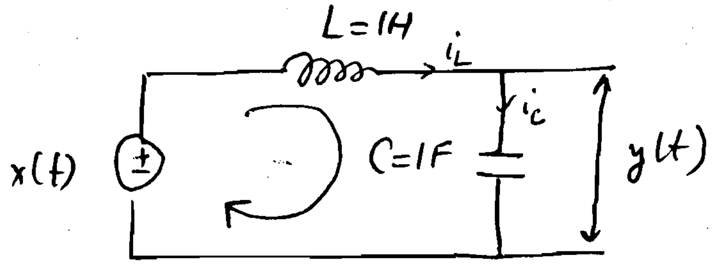
2.61)



$$x(t) - L\frac{di_L}{dt} - y(t) = 0$$

$$y(t) = v_c(t) = \frac{1}{C}\int i_c\,dt \longrightarrow \frac{dy}{dt} = \frac{1}{C} i_c = \frac{1}{C} i_L$$

$$\frac{d^2y}{dt^2} = \frac{1}{C}\frac{di_L}{dt}$$

$$x(t) - LC\frac{d^2y}{dt^2} - y = 0 \quad : \text{ input/output D.E.}$$

$$\boxed{LC\frac{d^2y}{dt^2} + y(t) = x(t)}$$

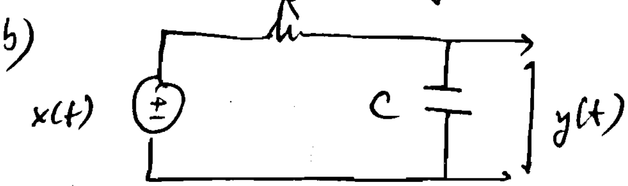
→ homogeneous eq: $\underbrace{LC\frac{d^2y}{dt^2} + y(t) = 0}_{\text{sol: } k_1 e^{j\omega_1 t} + k_2 e^{j\omega_2 t}}$

→ general equation: include a particular solution.

→ Sol. to the homogeneous equation is called "the natural response"

↳ (when input voltage is shortcircuited).

→ b)



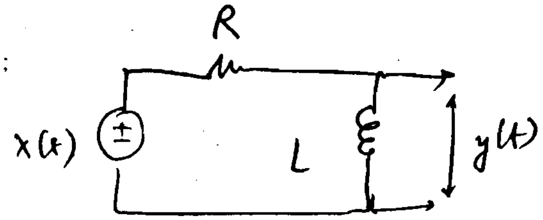
$$x(t) - i_c R - y(t) = 0$$

$$x(t) - RC\frac{dy}{dt} - y(t) = 0$$

$$\boxed{RC\frac{dy}{dt} + y = x(t)}$$

or $\frac{dy}{dt} + \frac{1}{RC} y = \frac{1}{RC} x \xrightarrow[y(0^-)=0]{} y(t) = \frac{1}{RC}\int_0^t x(\lambda)\, e^{-\frac{1}{RC}(t-\lambda)}\, d\lambda$

For curiosity:



$$x(t) - i_L R - y(t) = 0$$

$$y(t) = L\frac{di_L}{dt} \longrightarrow R i_L + L\frac{di_L}{dt} = x(t)$$

$$\text{or} \quad \frac{di_L}{dt} + \underbrace{\frac{R}{L}}_{a} i_L = \underbrace{\frac{1}{L}}_{b} x(t)$$

$i_L(0^-) = 0$

$$i_L(t) = \frac{1}{L}\int_0^t x(\lambda)\, e^{-\frac{R}{L}(t-\lambda)}\, d\lambda$$

$$y(t) = L\frac{di_L}{dt} = -\frac{R}{L}\int_0^t x(\lambda)\, e^{-\frac{R}{L}(t-\lambda)}\, d\lambda.$$

hom. eq: $\quad \dfrac{di_L}{dt} = -\dfrac{R}{L} i_L \rightarrow \dfrac{di_L}{i_L} = -\dfrac{R}{L}dt$

$$\ln i_L = -\frac{R}{L}t + \text{Const.}$$

$$i_L = k\, e^{-\frac{R}{L}t}$$

$$\boxed{i_L(t) = i_L(0)\, e^{-\frac{R}{L}t}}$$

1.36

$x(t) = e^{j\omega_0 t}$

Fundamental freq ω_0; fund. period $= \dfrac{2\pi}{\omega_0} = T_0$

$x[n] = x(nT) = e^{j\omega_0 nT}$

(a) $x[n]$ periodic only if $\boxed{\dfrac{T}{T_0} \text{ is a rational number}}$: Prove

$x[n+N] = x[n] \rightarrow$

$e^{j\omega_0 NT} = 1$ (integer) or $e^{j2\pi N\frac{T}{T_0}} = 1 \iff N\dfrac{T}{T_0} = \text{integer } m$

$\Rightarrow \boxed{\dfrac{T}{T_0} = \dfrac{m}{N}}$

(b) Suppose $x[n]$ is periodic $\left(\frac{T}{T_0} = \frac{p}{q}, \ p \,\&\, q \text{ are integers} \right)$

What are the $\underbrace{\text{fund. period}}_{\cancel{\text{Lowest}}\ N_0}$ & $\underbrace{\text{fund. frequency}}_{\text{Lowest } \tilde{w}_0}$ for $x[n]$?

$$e^{jw_0 T N_0} = 1 \quad \text{or} \quad e^{j 2\pi \frac{T}{T_0} N_0} = 1 \qquad \rightarrow \text{integer.}$$

$$\frac{T}{T_0} N_0 = m \rightarrow N_0 = \frac{T_0}{T} m = \frac{q}{p} m$$

<u>Examples:</u> $q = 7$; $p = 3 \longrightarrow$ Lowest $N_0 = 7$

$q = 28$; $p = 12 \longrightarrow$ lowest $N_0 = 7$

$$N_0 = \frac{28}{12} m = \frac{7}{3} m$$

Divide by greatest common divisor of $(p \,\&\, q)$

$$N_0 = \frac{\frac{p}{\gcd(p,q)}}{\cancel{\frac{q}{\gcd(p,q)}}} \cdot \underbrace{\frac{q}{\cancel{\gcd(p,q)}}}_{m \text{ to eliminate denominator}}$$

$$\boxed{N_0 = \frac{p}{\gcd(p,q)}}$$

c) $$\underbrace{\frac{N_0 T}{T_0}} = N_0 \frac{p}{q} = \frac{p}{\gcd(p,q)} \frac{p}{q} = \cancel{\frac{p}{\frac{q}{\gcd(p,q)}}}$$

Ch 3 Fourier Series Representation of Periodic Signals

HW3: 3.5; 3.12; 3.14; 3.22; 3.25; 3.28; 3.32; 3.46; 3.48; 3.52 a) b) c) d)

Periodic signals: (continuous – time)

Example: $\sin \omega t$, $\cos \omega t$, , $2\cos(10t) + \sin(4t)$

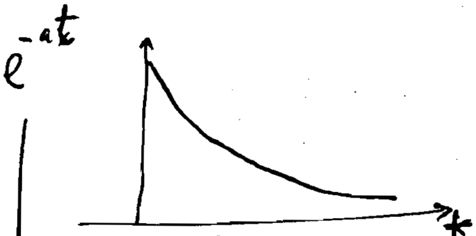
$\hookrightarrow Re[e^{j\omega t}]$

(they are combination of sinusoids)

General statement:

Is there any continuous & periodic function that is not a combination of sinusoids? No

e^{-at}



Not periodic

$e^{-jat} = \cos at - j\sin at$

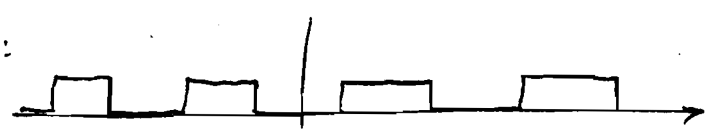
Periodic.

$\hookrightarrow$ Any continuous & periodic signal can be expressed as a sum (finite or infinite) of $\cos(k\omega_0 t)$ or $e^{jk\omega_0 t}$

k: integer

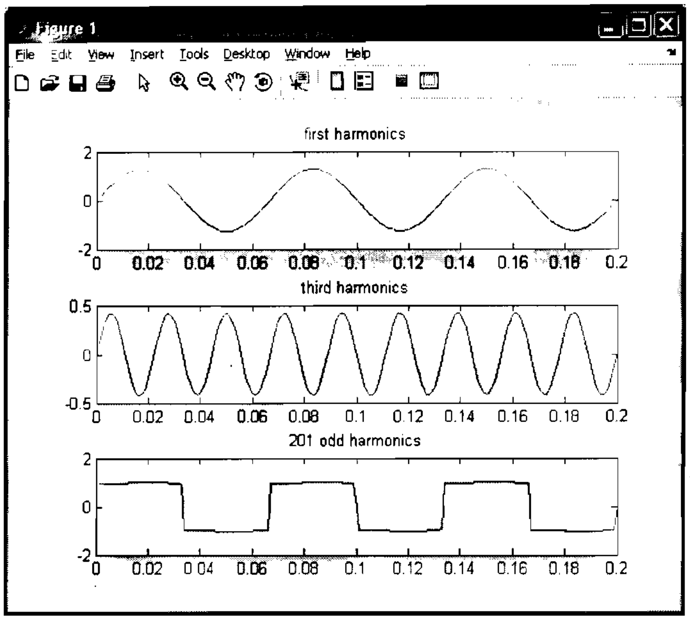
(ω_0: fundamental frequency).

E.g.:



Is a ∞ sum of odd harmonics: $\cos(\omega_0 t)$, $\cos(3\omega_0 t)$, $\cos(5\omega_0 t)$, etc ...

$\hookrightarrow$ $$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}} \quad (1)$$ Fourier Series Representation of a periodic signal $x(t)$



$w_0 = 2\pi \cdot 15$

$a_k = \dfrac{4}{(2k+1)\pi}$

```
%This example shows how to build a square wave from sinusoids
%Tomas Materdey, 2/06/06
clear all;
close all;
sec=1;
nf=200;%1000;%10;%=45;
f0=15;%9000000;
%deltaf=2*f;
sr=1000;%100000000;
deltat=1/sr;
ns=5000;%18000;%36000;%1000;
nt=ns*deltat/5;
sq=zeros(1,ns);
sq1=zeros(1,ns);
n=deltat:deltat:nt;
nr=deltat:deltat:nt/5;
clear sq;
sq=(4/pi)*sin(2*pi*f0*n);
sq0=sq;
subplot(3,1,1),plot(nr,sq(1:length(nr))),title('first harmonics')
sq1=(4/(3*pi))*sin(2*pi*3*f0*n);
subplot(3,1,2),plot(nr,sq1(1:length(nr))),title('third harmonics')
for i = 1:1:nf;
sq=sq+(4/((2*i+1)*pi))*sin(2*pi*(2*i+1)*f0*n+1/nf);
end
subplot(3,1,3)
plot(nr,sq(1:length(nr))), title(strcat(num2str(nf+1),' odd
harmonics'))
```

Corollary: a periodic signal $x(t)$ is identified by ω_0 and the series a_k.

How to find a_k given a signal $x(t)$? :

Use orthogonality of complex exponentials (or plane waves)

$$\hookrightarrow \underbrace{\int_0^T e^{jk\omega_0 t} e^{-jn\omega_0 t} dt}_{\text{product of 2 complex exponentials of frequencies } k\omega_0 \text{ \& } n\omega_0} = \begin{cases} T & \text{if } k=n \\ 0 & \text{if } k \neq n \end{cases}$$

Apply $\int_0^T dt\, e^{-jn\omega_0 t}$ to both sides of equation (1)

$$\int_0^T dt\, e^{-jn\omega_0 t} x(t) = \sum_{k=-\infty}^{\infty} a_k \underbrace{\int_0^T dt\, e^{j(k-n)\omega_0 t}}_{T\delta[k-n]}$$

$$= T a_n$$

$$\rightarrow \boxed{a_n = \frac{1}{T}\int_0^T dt\, e^{-jn\omega_0 t} x(t)} \text{ or } \boxed{a_k = \frac{1}{T}\int_0^T dt\, e^{-jk\omega_0 t} x(t)}$$

T is $\frac{2\pi}{\omega_0}$ or the fundamental period.

Properties of Fourier Series Representation (Table 3.1 pg 206)

Linearity:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$$

$x(t)$ & $y(t)$ are both periodic with fundamental freq ω_0

$\hookrightarrow$ What are the Fourier Series coefficients for $Ax(t) + By(t)$?

They are $Aa_k + Bb_k$ since:

$$Ax(t) + By(t) = \sum_{k=-\infty}^{\infty} (Aa_k + Bb_k)\, e^{jk\omega_0 t}$$

Time-shift:

Fourier series coefficients for $x(t)$ are a_k

What are the $\underbrace{\text{Fourier series coefficients for } x(t-t_0)}$?

↳ $$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x(t-t_0) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 (t-t_0)} = \sum_{k=-\infty}^{\infty} \boxed{a_k e^{-jk\omega_0 t_0}}\, e^{jk\omega_0 t}$$

- If we think of the F.S. repres. as a system, it is linear but not time invariant.
(but there is no time in the output anyway → can't say)

Frequency-shift

Time reversal

Time scaling

Multiplication

Differentiation

Integration

Parseval's relation:
$$\frac{1}{T}\int_T |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

35 :

$$x_1(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_1 t}$$ { $x(t)$ is identified by ω_1 & a_k

$$x_2(t) = \underbrace{x_1(1-t)}_{\omega_1} + \underbrace{x_1(t-1)}_{\omega_1}$$ { $x_2(t)$ is identified $\underset{=\omega_1}{\omega_2}$ & b_k

b_k : $$x_2(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_1(1-t)} + \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_1(t-1)}$$

goal: $x_2(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_1 t}$

$$= \sum_{k=-\infty}^{\infty} \Big(\underbrace{\underbrace{a_k e^{jk\omega_1}}_{\text{coeff}} \underbrace{e^{jk(-\omega_1)t}}_{e^{j(-k)\omega_1 t}}}_{①} + \underbrace{\underbrace{a_k e^{-jk\omega_1}}_{\text{coeff}} e^{jk\omega_1 t}}_{②} \Big)$$

	①	②
$k \in (-\infty, 0)$	$e^{j\lvert k\rvert\omega_1 t}$	$e^{-j\lvert k\rvert\omega_1 t}$
$k \in (0, +\infty)$	$e^{-j\lvert k\rvert\omega_1 t}$	$e^{j\lvert k\rvert\omega_1 t}$

$$= \underbrace{\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_1} \underbrace{e^{j(-k)\omega_1 t}}_{\text{would like to have } e^{jk\omega_1 t}}}_{k \to \text{ renamed by } -k \text{ (Math's trick)}} + \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega_1} \underbrace{e^{jk\omega_1 t}}$$

$$\sum_{k=+\infty}^{-\infty} a_{-k} e^{-jk\omega_1} e^{jk\omega_1 t}$$

$$\parallel$$

$$\sum_{k=-\infty}^{\infty} a_{-k} e^{-jk\omega_1} \underbrace{e^{jk\omega_1 t}}$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{(a_{-k} + a_k) e^{-jk\omega_1}}_{b_k} e^{jk\omega_1 t} \text{ same}$$

Now using table 3.1:

3.5.2 Time-shift: $x(t-t_0) \longrightarrow a_k e^{-jk\omega_0 t_0}$

$$x_2(t) = \underbrace{x_1(1-t)}_{\downarrow \; a_{-k}e^{jk\omega_0}} + \underbrace{x_1(t-1)}_{\downarrow \; a_k e^{-jk\omega_0}}$$

time-reversal & time-shift of +1 $\longrightarrow a_{-k}e^{jk\omega_0}$

$a_k e^{-jk\omega_0} \longleftarrow$ time-shift of 1

& 3.5.3 Time-reversal $x(-t) \longrightarrow a_{-k}$

or: let's define $\lambda \equiv t-1$ (another time variable) $\rightarrow x_2 = \underbrace{x_1(-\lambda)}_{\downarrow \; c_{-k}} + \underbrace{x_1(\lambda)}_{\downarrow \; c_k}$

(due to the time reversal property).

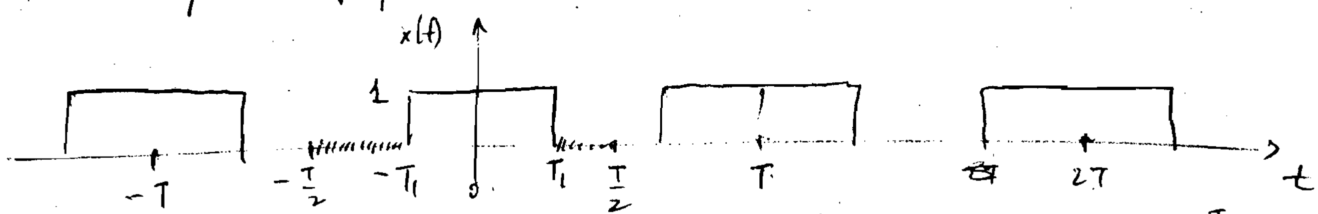
what is c_k? due to time-shift: $c_k = a_k e^{-jk\omega_0 \cancel{t}}$

so $c_{-k} = a_{-k} e^{jk\omega_0}$

→ Fourier Series Representation ~~for a continuous-time periodic signal~~:

Example 3.5: What are the a_k's for $x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < \frac{T}{2} \end{cases}$

$x(t)$ is train of rectangular pulses where $T_1 = \frac{1}{2}$ width of pulses; T is the period of pulses.



$$a_k = \frac{1}{T}\int dt\, e^{-jk\omega_0 t} x(t) = \frac{1}{T}\int_0^T dt\, e^{-jk\omega_0 t} x(t) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} dt\, e^{-jk\omega_0 t} x(t)$$

$$= \frac{1}{T}\int_{-T_1}^{T_1} dt\, e^{-jk\omega_0 t} = \frac{1}{T}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_{-T_1}^{T_1} = \frac{1}{-jk\omega_0 T}\underbrace{\left(e^{-jk\omega_0 T_1} - e^{+jk\omega_0 T_1}\right)}_{-2j\sin(k\omega_0 T_1)}$$

$$e^{ja} - e^{-ja} = 2j \sin a$$

$$\begin{cases} a_k = \dfrac{2 \sin(k\omega_0 T_1)}{\underset{\substack{\downarrow \\ \text{Fund. freq } \frac{2\pi}{T}}}{k\omega_0 T}} = \dfrac{\sin(k\omega_0 T_1)}{k\pi} \leftarrow \underline{\text{Good for } k \neq 0} \\ a_0 = \dfrac{1}{T}\displaystyle\int_{-T_1}^{T_1} dt \; e^{j0\omega_0 t} \text{-}1 = \dfrac{1}{T}\int_{-T_1}^{T_1} dt = \dfrac{2T_1}{T} \end{cases}$$

<u>Fourier Series Representation for a discrete-time periodic signal:</u>

$x[n]$, periodic : $x[n+N] = x[n]$

<u>Examples</u>: $\cos[\omega_0 n]$; $\cos[\omega n]$: $\cos[\omega(n+N)] = \cos[\omega n]$

$\rightarrow \omega N = m 2\pi$

$\omega = \dfrac{m 2\pi}{N}$

$\rightarrow \omega_0 = \dfrac{2\pi}{N} \quad (m=1)$

$\hookrightarrow e^{jk\omega_0 n} = e^{jk\frac{2\pi}{N}n}$

$e^{jk\omega_0 t}$

$\downarrow$

Not the same signal for different k's

F.S. $\downarrow$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$e^{jk\frac{2\pi}{N}n}$

$\downarrow$

If k is increased by N $\rightarrow$ same signal.

$$e^{j(k+N)\frac{2\pi}{N}n} = e^{jk\frac{2\pi}{N}n} \underbrace{e^{j2\pi n}}_{1}$$

F.S. $\downarrow$

$$x[n] = \sum_{\underbrace{k=\langle N\rangle}_{\text{only } N \text{ terms.}}} a_k e^{jk\frac{2\pi}{N}n}$$

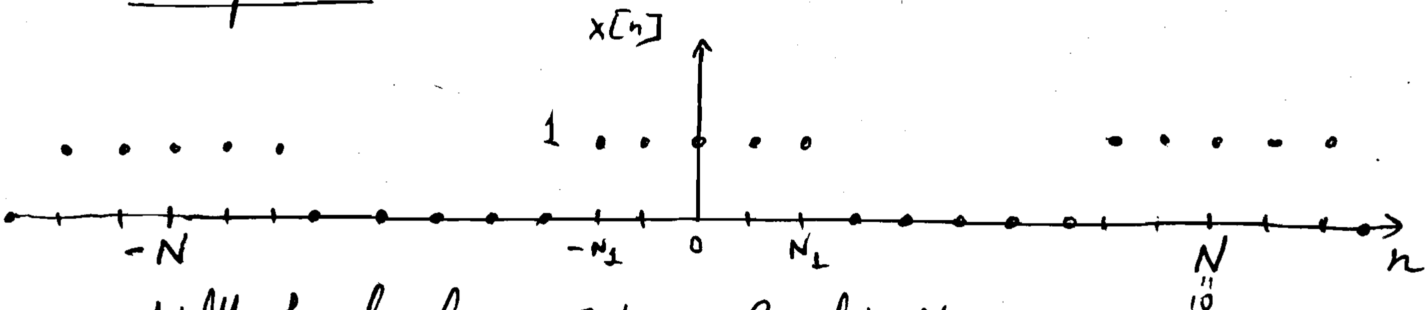
Fourier Series for a discrete-time periodic signal (Cont.):

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n} \qquad \text{(only } N \text{ terms)}$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jk\frac{2\pi}{N}n}$$

We did find a_k for a continuous-time train of rectangular pulses, now let's do the discrete-time counterpart:

Example 3.12:



Width of each pulse is $2N_1$. Period is N

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n] e^{-jk\frac{2\pi}{N}n} \underset{\substack{\text{center the}\\ \text{period around } n=0}}{=} \frac{1}{N}\sum_{n=-N_1}^{N_1} x[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\underbrace{\sum_{n=-N_1}^{N_1} e^{-jk\frac{2\pi}{N}n}}_{\text{geometric sum}}$$

$$\boxed{\sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha} \quad \text{Geometric sum}}$$

$$\sum_{n=-N_1}^{N_1} \underbrace{\left(e^{-jk\frac{2\pi}{N}}\right)}_{\equiv\ \alpha}^{n} \equiv \sum_{n=-N_1}^{N_1} \alpha^n = \sum_{m=0}^{2N_1} \alpha^{m-N_1} = \alpha^{-N_1}\sum_{m=0}^{2N_1}\alpha^m = \alpha^{-N_1}\frac{1-\alpha^{2N_1+1}}{1-\alpha}$$

change of dummy index $m = n + N_1$ } similar to a change of integration variable ‖ $\alpha^{m-N_1} = \alpha^m \alpha^{-N_1}$

$$\Rightarrow \quad a_k = \frac{1}{N} e^{jk\frac{2\pi N_1}{N}} \frac{1 - e^{-jk\frac{2\pi}{N}(2N_1+1)}}{1 - e^{-jk\frac{2\pi}{N}}}$$