Discreteness in time $\longrightarrow$ repetition in frequency

Repetition in time $\longrightarrow$ ~~repetition~~ discreteness in frequency.

$$\boxed{x(t) = e^{-at}u(t)} \quad (Re\{a\}>0) \longrightarrow \hat{X}(j\omega) = \int_0^\infty dt\, e^{-at} e^{-j\omega t} = \int_0^\infty dt\, e^{-(a+j\omega)t}$$

$$= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^\infty = \boxed{\frac{1}{a+j\omega}.}$$

$$\boxed{x(t) = t e^{-at}u(t)}, \quad (Re\{a\}>0) \longrightarrow \hat{X}(j\omega) = \int_0^\infty dt\, t e^{-(a+j\omega)t}$$

$$= -\frac{d}{d(a+j\omega)} \underbrace{\int_0^\infty dt \left(e^{-(a+j\omega)t}\right)}_{\frac{1}{a+j\omega}}$$

$$= \boxed{\frac{1}{(a+j\omega)^2}}$$

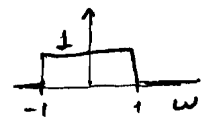
$$\boxed{x(t) = \frac{t^{n-1}}{(n-1)!} e^{-at}u(t)} \quad (Re\{a\}>0) \rightarrow \hat{X}(j\omega) = \boxed{\frac{1}{(a+j\omega)^n}}$$

4.10] a) Use Tables 4.1 & 4.2 to find $\hat{X}(j\omega)$ for $x(t) = t\underbrace{\left(\frac{\sin t}{\pi t}\right)^2}$

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

p329.

$\frac{\sin t}{\pi t} \rightarrow$



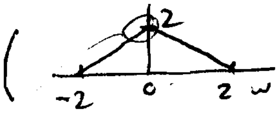
Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \|t\| < T_1 \\ 0, & T_1 < \|t\| \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t)\begin{cases} 1, & \|t\| < T_1 \\ 0, & \|t\| > T_1 \end{cases}$	$\frac{2\sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \|\omega\| < W \\ 0, & \|\omega\| > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \ \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \ \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$ $= \frac{1}{2\pi}\hat{X} * \hat{Y}$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$X(j\omega) = X^*(-j\omega)$ $\Re e\{X(j\omega)\} = \Re e\{X(-j\omega)\}$ $\Im m\{X(j\omega)\} = -\Im m\{X(-j\omega)\}$ $\|X(j\omega)\| = \|X(-j\omega)\|$ $\sphericalangle X(j\omega) = -\sphericalangle X(-j\omega)$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\Re e\{X(j\omega)\}$ $j\Im m\{X(j\omega)\}$
4.3.7	Parseval's Relation for Aperiodic Signals	$\int_{-\infty}^{+\infty}\|x(t)\|^2dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\|X(j\omega)\|^2d\omega$	

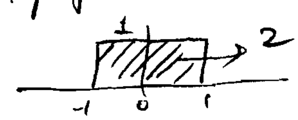
$$\left(\frac{\sin t}{\pi t}\right)^2 \to \frac{1}{2\pi}\left(\underbrace{\hat{X}}_{\hat{X}} * \underbrace{\hat{X}}_{\hat{X}}\right)$$

$$= \frac{1}{2\pi}\left(\quad \right)$$

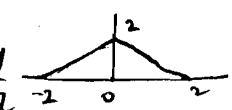


$$= \begin{cases} \frac{1}{2\pi}(-|w|+2) & \text{if } |w|<2 \\ 0 & \text{otherwise.} \end{cases}$$

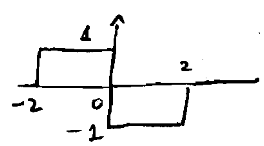
Area under the curve when the two rectangular (identical) pulses overlap exactly on top of each other



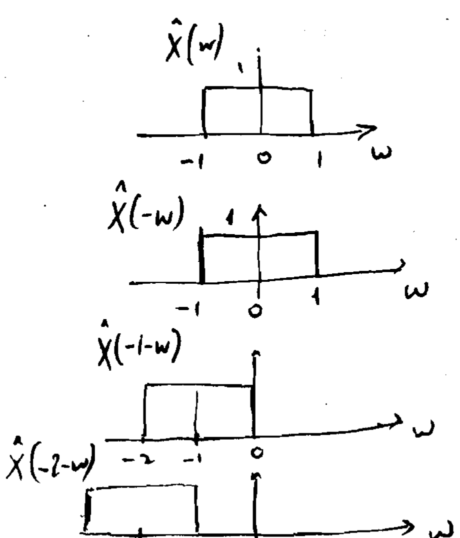
$$t\left(\frac{\sin t}{\pi t}\right)^2 \to j\frac{d}{dw}\left(\frac{1}{2\pi} \quad \right)$$



$$= \frac{j}{2\pi}$$



$$= \frac{j}{2\pi}\begin{cases} -\frac{w}{|w|} & |w|<2 \\ 0 & \text{otherwise.} \end{cases}$$



$$\hat{X}(w) * \hat{X}(w) = \int_{-\infty}^{\infty} d\theta\, \hat{X}(\theta)\, \hat{X}(w-\theta)$$

shift is the coordinate of the convolution

-2 is the last point on the left of 0 that convolution is non-zero.

3.46 $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ periodic, T_0 $\rightarrow$ $a_k = \frac{1}{T}\int_T dt\, x(t) e^{-jk\omega_0 t}$

$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$ periodic, T_0 (orthogonality of complex exponentials)

a) $z(t) = x(t) \cdot y(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$

$\rightarrow c_k = \sum_{n=-\infty}^{\infty} a_n b_{k-n}$ (Product in time $\leftrightarrow$ convolution in frequency, in this case of the Fourier series coefficients)

$$z(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t} \cdot \sum_{l=-\infty}^{\infty} b_l e^{jl\omega_0 t} = \sum_n \sum_l a_n b_l e^{j(n+l)\omega_0 t}$$

$$c_k = \frac{1}{T}\int_T dt\, z(t) e^{-jk\omega_0 t} = \frac{1}{\cancel{T}} \sum_n \sum_l a_n b_l \underbrace{\int_T dt\, e^{j(n+l-k)\omega_0 t}}_{= \cancel{T}\,\delta[n+l-k]}$$

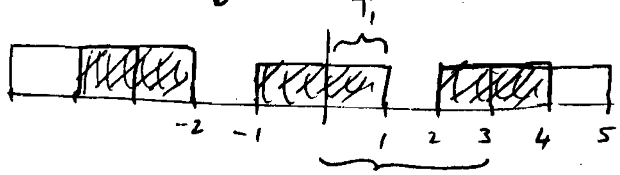
orthogonality of complex exponentials

On $\sum_l$ only term is $l = k-n$ $(l+n-k=0)$

$$\boxed{c_k = \sum_{n=-\infty}^{\infty} a_n b_{k-n}}\ \checkmark$$

b) $x_1(t) = \underbrace{\cos(20\pi t)}\cdot\underbrace{y(t)}$; $y(t)$ = train of rectangular pulses of width 2 & period 3

$T_0 = 3 \rightarrow \omega_0 = \frac{2\pi}{3}$



$\cos(30\omega_0 t)$

$\downarrow$

k

$\boxed{\text{Table 4.2 pg 324: } \cos(\omega_0 t) \longrightarrow b_1 = b_{-1} = \frac{1}{2}}$

$k=1$

$b_{30} = b_{-30} = \frac{1}{2}$

$\rightarrow \cos(20\pi t) = \frac{1}{2}[\delta(k-30) + \delta(k+30)]$

From notes pg 50 & 51 : $y(t) = \begin{cases} 1 & |t| < T_1 = 1 \\ 0 & T_1 = 1 < |t| < \frac{T}{2} = \frac{3}{2} \end{cases}$

$a_k = \frac{\sin(k\omega_0 T_1)}{k\pi}$; $a_0 = \frac{2T_1}{T}$

So the Fourier Series expansion for $\cos(20\pi t)$ with $T_0 = 3$ (or $\omega_0 = \frac{2\pi}{3}$)

is $\cos(20\pi t) = \frac{1}{2}\left[\delta(k-30) + \delta(k+30)\right]$

$$\rightarrow \quad c_k = \frac{1}{2}\left[\delta(k-30) + \delta(k+30)\right] * \frac{\sin\left(k\frac{2\pi}{3}1\right)}{k\pi}$$

$$\boxed{\begin{array}{l} \text{In general} \quad a_k * b_k = \sum_{n=-\infty}^{\infty} a_n b_{k-n} \\ \text{If } b_k = \delta(k-c) \rightarrow a_k * b_k = \sum_{n=-\infty}^{\infty} a_n \cdot \delta(k-n-c) = a_{k-c} \end{array}}$$

$k-n-c=0$ or $n=k-c$

$$= \frac{1}{2}\left[\frac{\sin\left[(k-30)\frac{2\pi}{3}\right]}{(k-30)\pi} + \frac{\sin\left[(k+30)\frac{2\pi}{3}\right]}{(k+30)\pi}\right]$$

$c_0 = 0$

$c_1 \neq 0$

$c_{30} = \frac{1}{3}$; $c_{-30} = \frac{1}{3}$