
y(t) = \begin{cases}
t<1: & \int_{-1}^{t-2} d\tau \, e^{-(t-2-\tau)} = 0 \\
1<t<4: & \int_{-1}^{t-2} d\tau \, e^{-(t-2-\tau)} = e^{-(t-2)} \left[\frac{e^{\tau}}{1} \right]_{-1}^{t-2} = 1 - e^{-(t-1)} \\
t>4: & \int_{-1}^{t-2} d\tau \, e^{-(t-2-\tau)} - \int_{2}^{t-2} d\tau \, e^{-(t-2-\tau)}
\end{cases}
$$

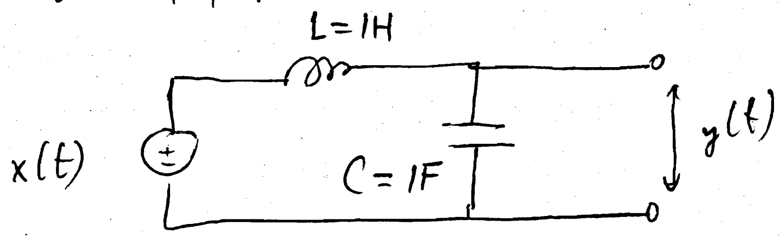
$$
= \int_{-1}^{2} d\tau \, e^{-(t-2-\tau)} = e^{-(t-2)} \left[e^{\tau} \right]_{-1}^{2}
$$

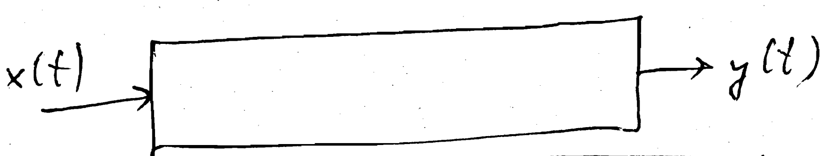
$$
= e^{-(t-2)} \left[e^{2} - e^{-1} \right]
$$

$$
= e^{-(t-4)} \left[1 - e^{-3} \right]
$$

Submit HW2 on 10/11/07

2.61



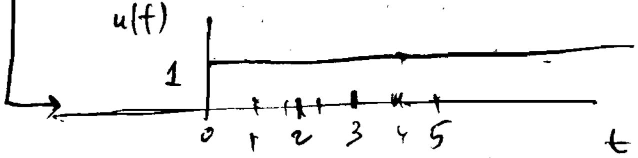


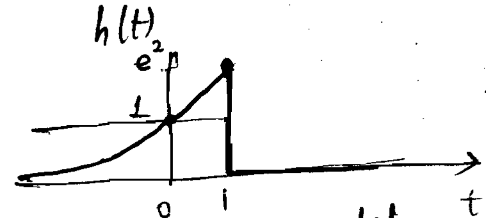
$$
\boxed{x(t) = LC \frac{d^2 y}{dt^2} + y(t)}
$$

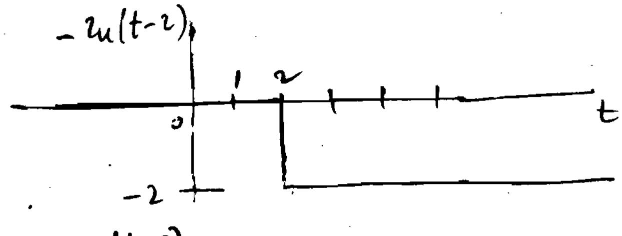
- - - -

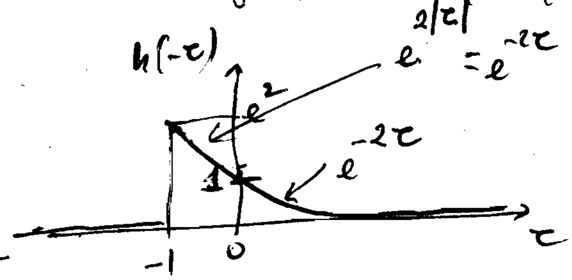
2.22 b)

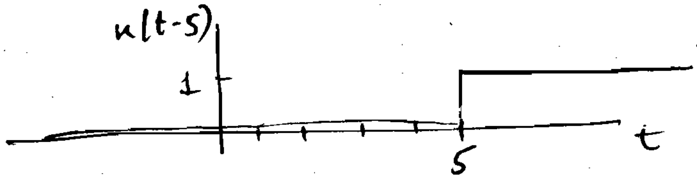
$$\left.\begin{aligned} x(t) &= u(t) - 2u(t-2) + u(t-5) \\ h(t) &= e^{2t}\,u(1-t) \end{aligned}\right\} \; y(t) = x * h$$

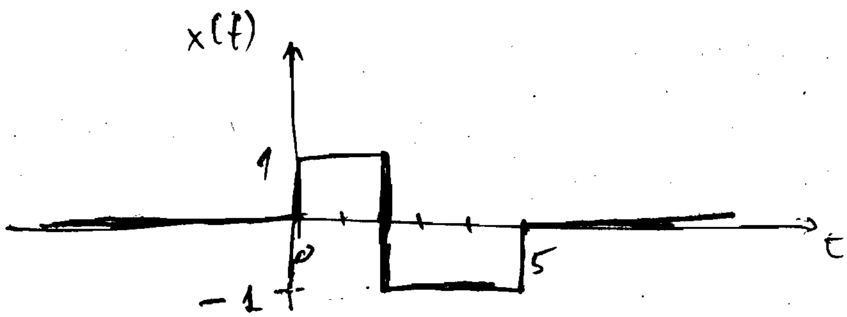




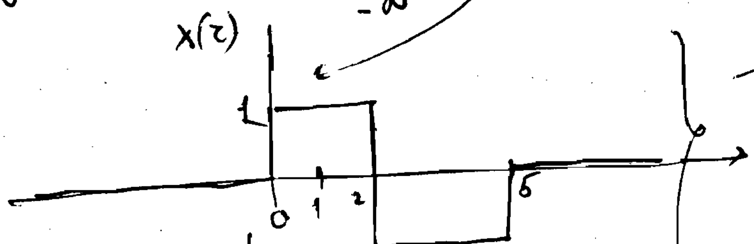






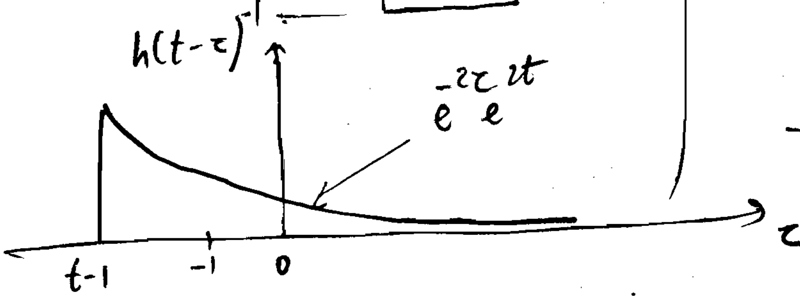


$$y(t) = x * h = \int_{-\infty}^{\infty} d\tau \; x(\tau)\, h(t-\tau)$$



→ Multiply them & integrate or (area under curve)

$t<0$



$$\rightarrow y(t) = e^{2t}\int_0^2 d\tau\, e^{-2\tau} - e^{2t}\int_2^5 d\tau\, e^{-2\tau}$$

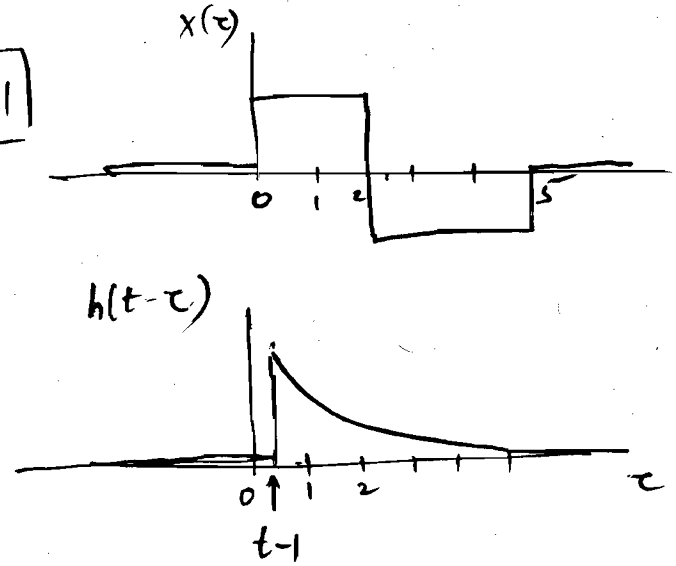
$$y(t) = e^{2t}\left[\int_0^2 d\tau\, e^{-2\tau} - \int_2^5 d\tau\, e^{-2\tau}\right]$$

$$y(t) = e^{2t}\left[\frac{e^{-4}-1}{-2} - \frac{e^{-10}-e^{-4}}{-2}\right] = \frac{1}{2}e^{2t}\left[1 - 2e^{-4} + e^{-10}\right]$$

For $1 \geq t > 0$: same integral for the convolution.

$$\Rightarrow y(t) = \frac{1}{2} e^{2t}\left[1 - 2e^{-4} + e^{-10}\right] \qquad t \leq 1$$

$\boxed{3 \geq t > 1}$

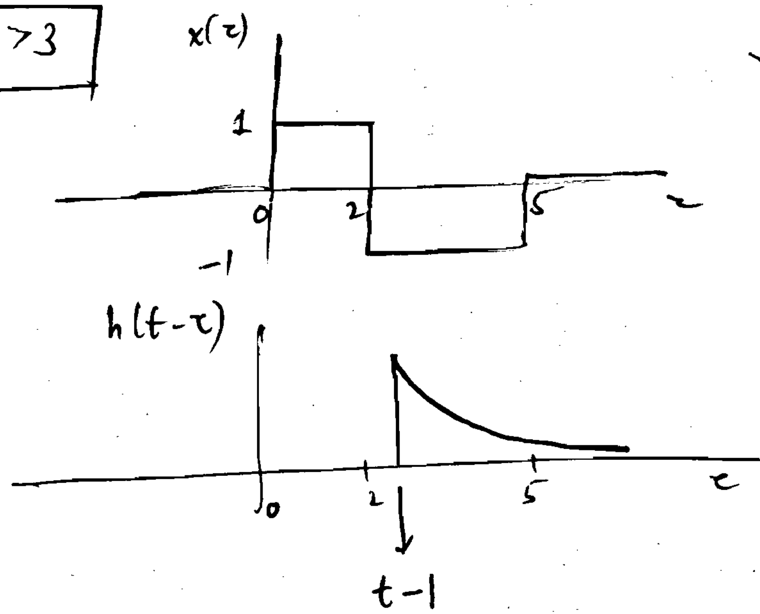


$$y = e^{2t}\left[\int_{t-1}^{2} d\tau\, e^{-2\tau} - \int_{2}^{5} d\tau\, e^{-2\tau}\right]$$

$$= e^{2t}\left[\frac{e^{-2(t-1)} - e^{-4}}{2} - \frac{e^{-4} - e^{-10}}{2}\right]$$

$$y(t) = \frac{1}{2} e^{2t}\left[e^{-2(t-1)} - 2e^{-4} + e^{-10}\right]$$

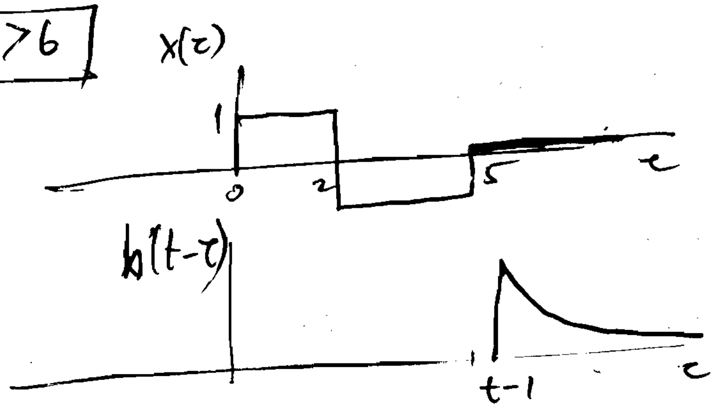
$\boxed{6 \geq t > 3}$



$$y(t) = e^{2t}\left[-\int_{t-1}^{5} e^{-2\tau}\, d\tau\right]$$

$$= \frac{1}{2} e^{2t}\left[e^{-10} - e^{-2(t-1)}\right]$$

$\boxed{t > 6}$



$$y(t) = 0$$

$$y(t) = \begin{cases} \frac{1}{2}e^{2t}\left[1 - 2e^{-4} + e^{-10}\right] & t \le 1 \\ \frac{1}{2}e^{2t}\left[e^{-2(t-1)} - 2e^{-4} + e^{-10}\right] & 1 < t \le 3 \\ \frac{1}{2}e^{2t}\left[e^{-10} - e^{-2(t-1)}\right] & 3 < t \le 6 \\ 0 & t > 6 \end{cases}$$

Math results:

1st order differential equation: $\longrightarrow$ $\boxed{\frac{dy}{dt} + ay = bx}$

(a, b are constants)

$\hookrightarrow$ General solution: $\boxed{y(t) = y(t_0)e^{-a(t-t_0)} + b\int_{t_0}^{t} d\lambda\, x(\lambda) e^{-a(t-\lambda)}}$

Why: using an integrating factor be^{at}: multiply this factor to both sides of the differential eq.

$$\underbrace{\frac{dy}{dt}be^{at} + ay\,be^{at}}_{\cancel{b}\frac{d}{dt}\left(ye^{at}\right)} = b^2 x e^{at}$$

$$\cancel{b}\frac{d}{dt}\left(ye^{at}\right) = b^{\cancel{2}} x e^{at}$$

Now integrate both sides of this eq. w.r.t. t. $\int_{t_0}^{t} dt$

$$ye^{at}\Big]_{t_0}^{t} = b\int_{t_0}^{t} dt\, xe^{at}$$

$$\underbrace{y(t)e^{at}}\, - \underbrace{y(t_0)e^{at_0}} = b\int_{t_0}^{t} d\lambda\, x(\lambda) e^{a\lambda}$$

$\downarrow$ divide by this factor. bring to RHS $\longrightarrow$ General solution shown above.