Trick: factor something out so the fraction carries exponentials with same exponent but opposite signs :

$$1 - e^{j\beta} = e^{j\frac{\beta}{2}} \underbrace{\left(e^{j\frac{\beta}{2}} - e^{-j\frac{\beta}{2}} \right)}_{2j\sin(\frac{\beta}{2})}$$

$$\Rightarrow a_k = \frac{1}{N} \underbrace{e^{\frac{jk2\pi N_1}{N}} \cdot \frac{e^{-j\frac{k2\pi}{N}(N_1+\frac{1}{2})}}{e^{-jk\frac{2\pi}{2N}}}}_{=1} \cdot \frac{\left(e^{j\frac{k2\pi}{N}(N_1+\frac{1}{2})} - e^{-j\frac{k2\pi}{N}(N_1+\frac{1}{2})} \right)}{\left(e^{jk\frac{2\pi}{2N}} - e^{-jk\frac{2\pi}{2N}} \right)}$$

$$= \frac{1}{N} \cdot \frac{\cancel{2j} \sin \frac{k2\pi}{N}(N_1+\frac{1}{2})}{\cancel{2j} \sin \frac{k2\pi}{2N}}$$

good for ?
$k \neq 0, 2N, 4N,$ etc...
(any k different than 0 and multiple of $2N$)

$\sin(m2\pi) = 0 \longrightarrow$

For $k=0$, and multiple of $2N$:

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} \underbrace{x[n]}_{1} \cdot \underbrace{e^{-jk\frac{2\pi}{N}n}}_{1} = \frac{1}{N} \sum_{n=-N_1} 1 = \frac{1}{N}(2N_1+1)$$

$k=0 : e^0 = 1$

$k=2N : e^{-j4\pi n} =$

$1+2+3+\dots+n$ = arithmetic sum

$1+n+n^2+\dots n^\alpha$ = geometric sum

$1+1+1+\dots 1$ = constant series sum

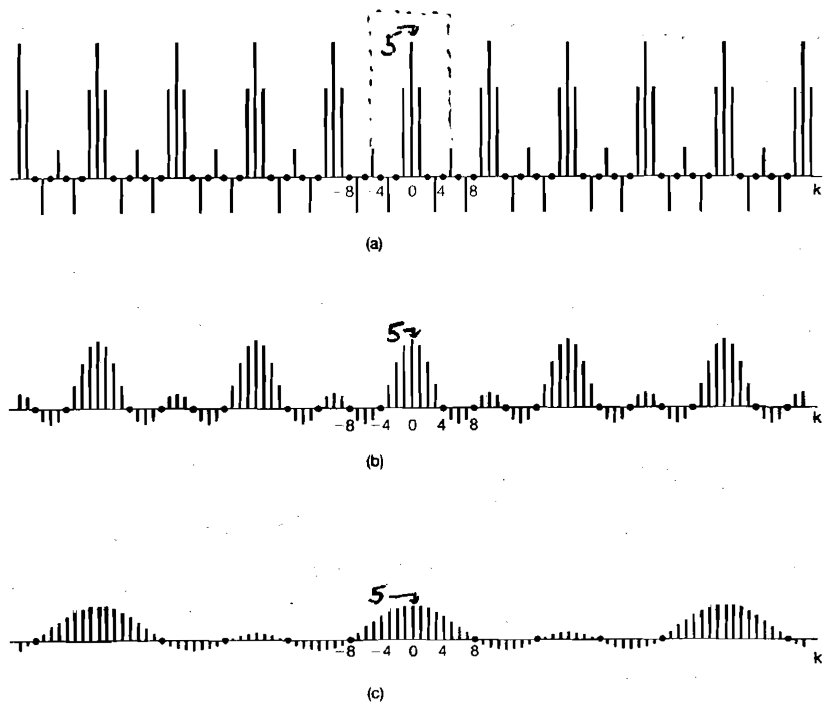


Figure 3.17 Fourier series coefficients for the periodic square wave of Example 3.12; plots of Na_k for $2N_1 + 1 = 5$ and (a) $N = 10$; (b) $N = 20$; and (c) $N = 40$.

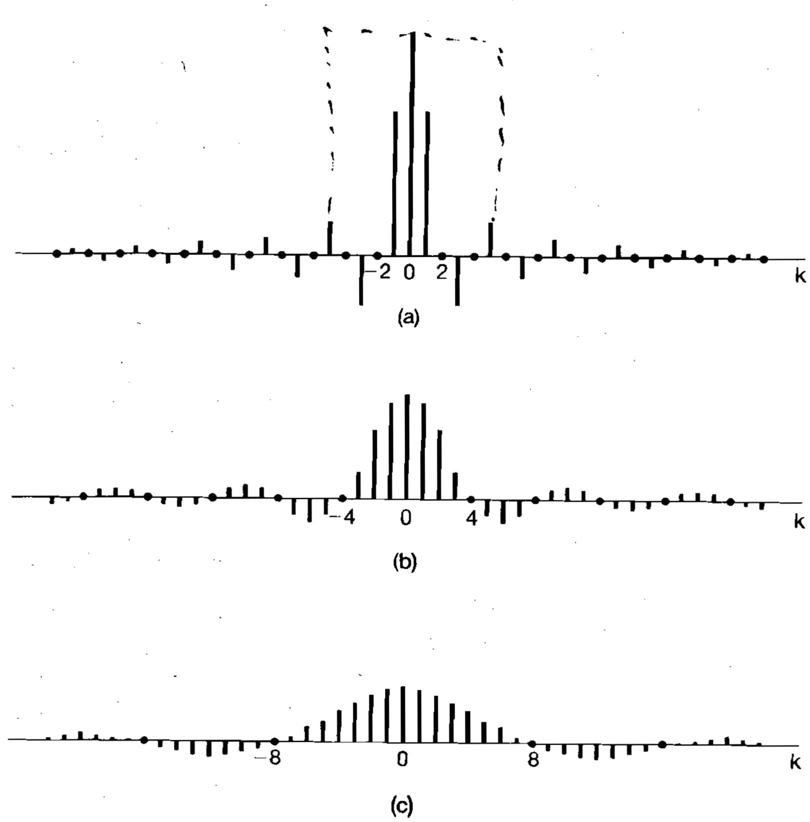


Figure 3.7 Plots of the scaled Fourier series coefficients Ta_k for the periodic square wave with T_1 fixed and for several values of T: (*a*) $T = 4T_1$; (*b*) $T = 8T_1$; (*c*) $T = 16T_1$. The coefficients are regularly spaced samples of the envelope $(2 \sin \omega T_1)/\omega$, where the spacing between samples, $2\pi/T$, decreases as T increases.

$\boxed{3.12}$ $x_1[n]$ & $x_2[n]$ both with period $N=4$

$x_1[n] \downarrow a_k$ $\quad$ $x_2[n] \downarrow b_k$ are Fourier series coefficients.

$a_0 = a_3 = 1$; $a_1 = a_2 = 2$ $\qquad$ $b_k = 1$, $k = 0, 1, 2, 3$

$\underbrace{\text{Multiplication property in Table 3.1}}$ find $\boxed{c_k}$ for $x_1[n]\,x_2[n]$

continuous time: $x(t)y(t) \longrightarrow \sum_{l=-\infty}^{\infty} a_l b_{k-l}$

discrete time: $x_1[n]x_2[n] \longrightarrow \sum_{\boxed{l=\langle N\rangle}} a_l b_{k-l} = a_0 b_k + a_1 b_{k-1} + a_2 b_{k-2} + a_3 b_{k-3}$

(Period of $x_1[n]\,x_2[n]$ is N)
True or false
↳ see attached Matlab's plot.

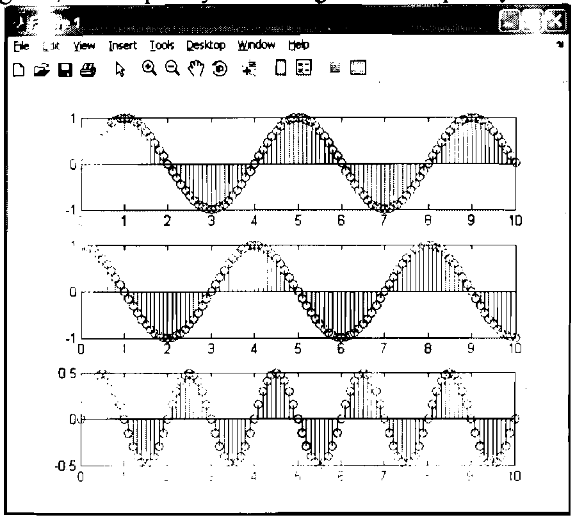
$= b_k + 2(b_{k-1} + b_{k-2}) + b_{k-3}$

$$c_k \begin{cases} c_0 = b_0 + 2(b_{-1} + b_{-2}) + b_{-3} = 1 + 2(1 + 1) + 1 = 6 \\ c_1 = b_1 + 2(b_0 + b_{-1}) + b_{-2} = 6 \\ c_2 = \\ c_3 = \end{cases}$$

$b_{-1} = b_3 \quad b_{-2} = b_2 \quad b_{-3} = b_0$

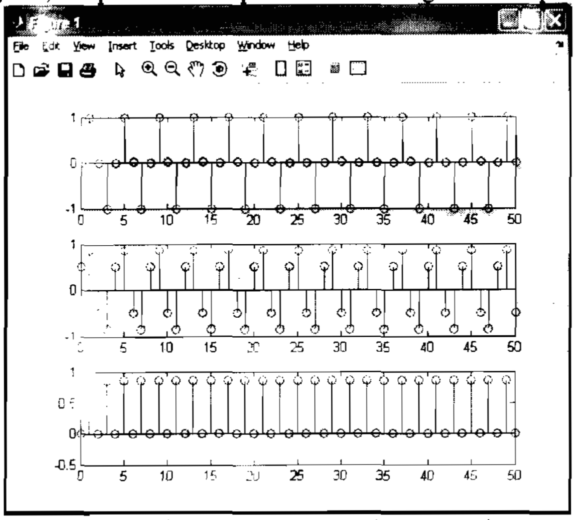
$\boxed{c_k = 6 \quad \forall k.}$

For continuous times signals, the frequency of two signals of frequency f is 2f or the period is halved

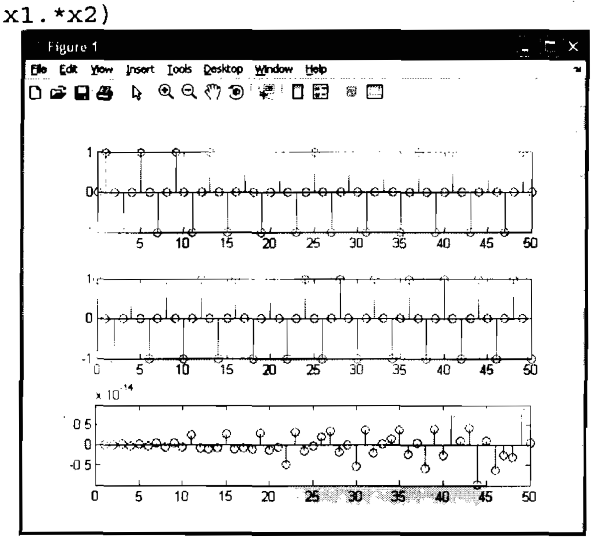



```
%is product of two discrete-time signals with periods of 4 also periodic
%with the same period?
period=4;
freq=1/period;
t=0:.1:10; %Increment of less than 1 to represent continuous-time signals
x1=sin(2*pi*freq*t);
x2=cos(2*pi*freq*t);
%x2=sin(2*pi*freq*t+30/180*pi);
figure(1),subplot(3,1,1), stem(t,x1)
figure(1),subplot(3,1,2), stem(t,x2)
figure(1),subplot(3,1,3), stem(t,x1.*x2)
figure(2), stem(t,x1.*x2)
```

For discrete-time signals, the period of the product of two signals with period 4 is not always 2

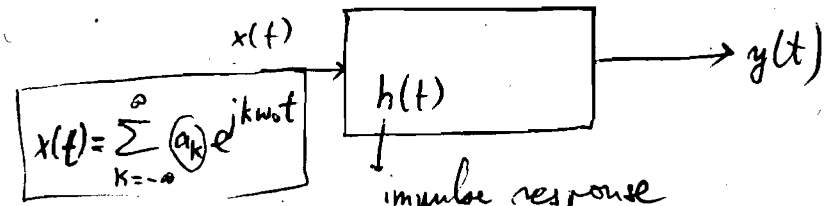



```
%is product of two discrete-time signals with periods of 4 also periodic
%with the same period?
period=4;
freq=1/period;
t=0:1:50; %increment of 1 to represent discrete-time signals
x1=sin(2*pi*freq*t);
%x2=cos(2*pi*freq*t);
x2=sin(2*pi*freq*t+30/180*pi);
figure(1),subplot(3,1,1), stem(t,x1)
figure(1),subplot(3,1,2), stem(t,x2)
figure(1),subplot(3,1,3), stem(t,x1.*x2)
figure(2), stem(t,x1.*x2)
```




```
%is product of two discrete-time signals with periods of 4 also periodic
%with the same period?
period=4;
freq=1/period;
t=0:1:50; %increment of 1 to represent discrete-time signals
x1=sin(2*pi*freq*t);
x2=cos(2*pi*freq*t);
%x2=sin(2*pi*freq*t+30/180*pi);
figure(1),subplot(3,1,1), stem(t,x1)
figure(1),subplot(3,1,2), stem(t,x2)
figure(1),subplot(3,1,3), stem(t,x1.*x2)
figure(2), stem(t,x1.*x2)
```

Fourier Series & Linear Systems:



$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}$$

$h(t)$ impulse response (can identify the system)

<u>if we know the Fourier Series coefficients for $x(t) \to a_k$, what are the corresponding coefficients for $y(t)$?</u>

If $y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$, then what are b_k's in terms of a_k's

1) Instead of inputing $x(t)$, it's the same if I input $a_k e^{jk\omega_0 t}$ $\forall k$ then sum over k (there is a 1-to-1 representation of a signal by its Fourier series): a_k is a constant, what is the output for $e^{jk\omega_0 t}$?

$$e^{jk\omega_0 t} * h(t) \underset{\substack{\downarrow \\ \text{definition} \\ \text{of convolution}}}{=} \int_{-\infty}^{\infty} d\tau\, h(\tau)\, e^{+jk\omega_0(t-\tau)}$$

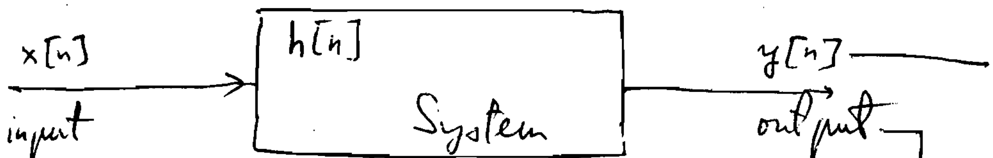
$$= e^{jk\omega_0 t} \underbrace{\int_{-\infty}^{\infty} d\tau\, h(\tau)\, e^{-jk\omega_0 \tau}}_{\substack{\text{Fourier Transform of } h(t) \\ \text{or } \hat{H}(jk\omega_0)}}$$

Output for $e^{jk\omega_0 t}$ is $e^{jk\omega_0 t}\hat{H}(jk\omega_0) \to y(t) = \sum_{k=-\infty}^{\infty} a_k \hat{H}(jk\omega_0) e^{jk\omega_0 t}$

$$\to \boxed{b_k = a_k \hat{H}(jk\omega_0)}$$

→ We proved that the F.S. coefficients for the output $y(t)$ were $b_k = a_k \hat{H}(jk\omega_0)$, where a_k were the F.S. coefficients for the input $x(t)$ and $\hat{H}$ is the Fourier Transform of the impulse response of the system. This was for continuous-time signals.

→ For discrete-time signals:



$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n} \quad (N\text{: period of } x[n])$$

$$y[n] = \sum_{k=\langle N\rangle} b_k e^{jk\frac{2\pi}{N}n} \leftarrow \text{linear systems}$$

→ It is equivalent to apply $x[n]$ or $a_k e^{jk\frac{2\pi}{N}n}$ $\forall k$ and then sum over k. Since a_k is a constant:

$$e^{jk\frac{2\pi}{N}n} * h[n] = \sum_{m=-\infty}^{\infty} h[m]\, e^{jk\frac{2\pi}{N}(n-m)}$$

Since this involves $h[m]$

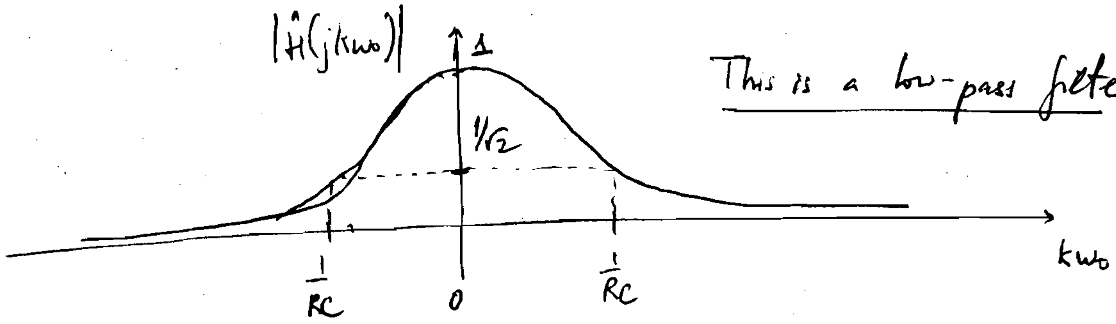
$$= e^{jk\frac{2\pi}{N}n} \sum_{m=-\infty}^{\infty} h[m]\, e^{-jk\frac{2\pi}{N}m}$$

$$y[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n} * h[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n} \underbrace{\sum_{m=-\infty}^{\infty} h[m] e^{-jk\frac{2\pi}{N}m}}_{\text{Discrete-time Fourier Transform of } h[n] \text{ or } \hat{H}(e^{j\omega})\ (\omega = \frac{2\pi}{N})}$$

Linear system.

$$= \sum_{k=\langle N\rangle} \underbrace{a_k \hat{H}(e^{j\omega})}_{b_k} e^{jk\frac{2\pi}{N}n}$$

$$\Rightarrow \boxed{b_k = a_k \hat{H}(e^{j\omega})}$$



This is a low-pass filter

$\Rightarrow$ In general: a system governed by $\frac{dy}{dt} + a y(t) = b x(t)$ is a low-pass filter.

Discrete-time: a discrete-time low-pass filter would be represented by the discrete-time version of $\frac{dy}{dt} + ay = bx$:

Using finite-difference replacement for time derivative:

$$\frac{dy}{dt} = \lim_{\Delta \to 0} \frac{y(t) - y(t-\Delta)}{\Delta} \simeq \frac{y(t) - y(t-\Delta)}{\Delta}$$

$$\text{or } \frac{y[n] - y[n-1]}{\Delta}$$

$$\longrightarrow \frac{y[n] - y[n-1]}{\Delta} + a\, y[n] = b\, x[n]$$

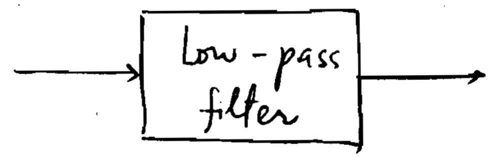
$$\left(\frac{1}{\Delta} + a\right) y[n] - \frac{1}{\Delta} y[n-1] = bx[n] \quad \text{or} \quad \overset{\beta}{\boxed{\frac{1}{b}\left(\frac{1}{\Delta} + a\right)}}\, y[n] \overset{\alpha}{\boxed{-\frac{1}{b\Delta}}}\, y[n-1] = x[n]$$

$$\Rightarrow \boxed{\beta\, y[n] + \alpha\, y[n-1] = x[n]}$$ In general represents a discrete-time low-pass filter

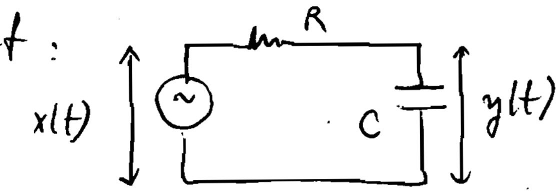
Complex numbers: $\hat{z} = a + jb \rightarrow z \equiv |\hat{z}| = \sqrt{\hat{z}\,\hat{z}^*} = \sqrt{(a+jb)(a-jb)} = \sqrt{a^2+b^2}$

Linear systems: Filters.

What is a low-pass filter?



Continuous-time: e.g. RC circuit:



$$RC\frac{dy}{dt} + y(t) = x(t)$$

(input/output D.E. representing our system)

Solution: $y(t) = \frac{1}{RC}\int_0^t d\lambda \; x(\lambda)\; e^{-\frac{1}{RC}(t-\lambda)}$

$$= \boxed{\frac{1}{RC}\int_0^\infty d\lambda \; x(\lambda)\; e^{-\frac{1}{RC}(t-\lambda)}\, u(t-\lambda) \\ = \int_0^\infty d\lambda \; x(\lambda)\; h(t-\lambda)} = x(t) * h(t)$$

Comparisons: $\boxed{h(t) = \frac{1}{RC}\, e^{-\frac{1}{RC}(t)}\, u(t)}$

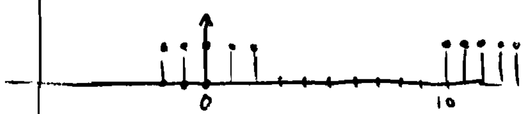
What is the Fourier Transform of this impulse-response:

$$\boxed{\hat{H}(jk\omega_0)} \equiv \int_\infty^\infty dt \; h(t)\, e^{-jk\omega_0 t} = \frac{1}{RC}\int_0^\infty dt \; e^{-(jk\omega_0 + \frac{1}{RC})t}$$

$$= \left[\frac{e^{-(jk\omega_0 + \frac{1}{RC})t}}{-(1 + jk\omega_0 RC)}\right]_0^\infty = \frac{0 - 1}{-(1 + jk\omega_0 RC)} = \boxed{\frac{1}{1 + jk\omega_0 RC}}$$

3.28) a) Fig. P3.28 a) : Pulse width $N_1 = 4$; train of rectangular pulses of period $N = 7$ → find : F.S. coeffs a_k :

From notes:



$$a_k = \begin{cases} \dfrac{1}{N} \cdot \dfrac{\sin \frac{k2\pi}{N}(N_1 + \frac{1}{2})}{\sin \frac{k2\pi}{2N}} & : k \neq 0, 2N, 4N, \text{etc.} \\[2ex] \dfrac{2N_1+1}{N} & : k = 0, 2N, 4N, \text{etc.} \end{cases}$$

$$a_k = \begin{cases} \dfrac{1}{7} \dfrac{\sin(\frac{5}{7}\pi k)}{\sin(\frac{\pi k}{7})} e^{-jk\frac{2\pi}{7}2} & k \neq 0, 14, 28, \text{etc.} \\[2ex] \dfrac{9}{7} \cdot e^{-jk\frac{4\pi}{7}} & k = 0, 14, 28, \text{etc.} \end{cases}$$

Table 3-1 for a time shift of 2

$x(t-t_0)$

$\hookrightarrow a_k e^{-jk\frac{2\pi}{N}t_0}$