Trick: factor something out so the fractions cancel, exponentials with same exponent but opposite signs:

\[
1 - e^{j \theta} = e^{j \frac{\theta}{2}} \left( e^{j \frac{\theta}{2}} - e^{-j \frac{\theta}{2}} \right) \sim \sin(\frac{\theta}{2})
\]

\[
\Rightarrow a_k = \frac{1}{N} \mathcal{F} \left\{ \sum_{n=1}^{N} \left( e^{j \frac{k \pi n}{N}} - e^{-j \frac{k \pi n}{N}} \right) \right\} = \frac{1}{N} \frac{\sin \frac{k \pi (N + 1)}{N}}{\sin \frac{k \pi}{2N}}
\]

\[
\quad = \frac{1}{N} \frac{\sin \frac{k \pi (N + 1)}{N}}{\sin \frac{k \pi}{2N}}
\]

\[
\sin \left( \frac{amn}{b} \right) = 0 \quad \text{good for?}
\]

\[
k \neq 0, 2N, 4N, \text{etc}...
\]

\[
\text{(any } k \text{ different than 0 and multiple of } 2N)
\]

For \( k = 0 \), and multiple of \( 2N \):

\[
a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} x[n] \cdot e^{-j \frac{k \pi n}{N}} = \frac{1}{N} \sum_{n=-N_1}^{N_1} 1 = \frac{1}{N} (2N_1 + 1)
\]

\[
k = 0: e^0 = 1
\]

\[
k = 2N: e^{-j \frac{2N \pi}{2N}} = 1
\]

\[
1 + 2 + 3 + \ldots + n = \text{arithmetic sum}
\]

\[
1 + n + n^2 + \ldots + n^a = \text{geometric sum}
\]

\[
1 + 1 + 1 + \ldots + 1 = \text{constant series sum}
\]
Sec. 3.6 Fourier Series Representation of Discrete-Time Periodic Signals

Figure 3.7 Plots of the scaled Fourier series coefficients $T_{a_k}$ for the periodic square wave with $T_i$ fixed and for several values of $T$: (a) $T = 4T_i$; (b) $T = 8T_i$; (c) $T = 16T_i$. The coefficients are regularly spaced samples of the envelope $(2 \sin \omega T_i)/\omega$, where the spacing between samples, $2\pi/T$, decreases as $T$ increases.
3.12 \[ x_1[n] \text{ and } x_2[n] \text{ both with period } N = 4 \]
\[
\downarrow \quad \downarrow
\]
\[
a_k \quad b_k
\]
are Fourier series coefficients.

\[ a_0 = a_3 = 1 \; ; \; a_1 = a_2 = 2 \]
\[ b_k = 1, \; k = 0, 1, 2, 3 \]

**Multiplication property in Table 3.1: find \( C_k \) for \( x_1[n]x_2[n] \)**

Continuous time: \( x(t) \cdot y(t) \rightarrow \sum_{l=-\infty}^{\infty} a_l b_{k-l} \)

Discrete time: \( x_1[n]x_2[n] \rightarrow \sum_{l=\langle N \rangle}^{\langle N \rangle} a_l b_{k-l} = a_0 b_k + a_1 b_{k-1} + a_2 b_{k-2} + a_3 b_{k-3} \)

(Period of \( x_1[n]x_2[n] \) is \( N \))

True or false \( \rightarrow \) see attached Matlab's plot.

\[ = b_k + 2(b_{k-1} + b_{k-2}) + b_{k-3} \]

\[ C_k \begin{cases} 
  c_0 = b_0 + 2(b_1 + b_2) + b_3 = 1 + 2(1 + 1) + 1 = 6 \\
  c_1 = b_1 + 2(b_0 + b_1) + b_2 = 6 \\
  c_2 = b_2 = b_3 = b_0 \\
  c_3 =
\end{cases} \]

\[ C_k = 6 \; \forall \; k. \]
For continuous times signals, the frequency of two signals of frequency $f$ is $2f$ or the period is halved.

\[
\text{period} = 4; \\
\text{freq} = 1/\text{period}; \\
t = 0:.1:10; \quad \text{Increment of less than 1 to represent continuous-time signals} \\
x1 = \sin(2\pi f \cdot t); \\
x2 = \cos(2\pi f \cdot t); \\
x2 = \sin(2\pi f \cdot t + 30/180 \cdot \pi); \\
\text{figure(1)}, \text{subplot}(3,1,1), \text{stem}(t, x1) \\
\text{figure(1)}, \text{subplot}(3,1,2), \text{stem}(t, x2) \\
\text{figure(1)}, \text{subplot}(3,1,3), \text{stem}(t, x1 \cdot x2) \\
\text{figure(2)}, \text{stem}(t, x1 \cdot x2)
For discrete-time signals, the period of the product of two signals with period 4 is not always 2

%Is product of two discrete-time signals with periods of 4 also periodic
%with the same period?
period=4;
freq=1/period;
t=0:1:50; %increment of 1 to represent discrete-time signals
x1=sin(2*pi*freq*t);
x2=cos(2*pi*freq*t);
x2=sin(2*pi*freq*t+30/180*pi);
figure(1), subplot(3,1,1), stem(t,x1)
figure(1), subplot(3,1,2), stem(t,x2)
figure(1), subplot(3,1,3), stem(t,x1.*x2)
figure(2), stem(t,x1.*x2)
Fourier Series & Linear Systems:

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t} \]

Impulse response (can identify the system)

If \( y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jkw_0 t} \), then what are the corresponding coefficients for \( y(t) \)?

4) Instead of inputting \( x(t) \), it's the same if I input \( e^{jkw_0 t} \). Then sum over \( k \) (there is a 1-to-1 representation of a signal by its Fourier series): \( a_k \) is a constant, what is the output for \( e^{jkw_0 t} \)?

\[
e^{jkw_0 t} * h(t) = \int_{-\infty}^{\infty} dc \ h(c) e^{jkc - jkw_0 c}
\]

Definition of Convolution

\[
e^{jkw_0 t} \cdot \hat{H}(jkw_0)
\]

Output for \( e^{jkw_0 t} \) is \( e^{jkw_0 t} \cdot \hat{H}(jkw_0) \rightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k \hat{H}(jkw_0) e^{jkw_0 t} \)
We proved that the F.S. coefficients for the output $y(t)$ were $b_k = a_k \hat{H}(jkw_0)$, where $a_k$ were the F.S. coefficients for the input $x(t)$ and $\hat{H}$ is the Fourier Transform of the impulse response of the system. This was for continuous-time signals.

For discrete-time signals:

For discrete-time signals, it is equivalent to apply $x[n]$ or $a_k e^{j k \frac{2\pi}{N} n}$ $\forall k$ and then sum over $k$. Since $a_k$ is a constant:

$$e^{j \frac{2\pi}{N} n} \ast h[n] = \sum_{m=-\infty}^{\infty} h[m] e^{j \frac{2\pi}{N} (n-m)}$$

Since this involves $h[m]$$e^{j \frac{2\pi}{N} n} = e^{j \frac{2\pi}{N} n} \sum_{m=-\infty}^{\infty} h[m] e^{-j \frac{2\pi}{N} m}$

$$y[n] = \sum_{k=\langle N \rangle} a_k e^{j \frac{2\pi}{N} n} \ast h[n] = \sum_{k=\langle N \rangle} a_k e^{j k \frac{2\pi}{N} n} \sum_{m=-\infty}^{\infty} h[m] e^{-j \frac{2\pi}{N} m}$$

$$= \sum_{k=\langle N \rangle} a_k \hat{H}(e^{jw}) e^{j k \frac{2\pi}{N} n}$$

$$\Rightarrow b_k = a_k \hat{H}(e^{jw})$$
In general: a system governed by \( \frac{dy}{dt} + ay(t) = bx(t) \) is a low-pass filter.

**Discrete-time**: a discrete-time low-pass filter would be represented by the discrete-time version of \( \frac{dy}{dt} + ay = bx \):

Using finite-difference replacement for time derivative:

\[
\frac{dy}{dt} \approx \frac{y(t) - y(t-\Delta)}{\Delta}
\]

\[
y[n] - y[n-1] + ay[n] = bx[n]
\]

\[
(\frac{1}{\Delta} + a)y[n] - \frac{1}{\Delta}y[n-1] = bx[n]
\]

\[
\beta y[n] + \alpha y[n-1] = x[n]
\]

In general represents a

**discrete-time low-pass filter**

**Complex numbers**: \( z = a + jb \) → \( z = |z| e^{j\theta} = \sqrt{a^2 + b^2} e^{j\theta} = (a + jb)(\bar{a} - jb) = |z|^2 \)
Linear systems: Filters.

What is a low-pass filter?

Continuous-time: e.g. RC circuit:

\[ RC \frac{dy}{dt} + y(t) = x(t) \]

(impulse/output D.E. representing our system)

Solution: \[ y(t) = \frac{1}{RC} \int_{0}^{t} d\lambda \ x(\lambda) \ e^{-\frac{t-\lambda}{RC}} \]

\[ = \frac{1}{RC} \int_{0}^{\infty} d\lambda \ x(\lambda) \ e^{-\frac{t-\lambda}{RC}} u(t-\lambda) = x(t) * h(t) \]

\[ = \int_{0}^{\infty} d\lambda \ x(\lambda) \ h(t-\lambda) \]

Comparisons: \[ h(t) = \frac{1}{RC} e^{- \frac{t}{RC}} u(t) \]

What is the Fourier Transform of this impulse response:

\[ \hat{h}(j\omega) = \int_{0}^{\infty} dt \ h(t) \ e^{-j\omega t} = \frac{1}{RC} \int_{0}^{\infty} dt \ e^{-\left(\frac{j\omega + \frac{1}{RC}}{RC}\right)t} \]

\[ = \left[ \frac{-\left(\frac{j\omega + \frac{1}{RC}}{RC}\right)t}{\left(-1 - j\omega RC\right)} \right]_{0}^{\infty} = \frac{0 - 1}{-(1+j\omega RC)} = \frac{1}{1+j\omega RC} \]
3.28 a) Fig. P3.28 a) Pulse width \( N_o = 4 \); train of rectangular pulses of period \( N = 7 \). Find F.S. coeff. \( a_k \):

\[
\begin{align*}
\text{From notes: } a_k &= \begin{cases} 
\frac{1 - \sin \frac{k\pi}{N}}{N} \sin \frac{k\pi}{2N}, & k \neq 0, 2N, 4N, \ldots \\
\frac{2N + 1}{N}, & k = 0, 2N, 4N, \ldots
\end{cases}
\end{align*}
\]

\[a_k = \begin{cases} 
\frac{1}{N} \sin \left( \frac{\pi}{N} \right) \left\{ \begin{array}{l}
\frac{\sin \left( \frac{\pi}{N} k \right)}{\sin \left( \frac{\pi}{N} \right)}, \quad k \neq 0, 14, 28, \ldots \\
\frac{1}{7} \left( e^{-j \frac{k\pi}{7}} - e^{-j \frac{k\pi}{14}} \right), \quad k = 0, 14, 28, \ldots
\end{array} \right. 
\end{cases} \]

Table 3.1 for a time shift of \( 2 \):

\[x(t-10) e^{-j \frac{k\pi}{N}} \]