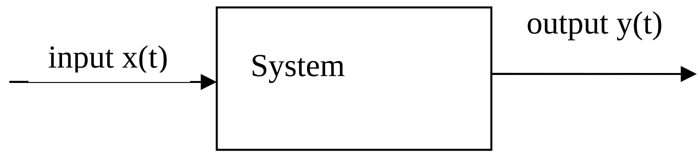
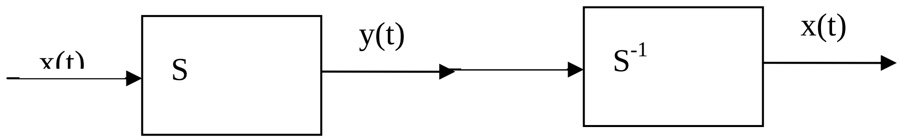
Systems: a system is represented with a box whose content we don’t know:
A system is identified by the equation that relates y(t) to x(t), e.g.:

$y(t)= 2x(t) - x^2(t+1)$
This is the “input-output equation”.

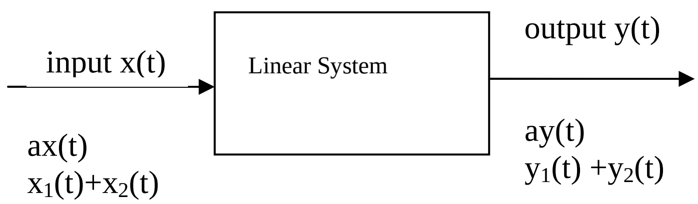


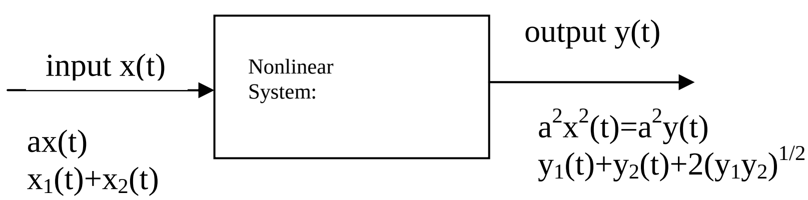
Properties:

1) **Memory:** has memory when current output depends on past inputs
2) **Invertibility:** when a system S has an inverse system S^{-1} such that

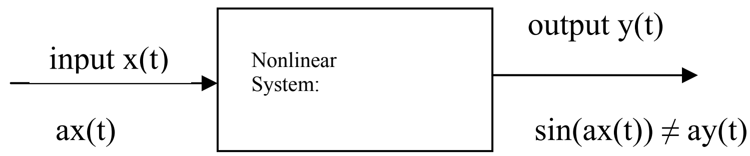


3) **Causality**: a system is causal if outputs only depend on current and or past inputs
4) **Stability**: a system is stable if small inputs will not produce extremely large outputs
5) **Time-invariance**: when a same input produces the same output regardless of when it is applied to the system
6) **Linearity**:

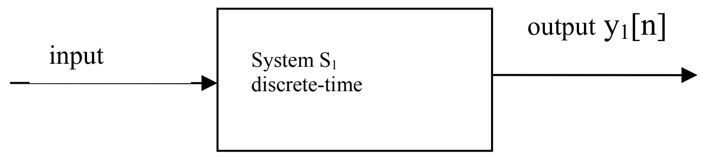




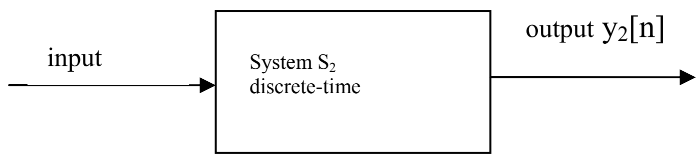
Another example of a non-linear system is



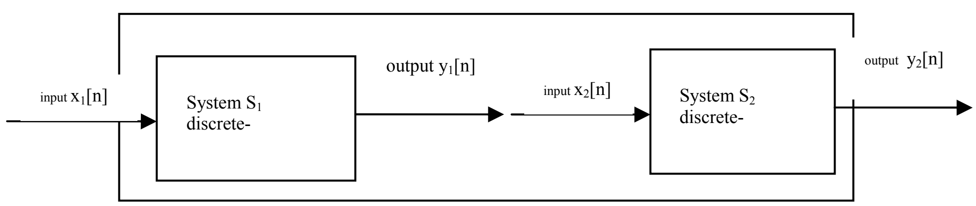
1.15

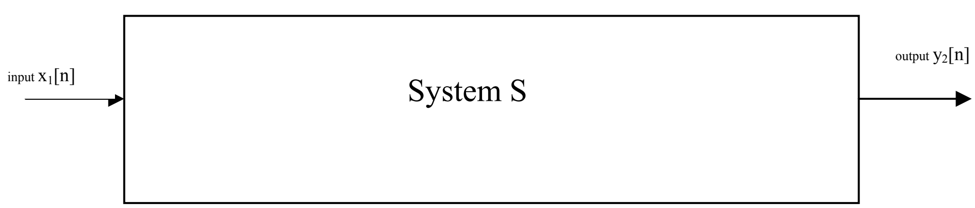


Input/output difference equation for S_1 is $y_1[n] = 2x_1[n] + 4\, x_1[n-1]$



Input/output difference equation for S_2 is $y_2[n]=x_2[n-2] + 1/2\ x_2[n-3]$
Combine S_1 with S_2 in series to obtain the combined system S, provide the input/output difference equation for S





$y_2[n]=x_2[n-2] + 1/2\ x_2[n-3]$
What is the relationship between x_2 and y_1? They are equal! (since S_1 and S_2 are connected in series, the output of the first system is the input to the second system)
$y_2[n]=y_1[n-2] + 1/2\ y_1[n-3]$; now using $y_1[n] = 2x_1[n] + 4\, x_1[n-1]$
$y_2[n]= 2x_1[n-2] + 4x_1[n-3] + x_1[n-3] + 2x_1[n-4] \quad \rightarrow \quad y_2[n]= = 2x_1[n-2] + 5x_1[n-3] + 2x_1[n-4]$
This is the input/output equation for the combined system S

1.16 A discrete-time system is given by its input-output equation as: $y[n] = x[n]\, x[n-2]$

a) Is the system memoryless? No, it depends on past inputs
b) What is the output when $x[n]=A\delta[n]$ (A is any real or complex number)?
$\delta[n]$ is centered at 0 (at 0 it is 1, everywhere else it is 0) ; $\delta[n-2]$ is centered at 2 (at 2 it is 1; everywhere else it is 0)
$y[n] = A\ \delta[n]\ A\ \delta[n-2] = 0$

c) Is the system invertible?
If it is, we should be able to find S^{-1} such that when combine in series with S, for each input x[n] we get back x[n]. Can we find such a system?, i.e. a system such that if x[n] x[n-2] is applied, it produces x[n]. No, this system is not invertible.

1.27 a) $y(t) = x(t-2) + x(2-t)$
1)Memory? Yes
2) Time invariance? let's shift the time by a: $t \rightarrow t+a$
$x(t+a-2) + x(2-t-a) = y(t+a)$; Yes it is
3) Linear? Yes
4) Causal?
$y(0)= x(-2) + x(2)$ not causal
$y(1)= x(-1) + x(1)$ causal
$y(2)= x(0)+ x(0)$ causal
System is causal for $t>=1$
5) Stable? Yes

Ch 2 Linear Time-Invariant Systems

<u>HW2:</u> 2.7; 2.8; 2.11; 2.17; 2.22; 2.24; 2.31; 2.40; 2.47; 2.61

<u>Convolution:</u> "$*$" is a mathematical operation that allows us to calulate $y[n]$ from $x[n]$ and $h[n]$

$$x[n] \longrightarrow \boxed{\begin{matrix}\text{System}\\ h[n]\end{matrix}} \longrightarrow y[n]$$

impulse-response: $h[n]$ is the output for an impulse input by the system. The impulse-response is used to characterize the system. $\delta[n] \longrightarrow \boxed{\text{System}} \longrightarrow h[n]$

$$y[n] = x[n] * h[n]$$ ("$y[n]$ is the convolution of $x[n]$ with $h[n]$")

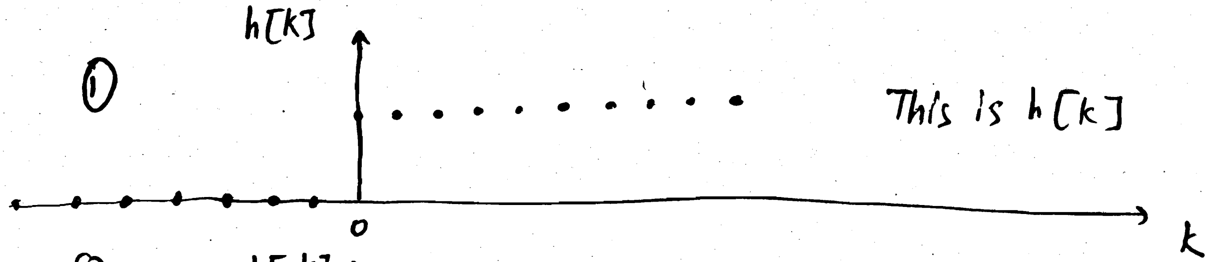
$$= \sum_{k=-\infty}^{\infty} x[k] \underbrace{.}_{\text{product}} h[n-k]$$

<u>Example:</u> Find $y[n]$ from $x[n] = \underbrace{\alpha^n}_{\text{a constant to the power of } n} . \underbrace{u[n]}_{\text{step function}}$ & $h[n] = u[n]$

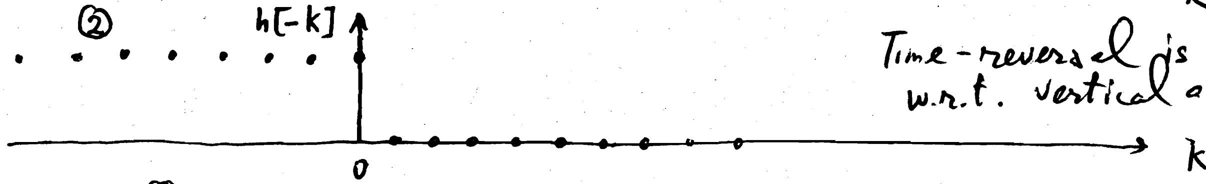
This system is characterized by the impulse response $h[n]$ = a step function :→ e.g. a switch. We are asked to find the output produced by the switch for an input that starts when the switch is on and increases as α^n.
We will get the output $y[n]$ by doing the convolution between $\underline{x[n]}$ and $\underline{h[n]}$:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

We need to multiply $\underbrace{x[k]}_{\substack{\downarrow \\ x[n] \text{ and} \\ \text{doing } n\to k}}$ to $\underbrace{h[n-k]}_{\substack{\downarrow \\ \text{time reversal on } h[k] \\ \text{and a time shift of } n}}$

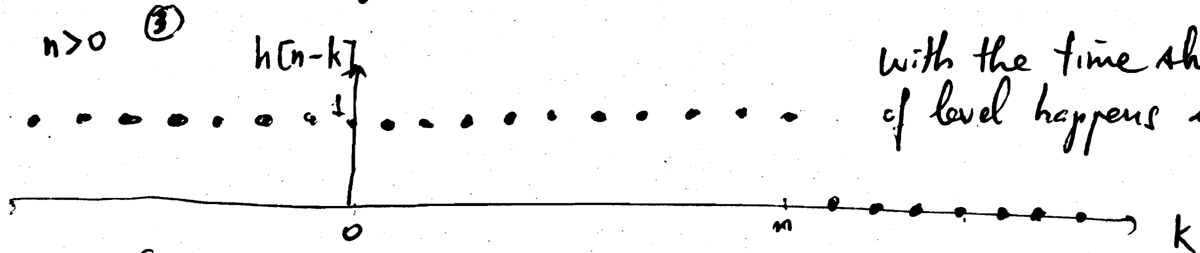


This is $h[k]$



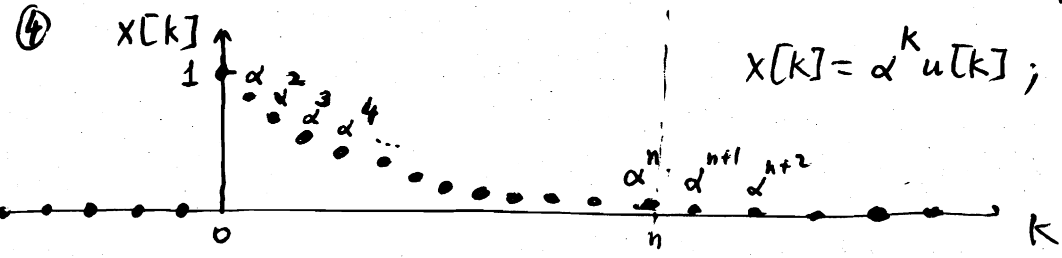
Time-reversal is a reflection w.r.t. vertical axis

$n>0$

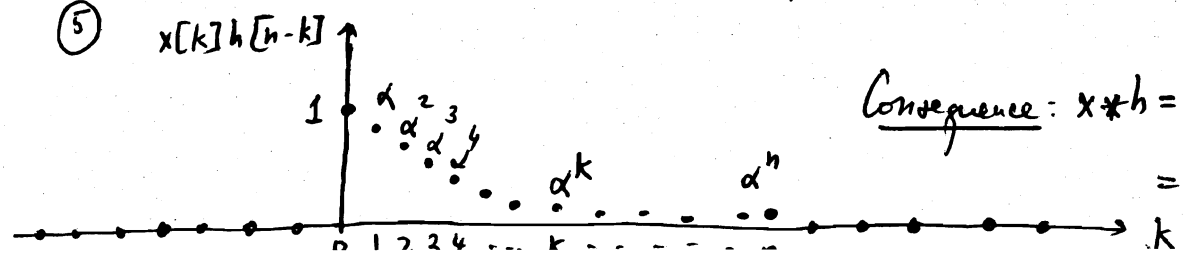


with the time shift: the change of level happens at n

$\boxed{\alpha<1}$



$x[k] = \alpha^k u[k]$;



Consequence: $x*h = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=0}^{n} \alpha^k$

$$n>0\ ;\ \alpha<1\ :\ y[n]=x[n]*h[n]=\sum_{k=0}^{n}\alpha^k$$

Math result

Sum of a geometric series: $\displaystyle\sum_{k=0}^{n}\alpha^k=\alpha^0+\alpha^1+\alpha^2+\alpha^3+\dots\alpha^{n-1}+\alpha^n=S$

$$\rightarrow\ \underbrace{\alpha^1+\alpha^2+\alpha^3+\dots\ \alpha^{n-1}+\alpha^n}=S-1$$

$$\alpha\underbrace{(1+\alpha+\alpha^2+\dots\alpha^{n-2}+\alpha^{n-1})}_{S-\alpha^n}=S-1$$

$$\alpha(S-\alpha^n)=S-1$$

solve for S: $\quad 1-\alpha^{n+1}=S-\alpha S$

$$S=\frac{1-\alpha^{n+1}}{1-\alpha}$$

$$\boxed{\sum_{k=0}^{n}\alpha^k=\frac{1-\alpha^{n+1}}{1-\alpha}}$$

$$\rightarrow\ n>0\ ;\ \alpha<1\qquad y[n]=x[n]*h[n]=\frac{1-\alpha^{n+1}}{1-\alpha}\ \text{(output)}$$

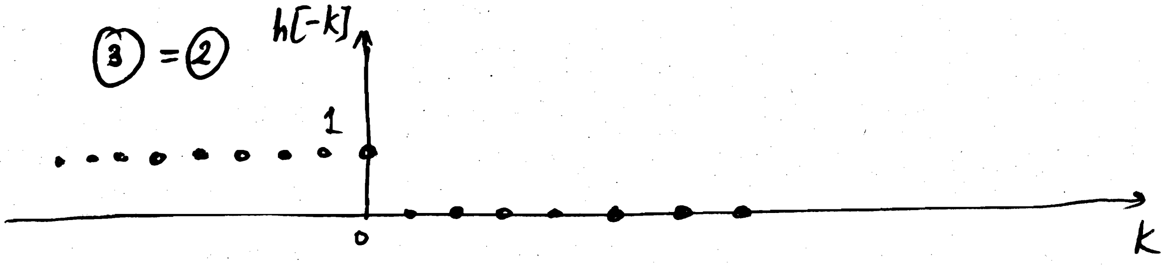
$$x[n]=\alpha^n u[n]\longrightarrow\boxed{\begin{array}{c}\text{System}\\ h[n]=u[n]\end{array}}\longrightarrow\begin{cases}n>0 & y[n]=\dfrac{1-\alpha^{n+1}}{1-\alpha}\\ n=0 & \\ n<0 & \end{cases}$$

What is $y[n]$ for $n=0$ and $n<0$?

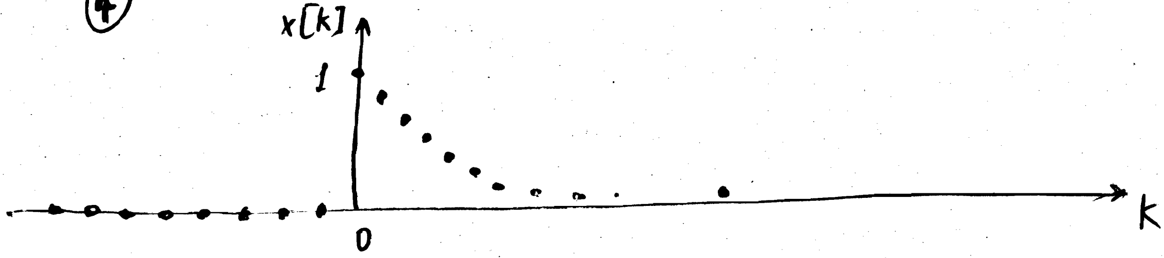
1) $y[0]=x[n]*h[n]\big/_{n=0}$

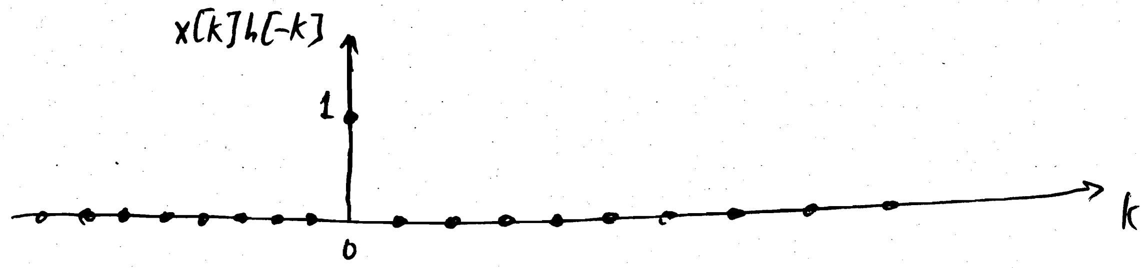
Repeating previous graphs for $n = 0$

③ = ②



④



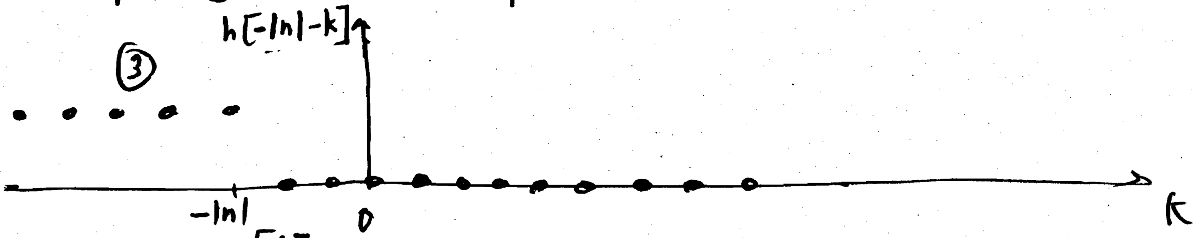


$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[-k] = 1$$

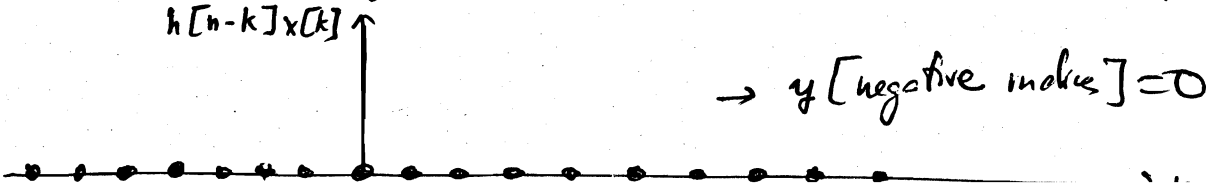
2) $y[\text{negative indices}] = ?$

Repeating previous graphs for $n<0$

③

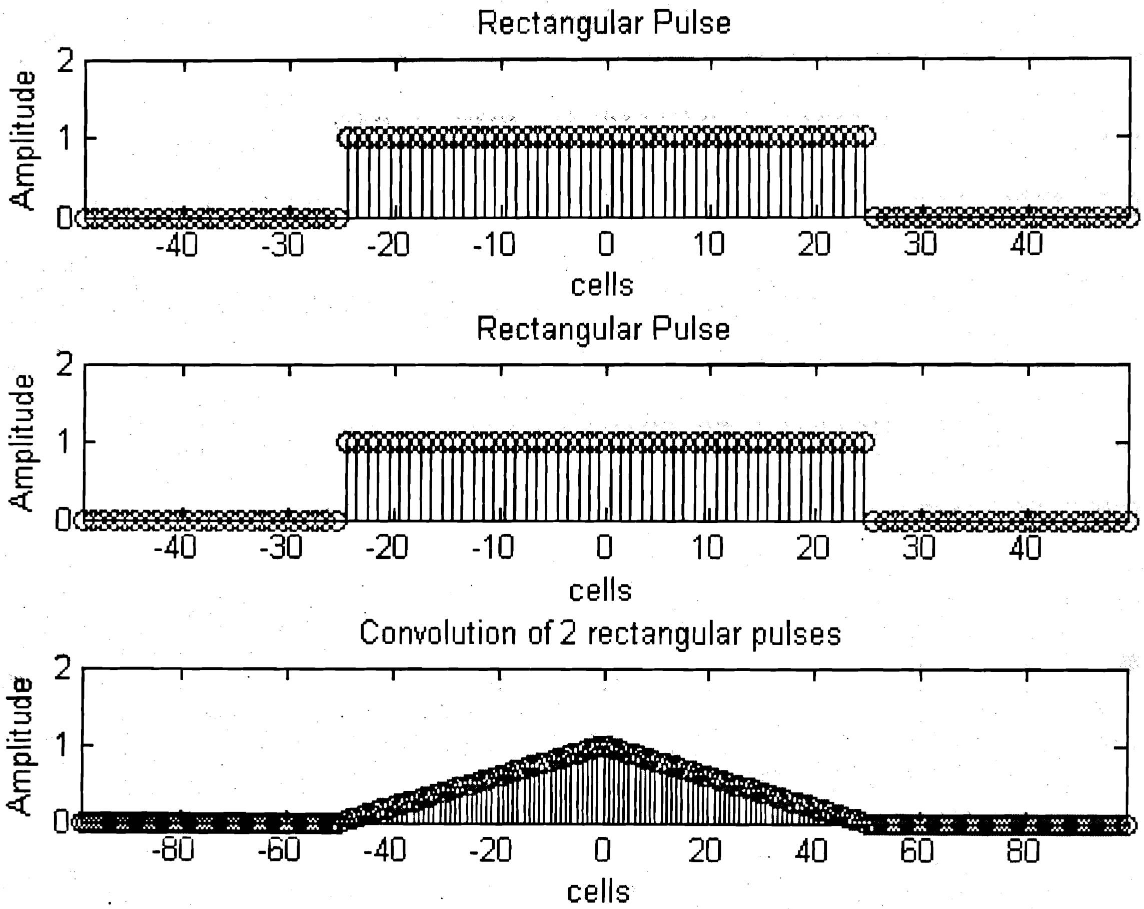


④



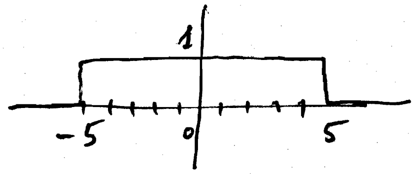
$\rightarrow y[\text{negative indices}] = 0$

$$x[n] = \alpha^n u[n] \longrightarrow \boxed{\begin{array}{c}\text{System}\\ h[n] = u[n]\end{array}} \xrightarrow{y[n]} \left\{\begin{array}{ll} 0 & n<0 \\ 1 & n=0 \\ \dfrac{1-\alpha^{n+1}}{1-\alpha} & n>0 \end{array}\right\} = \frac{1-\alpha^{n+1}}{1-\alpha}\, u[n]$$



Let's work out this Matlab result manually:

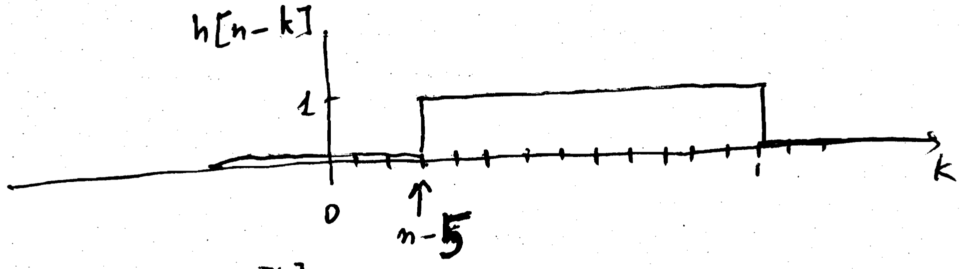
$x[n] = u[n+5] - u[n-5]$

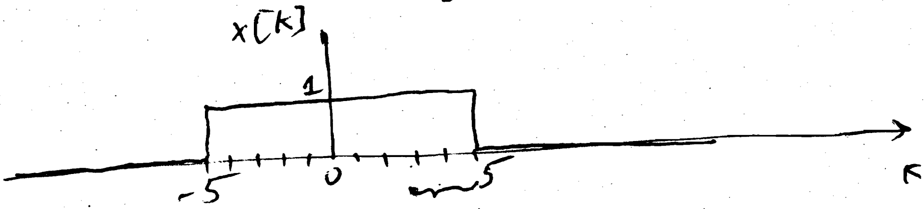


$h[n] = x[n]$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \begin{cases} & n>0 \\ & n=0 \\ & n<0 \end{cases}$$

$n>0$: $h[n-k]$: reflecting $x[k]$ w.r.t. vertical axis then shift n steps to the right





$$y[n] = \sum_{n-5}^{5} 1$$

$= 5-(n-5)+1$

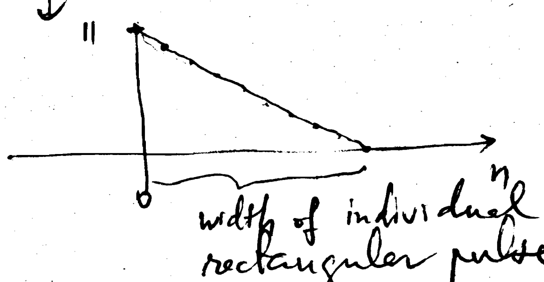
$= 11-n$

($n>0$; $n-k<5$ or $n<k+5$

$-5<k<5 \rightarrow 0<k+5<10$

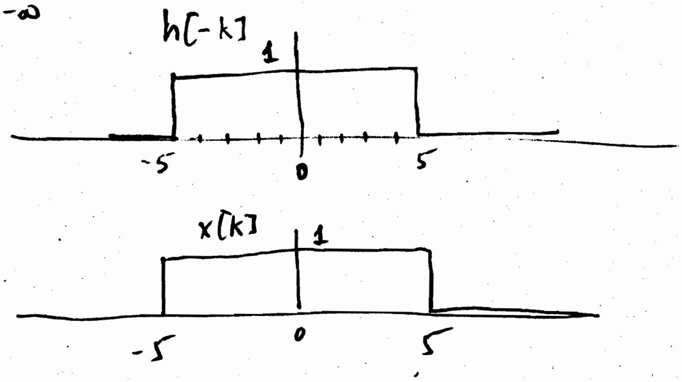
$\rightarrow 0<n<10$! since if $n>10$ there is no overlap b/w these two pulses.).

And 10, which is the width of individual rectangular pulses, is half the width of ~~the~~ their convolution! (see Matlab figure)



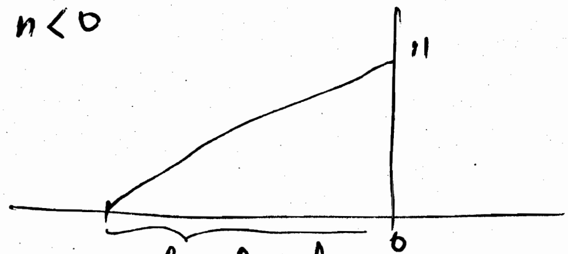
↓ will get triangular pulse with reflection (see MatLab figure).

$\underline{n=0}$: $y[0] = \sum_{k=-\infty}^{\infty} h[-k]\, x[k] = \sum_{-5}^{5} 1 = 5-(-5)+1 = 11$



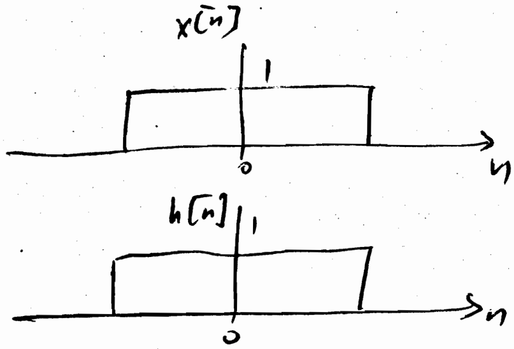
This is the max. value for the convolution!

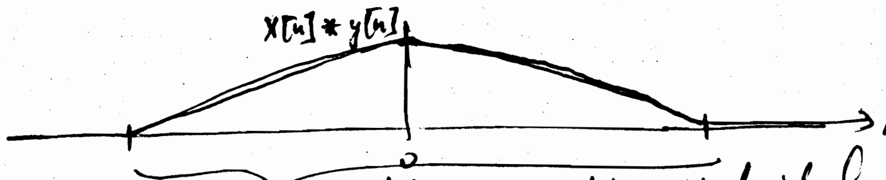
Due to the symmetry:



width of ind. rectangular pulse

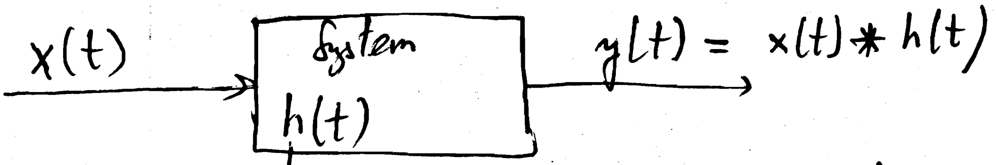
→





width = 2 width of individual rectangular pulses.

Convolution of continuous time signals:



↳ impulse-response is the output to $x(t) = \delta(t)$:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} d\lambda\; x(\lambda)\, h(t-\lambda) = \int_{-\infty}^{\infty} d\lambda\; x(t-\lambda)\, h(\lambda)$$

change of variable

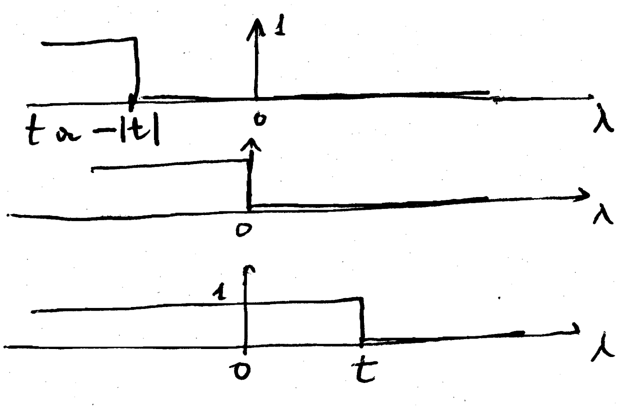
$$= h(t) * x(t)$$

- Convolution operation is commutative!
  ↳ Time-reversal and shift can be applied to either h or x leading to the same result
- λ is the continuous-time counterpart of the discrete-time index k.

<u>Example</u>: find $y(t)$ if $x(t) = e^{-at} u(t)$ and $h(t) = u(t)$

$$y(t) = \int_{-\infty}^{\infty} d\lambda\; e^{-a\lambda} \underbrace{u(\lambda)}_{\substack{0 \text{ for } \lambda<0 \\ 1 \text{ for } \lambda \geq 0}} u(t-\lambda) = \int_{0}^{\infty} d\lambda\; e^{-a\lambda}\, u(t-\lambda)$$

$$u(t-\lambda) = \begin{cases} & t<0 \\ & t=0 \\ & t>0 \end{cases}$$



$$\rightarrow y(t) = \begin{cases} t<0 & 0 \\ t=0 & 0 \\ t>0 : & \int_0^t d\lambda\, e^{-a\lambda} \cdot 1 = \left[\frac{e^{-a\lambda}}{-a}\right]_0^t = \frac{1-e^{-at}}{a} \end{cases}$$

$$\boxed{y(t) = \frac{1-e^{-at}}{a}\, u(t)}$$

Try doing this convolution for different signals and systems (HW2).