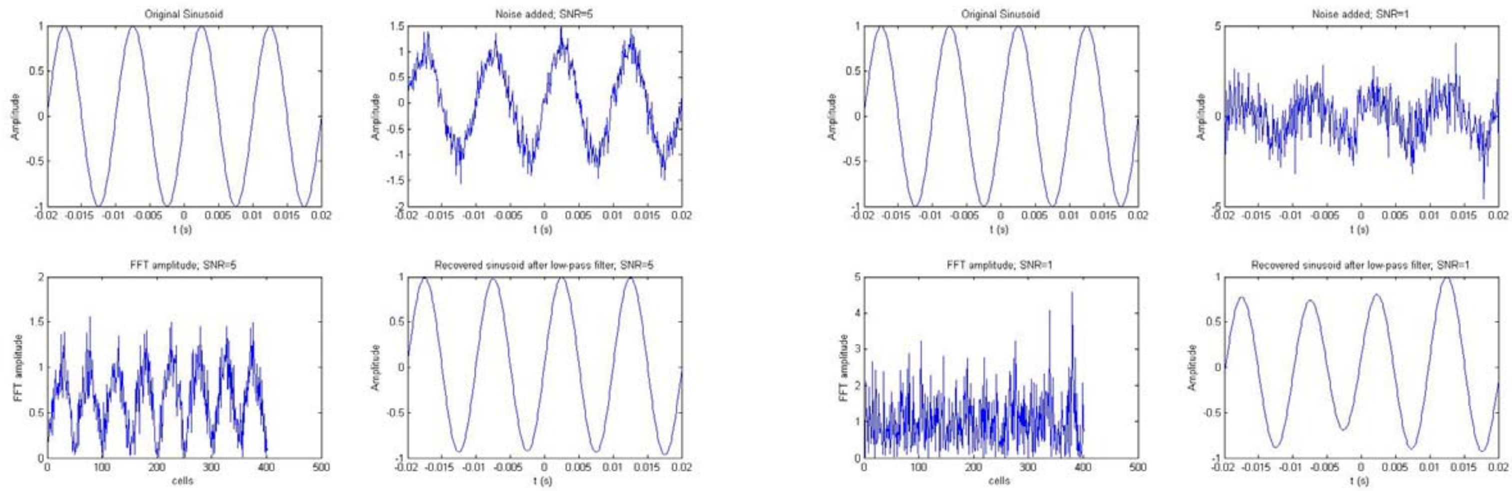

The Fourier Transform plays an essential role in the signal processing application of noise elimination

Fall ‘07

Engin 321

Linear System Theory I

Class #5554, 3 credits

Tu. & Th. 10:00-11:15am, S-3-126

Prof. Tomas Materdey

The concepts of signals and systems arise in all areas of technology. This course provides an introduction to the analysis of linear systems in the time- and frequency-domain, e.g., what is the output of a system if we know the input and the transfer function of a system. Students will use the convolution theorem, and the continuous-time and discrete-time Fourier and Laplace transforms in different applications. They will also learn to write simple Matlab™ codes as related to signal processing as illustrated in the figures above and below. The prerequisites are Math 140 (Calculus I) and Engin 232 (Circuit Analysis II) or by permission of instructor.

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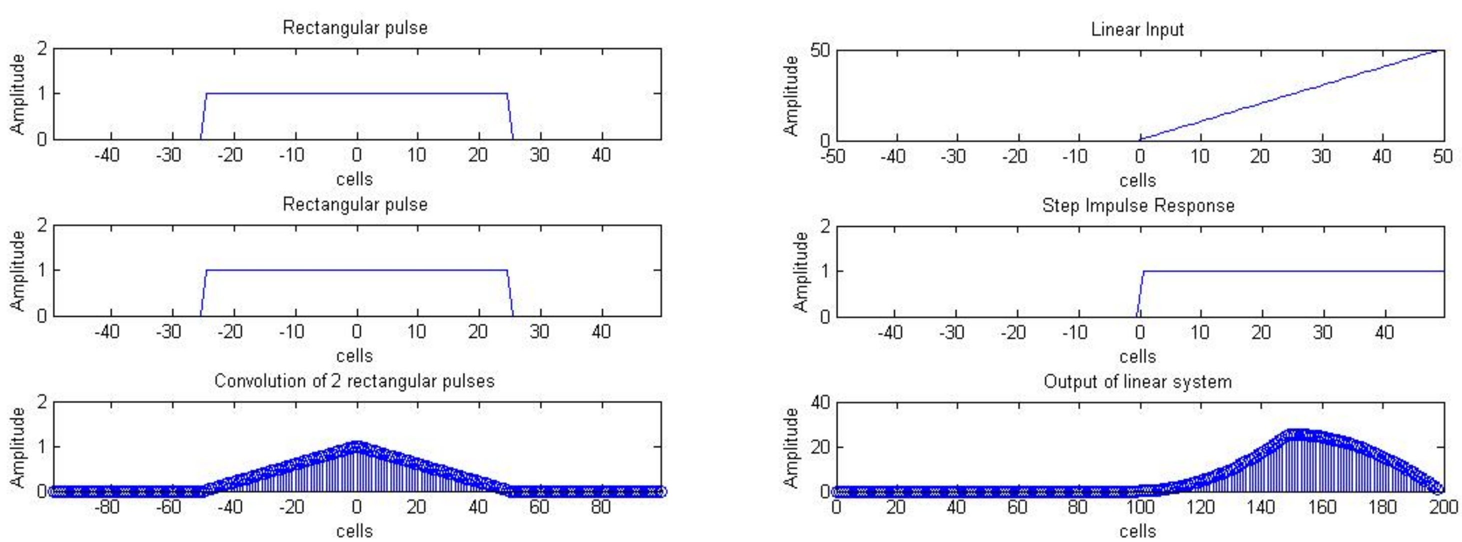
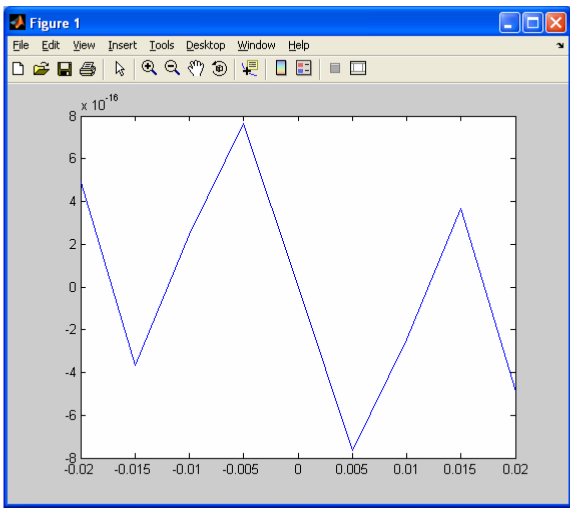


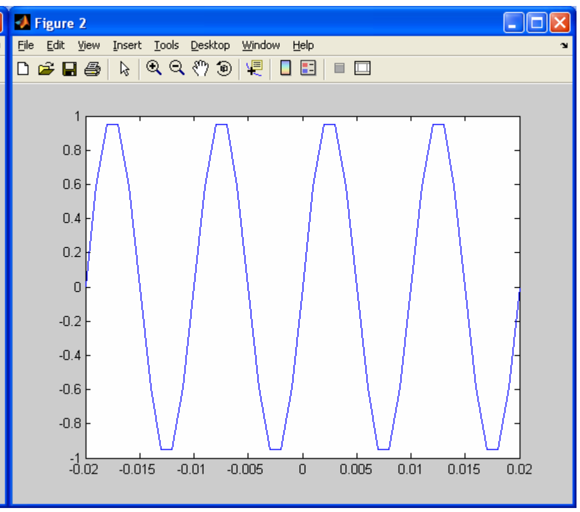
Applications of the convolution to obtaining output signals of linear systems of known impulse response.
Graphs generated using Matlab

Introduction to signals and noise elimination using Matlab

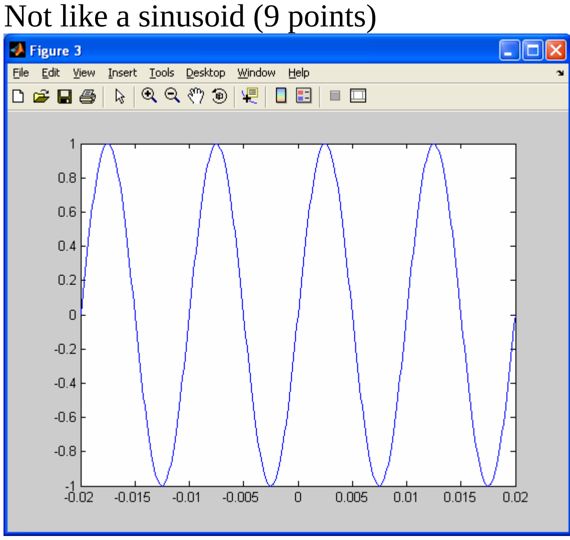
```
%This code will generate a signal, add noise, show the Fourier transform,
%then reconstruct the signal.

% Sinusoid generation
t=-0.02:0.005:0.02; %time series t (sec): we use 9  points
freq=100; %period is 0.01s
f=sin(2*pi*freq*t);
figure(1),plot(t,f)
t1=-0.02:0.001:0.02;%time series t1 (sec): we use 41 points
f1=sin(2*pi*freq*t1);
figure(2), plot(t1,f1)
t2=-0.02:0.0001:0.02;%time series t2 (sec): we use 401 points
f2=sin(2*pi*freq*t2);
figure(3), plot(t2,f2)
```



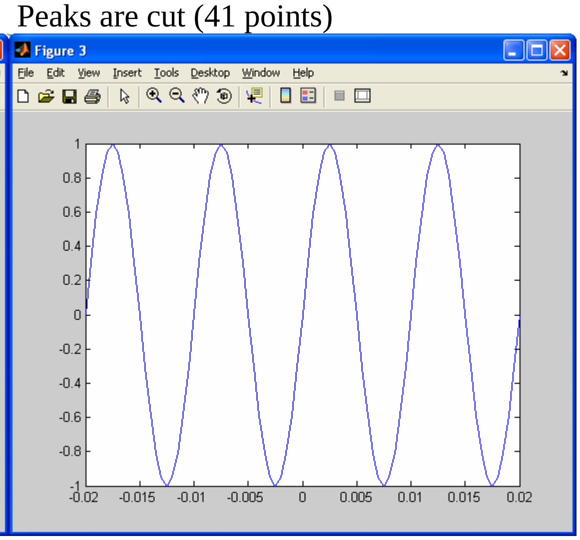
Not like a sinusoid (9 points)



Peaks are cut (41 points)



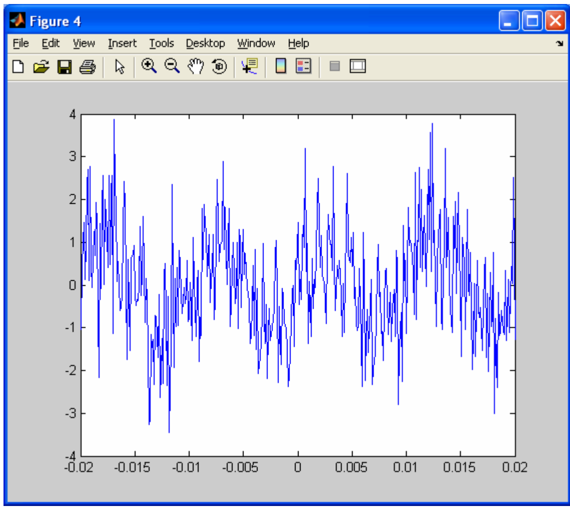
Peaks are included (401 points)

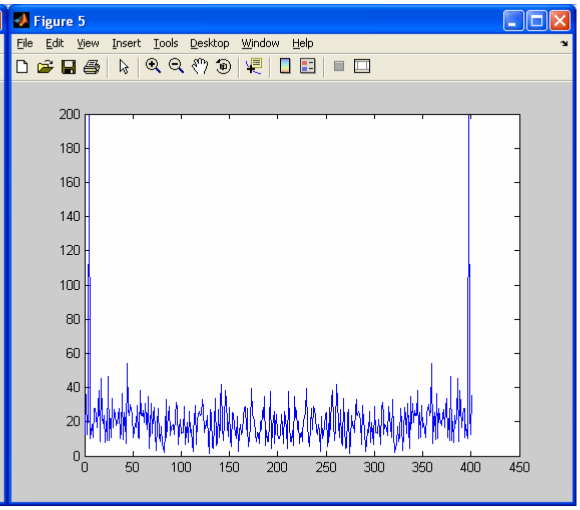


Peaks are included (81 points)*

*Although 81 points shows a good sinusoid; 401 points is needed for the noise elimination shown below.

```
%add noise to the sinusoid in figure 3
f2n=f2+1*randn(1,length(t2)); %randn gives a gaussian noise (white noise)
figure(4), plot(t2,f2n)

%show frequency spectrum for the signal in figure 4 (with noise) using fft
ff2n=fft(f2n);
figure(5), plot(abs(ff2n))
```



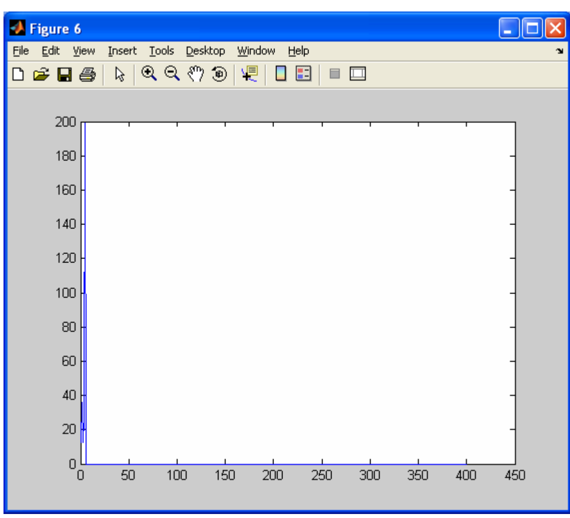


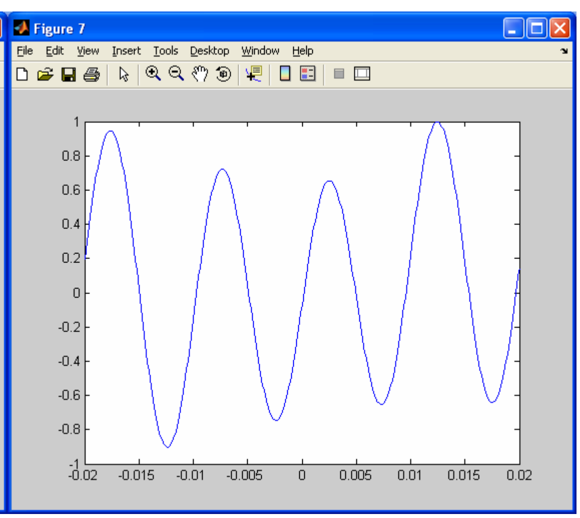
Added white noise of amplitude 1

Spectrum* of signal with noise in figure 4

*Spectrum: graph with different frequencies with their amplitudes
Noise oscillatates a lot, we cut out the high-frequency portion of the spectrum before doing inverse Fourier Transform to recover the signal without noise (we apply a low-pass filter)

```
%do lowpass filter
band=floor(length(t2)/100)+1;%10;
filt_one=ones(1,band);
filt_zero=zeros(1,length(t2)-band);
filt=[filt_one filt_zero];
ff2n=ff2n.*filt;
figure(6), plot(abs(ff2n))

%show inverse Fourier Transform
iff2n=ifft(ff2n);
figure(7) ,plot(t2,real(iff2n)/max(real(iff2n)))
```



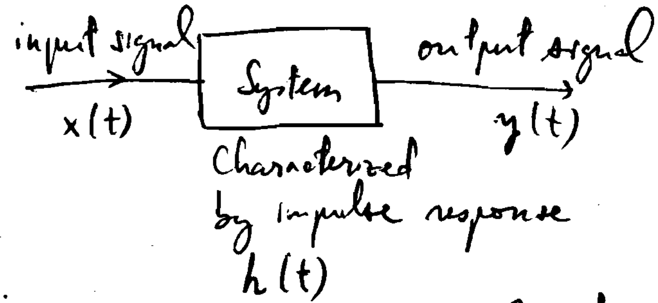


spectrum with low-pass filter applied

Recovered signal without noise (Inverse FFT)

Linear System Theory I

Introduction to Signals & Systems



Signal: a function of time

- continuous-time: $x(t) = \sin(2\pi f t)$ (t: time variable), $x(t) = \exp(-at)$ etc.
- discrete-time: $x[n] = \sin[2\pi f n]$, $x[n] = \exp[-an]$ etc. (n: time index)

$t = n\,\Delta t$

t: time variable, n: time index, Δt: time increment

Properties:

1) Behavior under a transformation of the time variable:

$$t \longrightarrow t' = \underbrace{a}_{\text{scaling factor}} t + \underbrace{b}_{\text{time shift}} \qquad \text{(linear transformation)}$$

2) Periodicity (e.g. sinusoids are periodic):

$$x(t+T) = x(t) \qquad (T\text{: period of the signal})$$

$$x[n+N] = x[n] \qquad (N\text{: period of the discrete-time signal})$$

→ What is the period T for $x(t) = e^{j\omega t}$?

j → complex factor ($j = \sqrt{-1}$: imaginary number)

ω → angular frequency

t → time variable

In general $\underbrace{e^{j\theta}}_{\text{complex number}} = \underbrace{\cos\theta}_{\text{real part}} + \underbrace{j\sin\theta}_{\text{imaginary part}}$

$$\begin{cases} Re\left[e^{j\theta}\right] = \cos\theta \\ Im\left[e^{j\theta}\right] = \sin\theta \end{cases}$$

If $x(t)$ is periodic : $x(t+T) = x(t)$

$$\underbrace{e^{j\omega(t+T)}} = e^{j\omega t}$$

$$\cancel{e^{j\omega t}} \cdot e^{j\omega T} = \cancel{e^{j\omega t}} \rightarrow \underbrace{e^{j\omega T}}_{\text{complex}} = \underbrace{1}_{\text{real}}$$

$$\rightarrow \cos\omega T + \underbrace{j\sin\omega T}_{=0} = 1$$

$$\begin{cases} \sin\omega T = 0 \\ \cos\omega T = 1 \end{cases} \rightarrow \omega T = 2\pi \quad \boxed{T = \frac{2\pi}{\omega}}$$

→ What is the period T for $x(t) = e^{j3\pi t}$?

$$T = \frac{2\pi}{\omega} = \frac{2\cancel{\pi}}{3\cancel{\pi}} = \frac{2}{3}\,s$$

→ What is the period N for $e^{j3\pi n}$?

- Discrete-time signals: $t = n\Delta t$: n are integers → N also integers

- $$\underbrace{e^{j3\pi(n+N)}}_{\cancel{e^{j3\pi n}} \cdot e^{j3\pi N}} = e^{j3\pi n}$$

$$\cancel{e^{j3\pi n}} \cdot e^{j3\pi N} = \cancel{e^{j3\pi n}} \rightarrow e^{j3\pi N} = 1 \begin{cases} \cos 3\pi N = 1 \\ \sin 3\pi N = 0 \end{cases}$$

$$3\pi N = 2\pi, 4\pi, 6\pi, 8\pi, \text{etc} -$$
$$= m2\pi \quad (m = \text{an integer})$$

$$\rightarrow N = m\frac{2\cancel{\pi}}{3\cancel{\pi}} = \frac{2}{3}m \quad \left(: \frac{2}{3}; \frac{4}{3}; 2 \ldots\right)$$

$$\boxed{N=2}$$

→ What is the period T for e^{j3t} ? $T = \frac{2\pi}{3}$ s

→ What is the period N for e^{j3n} ?

$3N = m\,2\pi$: m integer

$N = m\frac{2\pi}{3}$: never happens to be an integer.

→ Not periodic.

HW1: 1.10; 1.11; 1.36; 1.39;
1.15; 1.16; 1.27; 1.28; 1.31; 1.46 } Due 9/13

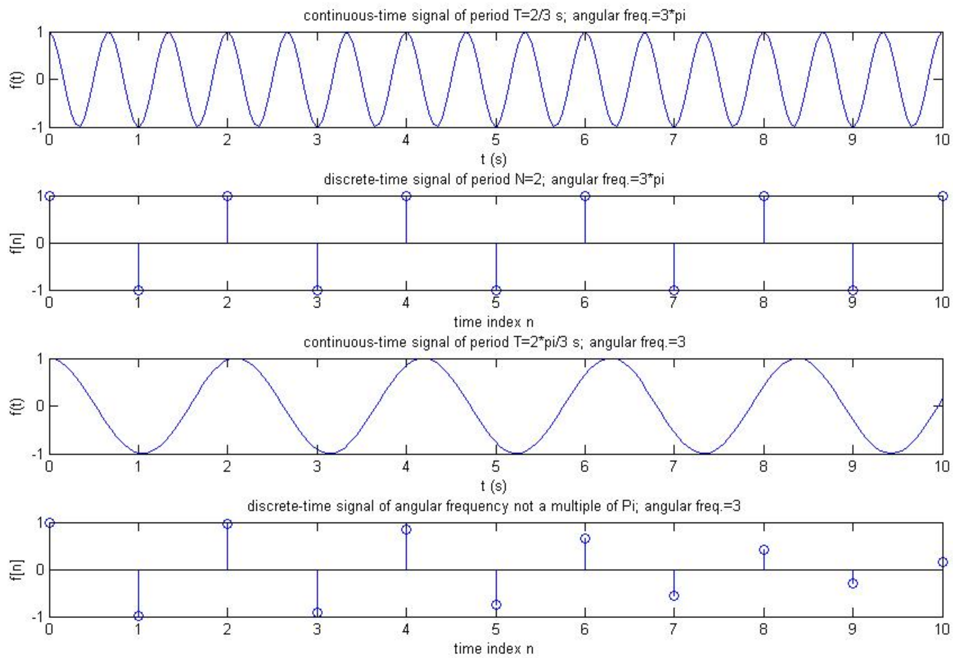
Differences between a continuous-time and a discrete-time signals

Sinusoids	Continuous–time: always periodic
	Discrete-time: not always periodic (only when ω is a multiple of π)*

* $e^{j\,3\pi n}$ was periodic (N=2); e^{j3n} was not periodic

To visualize this first difference between continuous-time and discrete-time signals, write Matlab code and see figure 1 below.

```
%Periodic and non-periodic signals (continuous-time and discrete-time)
%9/11/07

%Continuous and discrete-time periodic sinusoids
tc=0:.05:10;
td=0:1:10;
omega=3*pi;
figure(1), subplot(4,1,1), plot(tc,cos(omega*tc))
title('continuous-time signal of period T=2/3 s; angular freq.=3*pi')
xlabel('t (s)');
ylabel('f(t)');
subplot(4,1,2), stem(td,cos(omega*td))
title('discrete-time signal of period N=2; angular freq.=3*pi')
xlabel('time index n');
ylabel('f[n]');

%Continuous periodic and discrete-time non-periodic sinusoids
tc=0:.05:10;
td=0:1:10;
omega=3; %angular frequency is not a multiple of Pi
figure(1), subplot(4,1,3), plot(tc,cos(omega*tc))
title('continuous-time signal of period T=2*pi/3 s; angular freq.=3')
xlabel('t (s)');
ylabel('f(t)');
subplot(4,1,4), stem(td,cos(omega*td))
title('discrete-time signal of angular frequency not a multiple of Pi;
angular freq.=3')
xlabel('time index n');
ylabel('f[n]');
```



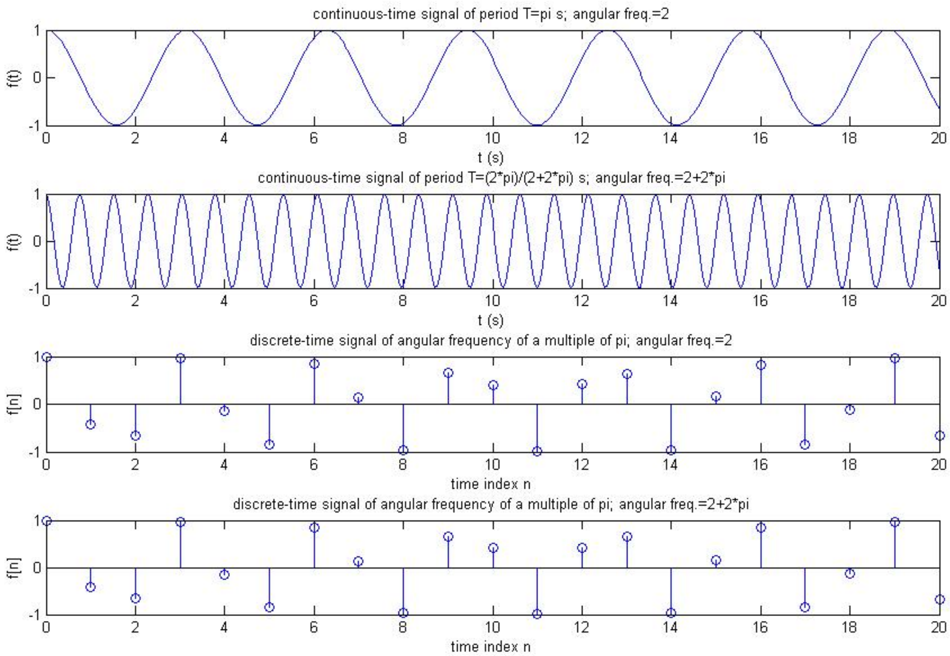
The second difference between continuous-time and discrete-time signals is e^{j2t} (angular frequency 2) is not the same as $e^{j(2+2\pi)t}$(angular frequency is $2+2\pi$); while e^{j2n} (discrete-time signal with angular frequency 2) is the same as $e^{j(2+2\pi)n}$(discrete-time signal with angular frequency is $2+2\pi$);

$$e^{j(2+2\pi)n} = e^{jn}e^{j2\pi n} = e^{jn}$$

To visualize this second difference, write Matlab code and see figure 2 below:

```
% Continuous-time signals with omega2=omega1+2*pi
tc=0:.05:20;
omega1=2;
figure(2), subplot(4,1,1), plot(tc, cos(omega1*tc))
title('continuous-time signal of period T=pi s; angular freq.=2')
xlabel('t (s)');
ylabel('f(t)');
omega2=2+2*pi;
subplot(4,1,2), plot(tc, cos(omega2*tc))
title('continuous-time signal of period T=(2*pi)/(2+2*pi) s; angular
freq.=2+2*pi')
xlabel('t (s)');
ylabel('f(t)');

% Discrete-time signals with omega2=omega1+2*pi
td=0:1:20;
omega1=2;
figure(2), subplot(4,1,3), stem(td, cos(omega1*td))
```

```
title('discrete-time signal of angular frequency of a multiple of pi;
angular freq.=2')
xlabel('time index n');
ylabel('f[n]');
omega2=2+2*pi;
subplot(4,1,4), stem(td, cos(omega2*td))
title('discrete-time signal of angular frequency of a multiple of pi;
angular freq.=2+2*pi')
xlabel('time index n');
ylabel('f[n]');
```



1.10 Fundamental period of $x(t)=2*\cos(10t+1) - \sin(4t-1)$

This is a combination of two sinusoids:
1)First sinusoid: $x_1(t+T_1)=x_1(t)$ or $10T_1=n_1\, 2\pi$ or $T_1=n_1 2\pi/10=\{2\pi/10, 4\pi/10, 6\pi/10, 8\pi/10, \pi, ...\}$
2)Second sinusoid: $x_2(t+T_2)=x_2(t)$ or $4T_2=n_2\, 2\pi$ or $T_2=n_2 2\pi/4=\{\pi/2, \pi, 3\pi/2.....\}$

When is the first time both signals repeat themselves? $T_1=T_2= \pi$, this is the period of the combined signal x(t)

1.11 Fundamental period of $x[n]=1 + \exp(j\, 4\pi n/7) - \exp(j\, 2\pi n/5)$
This is a combination of two complex exponentials (discrete-time)
1)First signal: $x_1[n+N_1]=x_1[n]$ or $4\pi N_1/7=2\pi n_1$ or $N_1=n_1\, 7/2=\{7, 14, 21, 28, 35, 42, ...\}$
2) Second signal: $x_2[n+N_2]=x_2[n]$ or $N_2=n_2\, 5=\{5, 10, 15, 20, 25, 30, 35, 40, ...\}$

Both signals come back to the same value at $N_1=N_2=35$, this is the period of the combined signal!

1.36 $x(t) = \exp(j\omega_0 t)$; fundamental frequency ω_0, fundamental period $T_0=2\pi/\omega_0$
Look at the discrete time signal with t=nT (T is the time increment):
$x[n]= \exp(j\, \omega_0 nT)$
(a) Show that x[n] is periodic if and only if T/T_0 is a rational number
A discrete-time sinusoid is periodic when the angular frequency is a multiple of π:
$\omega_0 T=k\,\pi$ or $2\,\pi\, T/T_0 = k\,\pi$ (k integer) or $T/T_0 = k/2$, this is a rational number since k is an integer

(b) Suppose x[n] is periodic, i.e., $T/T_0= p/q$ (p and q are integers); what is the fundamental period N_0 of x[n]?
$x[n+N_0]=x[n]$ or $\exp(j\, \omega_0 T(n+N_0)) = \exp(j\, \omega_0 Tn)$ or
$\exp(j\, \omega_0 Tn)\exp(j\, \omega_0 TN_0) = \exp(j\, \omega_0 Tn)$ or
$\exp(j\, \omega_0 T\, N_0) = 1$ or $\omega_0 N_0 T = 2\pi m$ (m integer) or $2\pi\, N_0\, T/T_0= 2\pi m$ (m integer)
or $N_0\, p/q = m$ or $\mathbf{N_0 = m\, q/p}$

For example: suppose q=7, p=3 $\rightarrow N_0= m\, 7/3$, fundamental period is 7 (remember problem 1.11)
For example: suppose q=28, p =12 $\rightarrow N_0= m\, 28/12 = m\, 7/3$ (irreducible fraction), fundemental period is 7

In conclusion N_0 is q is q/p is irreducible; mathematically $N_0 = q/\gcd(q,p)$
gcd is the greatest common divisor (check: in the case of q=28; p=12: we have $28/\gcd(28,12) = 28/4=7$

Since the fundamental period is $N_0 = q/\gcd(q,p)$; the fundamental frequency ω_0' for the discrete-time x[n] is
$\omega_0'=2\pi/N_0 = 2\pi\, q/\gcd(q,p)$

c) Assuming x[n] is periodic, i.e. p/q is a fractional number, how many periods T_0 of the original signal are needed to obtain samples that form one period of x[n]?

One period of x[n] in time index is N_0; in time is N_0T. How many periods of T_0 we have in this?

N_0T/T_0= q/gcd(q,p) T/T_0 = q/gcd(q,p) p/q = **p/gcd(q,p)**