b) $y[n] = \cos\left(\frac{5\pi}{2} n + \frac{\pi}{4}\right)$: $\frac{5\pi}{2} N = m 2\pi \rightarrow N = \frac{4m}{5} \rightarrow N = 4$

also b/c our system is linear!

(4 terms in input → 4 terms in output)

$$b_k = \frac{1}{4} \sum_{n=0}^{3} \cos\left(\frac{5\pi}{2} n + \frac{\pi}{4}\right) e^{-jk\frac{\pi}{2} n}$$

$$b_0 = \frac{1}{4} \sum_{n=0}^{3} \cos\left(\frac{5\pi}{2} n + \frac{\pi}{4}\right) = \frac{1}{8} \sum_{n=0}^{3} \left[e^{+j\left(\frac{5\pi}{2} n + \frac{\pi}{4}\right)} + e^{-j\left(\frac{5\pi}{2} n + \frac{\pi}{4}\right)} \right]$$

$$= \frac{1}{8} \left[e^{j\frac{\pi}{4}} \underbrace{\sum_{n=0}^{3} \left(e^{j\frac{5\pi}{2}}\right)^n}_{\frac{1 - e^{j\frac{5\pi}{2} 4}}{1 - e^{j\frac{5\pi}{2}}}} + e^{-j\frac{\pi}{4}} \underbrace{\sum_{n=0}^{3} \left(e^{-j\frac{5\pi}{2}}\right)^n}_{\frac{1 - e^{-j\frac{5\pi}{2} 4}}{1 - e^{-j\frac{5\pi}{2}}}} \right] = 0 \checkmark$$

$$\boxed{\sum_{n=0}^{N} a^n = \frac{1 - a^{N+1}}{1 - a}}$$

$$b_1 = \frac{1}{8} \sum_{n=0}^{3} \left[e^{j\left(\frac{4\pi}{2} n + \frac{\pi}{4}\right)} + e^{-j\left(\frac{6\pi}{2} n + \frac{\pi}{4}\right)} \right]$$

$$= \frac{1}{8} \left[e^{j\frac{\pi}{4}} \sum_{n=0}^{3} \left(e^{j2\pi}\right)^n + e^{-j\frac{\pi}{4}} \sum_{n=0}^{3} \left(e^{j3\pi}\right)^n \right]$$

$$= \frac{1}{8} \left[e^{j\frac{\pi}{4}} \underbrace{\boxed{\frac{1 - e^{j2\pi \times 4}}{1 - e^{j2\pi}}}}_{\rightarrow 1} + e^{-j\frac{\pi}{4}} \underbrace{\boxed{\frac{1 - e^{j3\pi 4}}{1 - e^{j3\pi}}}}_{\rightarrow 0} \right] = \cancel{\frac{1}{8}} e^{j\frac{\pi}{4}}$$

$$b_2 = \frac{1}{8} \sum_{n=0}^{3} \left[e^{j\left(\frac{3\pi}{2} n + \frac{\pi}{4}\right)} + e^{-j\left(\frac{7\pi}{2} n + \frac{\pi}{4}\right)} \right]$$

$$= \frac{1}{8} \left[e^{j\frac{\pi}{4}} \sum_{n=0}^{3} \left(e^{j\frac{3\pi}{2}}\right)^n + e^{-j\frac{\pi}{4}} \sum_{n=0}^{3} \left(e^{-j\frac{7\pi}{2}}\right)^n \right]$$

$$= \frac{1}{8} \left[e^{j\frac{\pi}{4}} \frac{1 - e^{j\frac{3\pi}{2} 4}}{1 - e^{j\frac{3\pi}{2}}} + e^{-j\frac{\pi}{4}} \frac{1 - e^{-j\frac{7\pi}{2} 4}}{1 - e^{-j\frac{7\pi}{2}}} \right] = 0 \checkmark$$

$$b_3 =$$

$$H\left(e^{jk\frac{\pi}{2}}\right) \begin{cases} k=0 & H\left(e^{j0}\right) = \dfrac{b_0}{a_0} = 0 \\ k=1 & H\left(e^{j\frac{\pi}{2}}\right) = \dfrac{b_1}{a_1} = \dfrac{\frac{1}{8} e^{j\frac{\pi}{4}}}{\frac{1}{4}} = \frac{1}{2} e^{j\frac{\pi}{4}} \\ k=2 & H\left(e^{j\pi}\right) = \dfrac{b_2}{a_2} = \dfrac{0}{\frac{1}{4}} = 0 \\ k=3 & H\left(e^{j\frac{3\pi}{2}}\right) = \dfrac{b_3}{a_3} = \end{cases}$$

Ch 4 Continuous-time Fourier Transform

HW4: 4.10, 4.14, 4.19, 4.23, 4.26, 4.28, 4.34, 4.44, 4.46, 4.53

$x(t) \longrightarrow$ Fourier Transform of $x(t)$: $\boxed{\hat{X}(j\underline{\underline{\omega}}) = \int_{-\infty}^{\infty} dt \; x(t) e^{-j\underline{\underline{\omega}}t}}$

continuous versus discrete values of $k\omega_0$ in F.S. coeffs a_k.

$\hat{X}(j\omega) \longrightarrow$ Inverse Fourier Transform of $\hat{X}(j\omega)$: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \; \hat{X}(j\omega) e^{+j\omega t}$

Properties: (Table 4.1 pg. 328)

Time shifting:

If $x(t) \longrightarrow \hat{X}(j\omega)$

$x(t-t_0) \longrightarrow \; ?$ Applying the definition:

$\int_{-\infty}^{\infty} dt \; x(t-t_0) \, e^{-j\omega t}$, to relate this to $\hat{X}(j\omega)$ we can do a change of variable $t' = t - t_0$ $\begin{cases} dt' = dt \\ t = t' + t_0 \end{cases}$

$$\int_{-\infty}^{\infty} dt' \; x(t') \, e^{-j\omega(t'+t_0)} = e^{-j\omega t_0} \underbrace{\int_{-\infty}^{\infty} dt' \, x(t') e^{-j\omega t'}}_{\hat{X}(j\omega)}$$

So $\boxed{x(t-t_0) \longrightarrow e^{-j\omega t_0} \hat{X}(j\omega)}$ (4.3.2)

<u>Convolution</u>:

If $x(t) \longrightarrow \hat{X}(j\omega)$ and $y(t) \longrightarrow \hat{Y}(j\omega)$, what is the Fourier Transform of the convolution of $x(t)$ and $y(t)$ or $x*y$

$$x*y \longrightarrow \int_{-\infty}^{\infty} dt\,(x*y)\,e^{-j\omega t} = \int_{-\infty}^{\infty} dt\, e^{-j\omega t} \underbrace{\int_{-\infty}^{\infty} d\tau\, x(\tau)\, y(t-\tau)}$$

(Mathematically) $$= \int_{-\infty}^{\infty} d\tau\, x(\tau) \underbrace{\int_{-\infty}^{\infty} dt\, e^{-j\omega t}\, y(t-\tau)}_{\text{Time shifting property: } e^{-j\omega\tau}\underbrace{\hat{Y}(j\omega)}_{\text{independent of } \tau}}$$

(Mathematically) $$= \hat{Y}(j\omega) \cdot \underbrace{\int_{-\infty}^{\infty} d\tau\, x(\tau)\, e^{-j\omega\tau}}_{\hat{X}(j\omega)}$$

So: $$\boxed{x*y \longrightarrow \hat{X}\cdot\hat{Y}} \quad (4.4)$$ or The Fourier Transform of a convolution is the product of the Fourier Transforms!

→ Why is this convolution property of Fourier Transform important for linear system analysis?

$$x(t) \longrightarrow \boxed{\begin{matrix}\text{System}\\ h(t)\end{matrix}} \longrightarrow y(t) = x*h$$

If we know $x(t)$ & the system, how would this property help calculate $y(t)$? Since $y(t) = x(t) * h(t)$ → an alternative to do this convolution in time is: get $\hat{X}$, get $\hat{H}$, do $\hat{X}\cdot\hat{H}$, then do inverse Fourier Transform on $\hat{Y} = \hat{X}\cdot\hat{H}$ to get $y(t)$.

$x(t)$ periodic, period $T \rightarrow$ fundamental freq is $\omega_0 = \frac{2\pi}{T}$

$\hookrightarrow a_k = \frac{1}{T}\int_0^T dt\ x(t)\, e^{-jk\omega_0 t}$: Fourier Series Coeff.

$\hookrightarrow \hat{X}(jk\omega_0) = \underbrace{\int_{-\infty}^{\infty} dt\ x(t)\, e^{-jk\omega_0 t}}$: Fourier Transform.

Disadvantage { $\hookrightarrow$ May not exist due to the $(-\infty, \infty)$ limits
Condition of existance: 1) $x(t)$ absolutely integrable
$\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$
2) $x(t)$ with finite number of max & min within any finite interval
3) $x(t)$ with finite number of discontinuties within any finite interval. (Finite discontinuities)

Advantage { ω does not have to be $k\omega_0$ as with Fourier Series. in another words, a non-periodic signal can have a Fourier Transform, while it doesnot have Fourier Series Coefficient.

Basic pairs of Fourier Transforms.

$x(t) = \cos\omega_0 t \longrightarrow \hat{X}(j\omega) = \int_{-\infty}^{\infty} dt\ \cos\omega_0 t\ e^{-j\omega t} = \int_{-\infty}^{\infty} dt \left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right) e^{-j\omega t}$

$$= \frac{1}{2}\Big[\underbrace{\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t}\,dt} + \underbrace{\int_{-\infty}^{\infty} e^{-j(\omega+\omega_0)t}\,dt} \Big]$$

$$= \frac{1}{2}\big[2\pi\,\delta(\omega-\omega_0) + 2\pi\,\delta(\omega+\omega_0) \big]$$

Orthogonality of complex exponentials :

$$\int_{-\infty}^{\infty} e^{j(k-n)\omega t}\, dt = 2\pi\, \delta\big((k-n)\omega\big) = \frac{2\pi}{\omega}\, \delta[k-n]$$
$$= T\, \delta[k-n]$$

$$\boxed{\delta(at) = \frac{1}{|a|}\delta(t)}$$
Scaling Property of δ-function

So $\quad \cos\omega_0 t \longleftrightarrow \frac{\pi}{2}\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$

$\sin\omega_0 t \longleftrightarrow \frac{\pi}{2j}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$

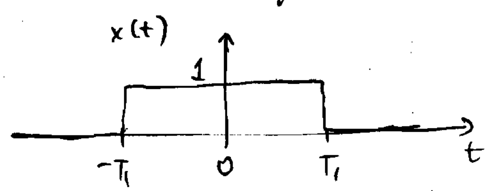
$= j\frac{\pi}{2}\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$

What is the Fourier Transform of a constant?

$$x(t) = 1 \longrightarrow \hat{X}(j\omega) = \int_{-\infty}^{\infty} dt\, e^{-j\omega t} = 2\pi\,\delta(\omega) \begin{cases} \omega = 0 \to \int_{-\infty}^{\infty} dt\, 1 = \infty \\ \omega \neq 0 \to \int_{-\infty}^{\infty} dt\, e^{-j\omega t} = 0 \end{cases}$$

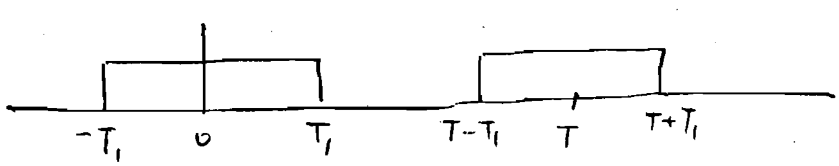
→ What is the Fourier Transform for a rectangular pulse?

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases}$$



$$\longrightarrow \hat{X}(j\omega) = \int_{-T_1}^{T_1} dt\, e^{-j\omega t} = \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-T_1}^{T_1} = \frac{e^{j\omega T_1} - e^{-j\omega T_1}}{j\omega}$$
$$= 2\,\frac{\sin\omega T_1}{\omega}$$

→ What is the Fourier Transform for a periodic rectangular pulse.



$$\rightarrow \hat{X} = \sum_{k=-\infty}^{\infty} \frac{2\sin k\omega_0 T_1}{k}\, \underbrace{\delta(\omega - k\omega_0)}$$

ω has to be discrete.

Based on conclusion on notes pg 54 : kind duality of time & frequency (duality = similarity in properties)

Discreteness in time $\longrightarrow$ repetition in frequency

Repetition in time $\longrightarrow$ ~~repetition~~ discreteness in frequency.