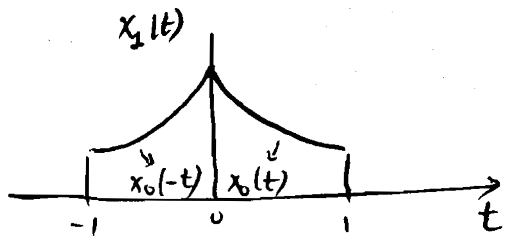
4.23

$$x_0(t) = \begin{cases} e^{-t} & 0 \le t \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$\hat{X}_0(j\omega) = \int_0^1 dt\; e^{-t} e^{-j\omega t} = \int_0^1 dt\; e^{-(1+j\omega)t} = \left.\frac{e^{-(1+j\omega)t}}{-(1+j\omega)}\right]_0^1$$

$$= \frac{1 - e^{-(1+j\omega)}}{1+j\omega}$$

(a)



$$x_1(t) = x_0(t) + x_0(-t)$$

$$\downarrow$$

$$\hat{X}_1(j\omega) = \hat{X}_0(j\omega) + \hat{X}_0(-j\omega)$$ (we used linearity & time reversal for Fourier Transforms)

$$\hat{X}_1(j\omega) = \frac{1 - e^{-(1+j\omega)}}{1+j\omega} + \frac{1 - e^{-(1-j\omega)}}{1-j\omega}$$

$$= \frac{1 - e^{-(1+j\omega)} \cancel{-j\omega} + j\omega e^{-(1+j\omega)} + 1 - e^{-(1-j\omega)} \cancel{+j\omega} - j\omega\, e^{-(1-j\omega)}}{1+\omega^2}$$

$$= \frac{2 - e^{-1}\left(e^{-j\omega} + e^{j\omega}\right) - j\omega e^{-1}\left(-e^{-j\omega} + e^{j\omega}\right)}{1+\omega^2}$$

$$= \frac{2 - e^{-1}\, 2\cos\omega - j\omega e^{-1} 2j \sin\omega}{1+\omega^2} = 2\,\frac{1 - e^{-1}\cos\omega + \omega e^{-1}\sin\omega}{1+\omega^2}$$

(b)

(c) : Time shift : (4.3.2)

(d) : Multiplication with t (4.3.6).

4.14

$x(t) \longrightarrow \hat{X}(j\omega)$

1) $x(t)$ real and $x(t) \geq 0$

2) $F^{-1}\{(1+j\omega)\hat{X}(j\omega)\} = A e^{-2t} u(t)$ (A independent of t)

3) $\int_{-\infty}^{\infty} d\omega \, |\hat{X}(j\omega)|^2 = 2\pi$.

$\rightarrow$ Determine $x(t)$.

$\rightarrow$ Parseval relation : $\underbrace{\int_{-\infty}^{\infty} dt \, |x(t)|^2}_{=1} = \frac{1}{2\pi} \underbrace{\int_{-\infty}^{\infty} d\omega \, |\hat{X}(j\omega)|^2}_{2\pi}$

$\rightarrow$ Can find A

$\rightarrow F\{2)\} \rightarrow (1+j\omega)\hat{X}(j\omega) = F\{A e^{-2t} u(t)\} = A \frac{1}{2+j\omega}$

Multiplication w/ exponential in time $\rightarrow$ Table 4.2

$$\rightarrow \hat{X}(j\omega) = \frac{A}{(1+j\omega)(2+j\omega)} = A\left\{\frac{1}{j\omega+1} - \frac{1}{j\omega+2}\right\}$$

Partial Fraction expansion.

$\downarrow F^{-1}$

$$x(t) = A\left\{e^{-t}u(t) - e^{-2t}u(t)\right\}$$

$$1 = \int_0^{\infty} dt \, A^2\left(e^{-t} - e^{-2t}\right)^2 = A^2 \int_0^{\infty} dt \left(e^{-2t} - 2e^{-t}e^{-2t} + e^{-4t}\right)$$

$$= A^2\left[\left(\frac{e^{-2t}}{-2}\right)_0^{\infty} - 2\left(\frac{e^{-3t}}{-3}\right)_0^{\infty} + \left(\frac{e^{-4t}}{-4}\right)_0^{\infty}\right] = A^2 \underbrace{\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right]}_{\frac{3}{4} - \frac{2}{3} = \frac{9}{12} - \frac{8}{12}}$$

$$\rightarrow A = \sqrt{12} \rightarrow \boxed{x(t) = \sqrt{12}\left\{e^{-t} - e^{-2t}\right\}u(t)}$$

4.28 (a) $x(t) \longrightarrow \hat{X}(j\omega)$

$$p(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}$$: periodic, fundamental freq is ω_0

Determine Fourier Transform for $y(t) = x(t) \cdot p(t)$.

In Table 4.1 : property 4.5 : $x(t) \cdot p(t) \longrightarrow \boxed{\frac{1}{2\pi} \hat{X}(j\omega) * \hat{P}(j\omega)}$

$$\hat{P}(j\omega) = F\left\{ \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t} \right\} \underset{\text{Linearity}}{=} \sum_{n=-\infty}^{\infty} a_n \underbrace{F\left(e^{jn\omega_0 t}\right)}_{\text{Table 4.2}} = \sum_{n=-\infty}^{\infty} a_n 2\pi \delta(\omega - n\omega_0)$$

$$\frac{1}{\cancel{2\pi}} 2\pi \; \hat{X}(j\omega) * \sum_{n=-\infty}^{\infty} a_n \delta(\omega - n\omega_0) = \sum_{n=-\infty}^{\infty} a_n \hat{X}(j(\omega - n\omega_0))$$

Recall (pg. 73) · $a_k * b_k$; if $b_k = \delta(k-c)$
$\longrightarrow = a_{k-c}$

(b) Sketch.

4.53 2D Fourier Analysis

$x(t_1, t_2) \longrightarrow \hat{X}(j\omega_1, j\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_1\, dt_2\, x(t_1, t_2)\, e^{-j(\omega_1 t_1 + \omega_2 t_2)}$

t_1, t_2 are independent

a) This is 2 successive integrals : first in t_1 with t_2 assumed to be constant, then in t_2.

$$\hat{X}(j\omega_1, j\omega_2) = \int_{-\infty}^{\infty} dt_2\, e^{-j\omega_2 t_2} \underbrace{\int_{-\infty}^{\infty} dt_1\, e^{-j\omega_1 t_1}\, x(t_1, t_2)}_{\substack{\text{is a function of } \omega_1 \text{ and } t_2 \\ \hat{X}(j\omega_1,\, t_2)}}$$

b) $$x(t_1,t_2) = F_{2D}^{-1}\left\{\hat{X}(j\omega_1,j\omega_2)\right\} = F_{1D}^{-1}\left\{F_{1D}^{-1}\left\{\hat{X}(j\omega_1,j\omega_2)\right\}\right\}$$

$$= F_{1D}^{-1}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega_1\, \hat{X}(j\omega_1,j\omega_2)\, e^{j\omega_1 t}\right\}$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} d\omega_2 \int_{-\infty}^{\infty} d\omega_1\, \hat{X}(j\omega_1,j\omega_2)\, e^{j(\omega_1 t+\omega_2 t)}$$

(c) (i) $$x(t_1,t_2) = e^{-t_1+2t_2}\, u(t_1-1)\, u(2-t_2)$$

$$\hat{X}(j\omega_1,j\omega_2) = F\left\{e^{-t_1} u(t_1-1)\right\} \cdot F\left\{e^{2t_2} u(2-t_2)\right\}$$

$$= \frac{e^{-(1+j\omega_1)}}{1+j\omega_1} \cdot \frac{e^{2(2-j\omega_2)}}{2-j\omega_2}$$

Tables.