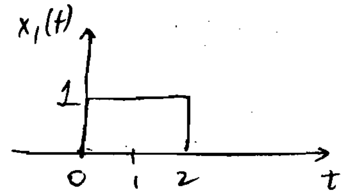
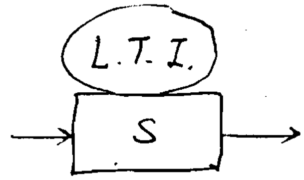
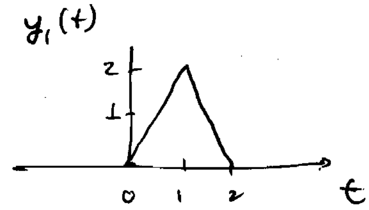
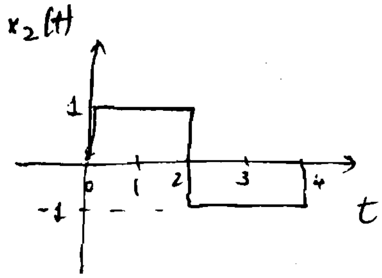
$a x_1$ $a y_1$

1.31 a)





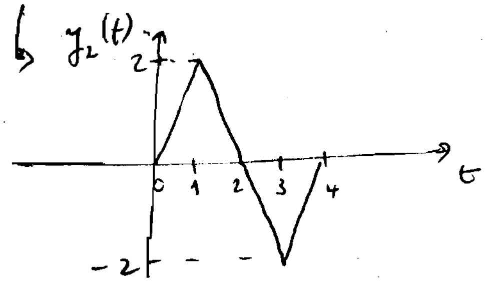




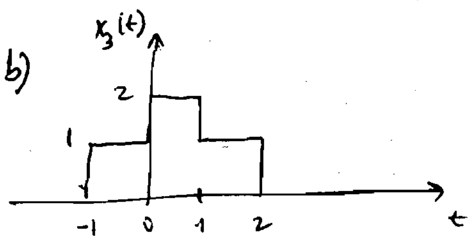
$$x_2(t) = x_1(t) - x_1(t-2)$$

$\downarrow$ L.T.I.

$$y_2(t) = y_1(t) - y_1(t-2)$$



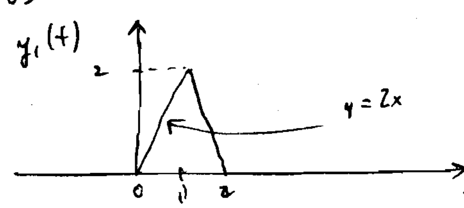
b)

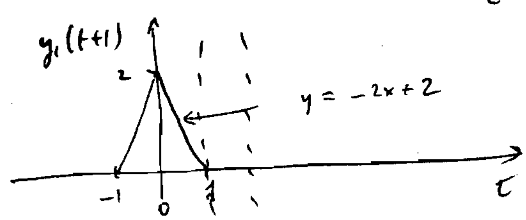


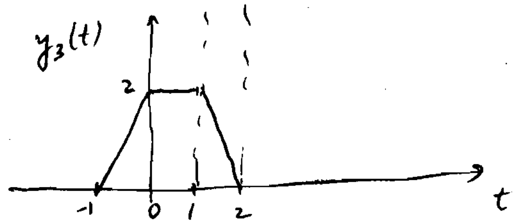
$$x_3(t) = x_1(t) + x_1(t+1)$$

$\downarrow$

$$y_3(t) = y_1(t) + y_1(t+1)$$







2.17)

$$x(t) \longrightarrow \boxed{S} \longrightarrow y(t)$$

a)

↳ input/output differential equation:

1st order D.E.

$\boxed{\frac{dy}{dt} + ay = bx}$ $\qquad$ $\boxed{\frac{dy}{dt} + 4y(t) = x(t)}$; $\underbrace{\text{initial rest}: x(0^-) = 0, \; y(0^-) = 0}$

a) What is $y(t)$ if $x(t) = e^{(-(1+3j)t}\, u(t)$?

→ Solve this differential equation: integrating factor: be^{at}, to be multiplied to both sides of the D.E.

$$\underbrace{\frac{dy}{dt} b e^{at} + ay\, b e^{at}} = b e^{at} x$$

$$\underbrace{b\left(\frac{dy}{dt} e^{at} + y a e^{at}\right)}$$

$$b\left(\frac{dy}{dt} e^{at} + y \frac{d}{dt} e^{at}\right)$$

$$\cancel{b} \frac{d}{dt}\left(y e^{at}\right)$$

This allows me to integrate the D.E: $\left[y e^{at}\right]_{t_0}^{t} = b \int_{t_0}^{t} x e^{at}\, dt$

$$\rightarrow y(t) e^{at} - \underbrace{y(t_0) e^{at_0}} = b \int_{t_0}^{t} x(\lambda) e^{a\lambda}\, d\lambda$$

$$\rightarrow y(t) \underbrace{e^{at}} = y(t_0) e^{at_0} + b \int_{t_0}^{t} x(\lambda) e^{a\lambda}\, d\lambda$$

$$\boxed{\frac{dy}{dt} + ay(t) = bx(t)} \rightarrow \boxed{y(t) = y(t_0) e^{-a(t-t_0)} + b \int_{t_0}^{t} x(\lambda) e^{-a(t-\lambda)}\, d\lambda}$$

In our problem: $t_0 = 0$ and $y(0^-) = 0$

$$\hookrightarrow y(t) = 1\int_0^t x(\lambda)\, e^{-4(t-\lambda)}\, d\lambda \quad ; \quad x(\lambda) = e^{+(-1+3j)\lambda}\, u(\lambda)$$

$$= \int_0^t e^{+(-1+3j)\lambda}\, e^{-4(t-\lambda)}\, d\lambda = e^{-4t}\int_0^t \underbrace{e^{+(3+3j)\lambda}}_{e^{(3+3j)\lambda}}\, d\lambda$$

$$= e^{-4t}\left[\frac{e^{(3+3j)\lambda}}{3+3j}\right]_0^t = e^{-4t}\left[\frac{e^{3(1+j)t} - 1}{3(1+j)}\right]$$

$$\boxed{y(t) = \frac{1}{3(1+j)}\left[e^{(-1+3j)t} - e^{-4t}\right], \quad t>0}$$

b) What is $y(t)$ if $x(t) = Re\left[e^{(-1+3j)t}\, u(t)\right]$

$$= Re\left[e^{-t}\, \underbrace{e^{3jt}}_{(\cos 3t + j\sin 3t)}\, u(t)\right] = e^{-t}\cos 3t\, u(t)$$

$x \to \boxed{S} \to y$

$\boxed{Re(x)} + j\, Im(x)$ $\qquad$ $\boxed{Re(y)} + j\, Im(y)$

$$\to y(t) = Re\left[\frac{1}{3(1+j)}\left[e^{(-1+3j)t} - e^{-4t}\right]\right]$$

$$= Re\left[\frac{(1-j)}{3\cdot 2}\left[e^{-t}e^{3jt} - e^{-4t}\right]\right]$$

$$= \frac{1}{6} Re\left[e^{-t}e^{3jt} - e^{-4t} - je^{-t}e^{3jt} + je^{-4t}\right]$$

$$= \frac{1}{6}\left[e^{-t}\cos 3t - e^{-4t} + e^{-t}\sin 3t\right] \cancel{u(t)}$$

$$\to \boxed{y(t) = \frac{1}{6}\left[e^{-t}\cos 3t - e^{-4t} + e^{-t}\sin 3t\right] u(t)}$$

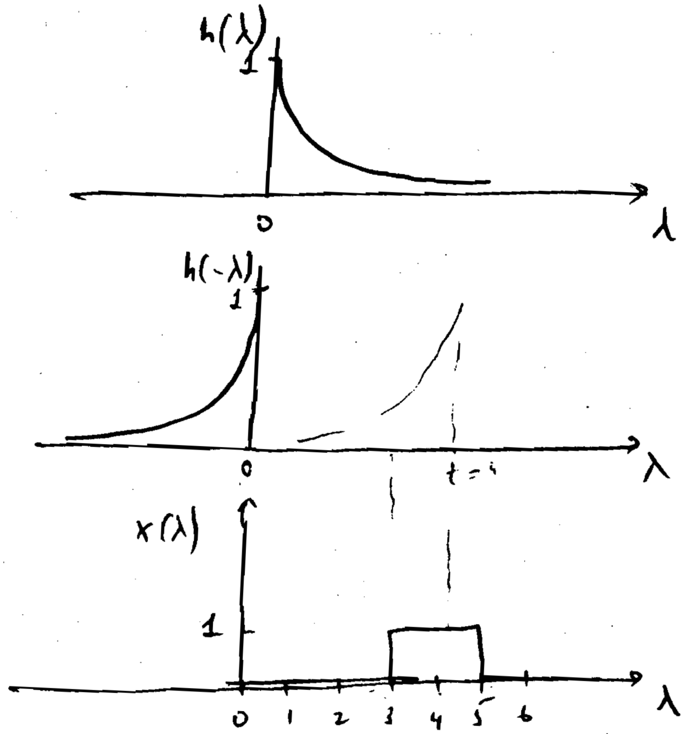
c) How is $g(t) = \frac{dx}{dt} * h$ & $y(t) = x * h$

$$\frac{dy}{dt} = \left\{ \begin{array}{lll} t<3 & : & 0 \\ 3 \le t \le 5 & \cdot & -\frac{(-3)}{3} e^{-3(t-3)} = e^{-3(t-3)} \\ 5<t & : & -\frac{3}{3}\left[e^{-3(t-5)} - e^{-3(t-3)} \right] = e^{-3(t-3)} - e^{-3(t-5)} \end{array} \right\} = g(t)$$

Property of convolution ! $\frac{d}{dt}(x * h) = \frac{dx}{dt} * h = x * \frac{dh}{dt}$

If we use the other expression for the convolution:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} d\lambda \; x(\lambda) h(t-\lambda)$$



$t<3$ no overlap.

$3 \le t \le 5$: overlapping interval is $(3, t)$ ✓

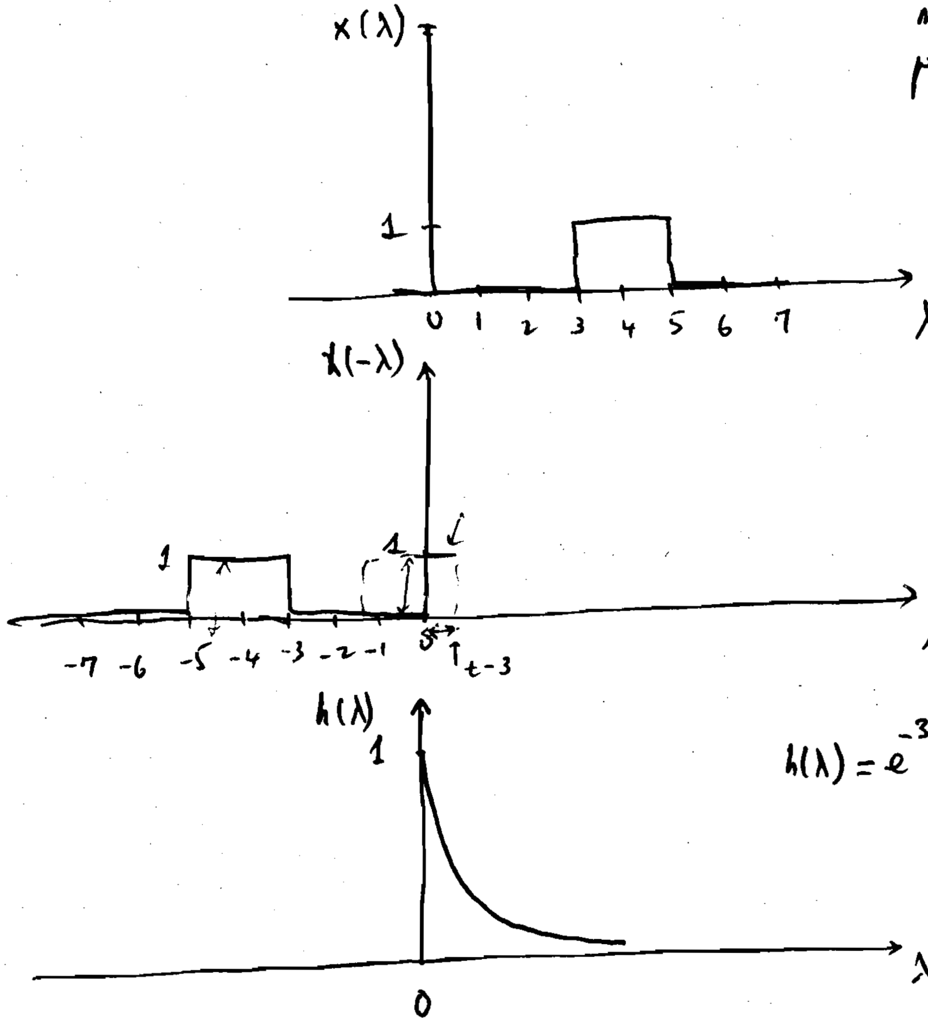
$5<t$: overlapping interval is $(3, 5)$

$$y(t) = \begin{cases} t<3 : & 0 \\ 3 \le t \le 5 : & \displaystyle\int_3^t d\lambda \; 1 . e^{-3(t-\lambda)} = e^{-3t}\left(\frac{e^{3\lambda}}{3}\right)_3^t = e^{-3t}\left[\frac{e^{3t} - e^{9}}{3}\right] \\ & \displaystyle = \frac{1 - e^{-3(t-3)}}{3} \\ 5<t : & \displaystyle\int_3^5 d\lambda \; 1 . e^{-3(t-\lambda)} = e^{-3t}\left[\frac{e^{3\lambda}}{3}\right]_3^5 = e^{-3t}\left[\frac{e^{15} - e^{9}}{3}\right] \\ & \displaystyle = \frac{e^{-3(t-5)} - e^{-3(t-3)}}{3} \end{cases}$$

Same as before!

2.11 a) $x(t) = u(t-3) - u(t-5)$; $h(t) = e^{-3t} u(t)$

$$y(t) = x(t) * h(t) = \underbrace{\int_{-\infty}^{\infty} d\lambda\, h(\lambda)\, x(t-\lambda)}_{\text{multiplying point by point } h(\lambda) \text{ \& } x(t-\lambda)} = \int_{-\infty}^{\infty} d\lambda\, x(\lambda)\, h(t-\lambda)$$



$h(\lambda) = e^{-3\lambda} u(\lambda)$

Intervals for t:

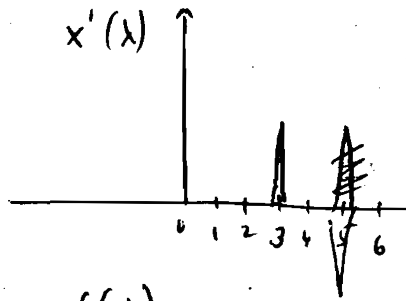
$t < 3$ - no overlap

$3 \leq t \leq 5$: overlapping interval is $(0, t-3)$

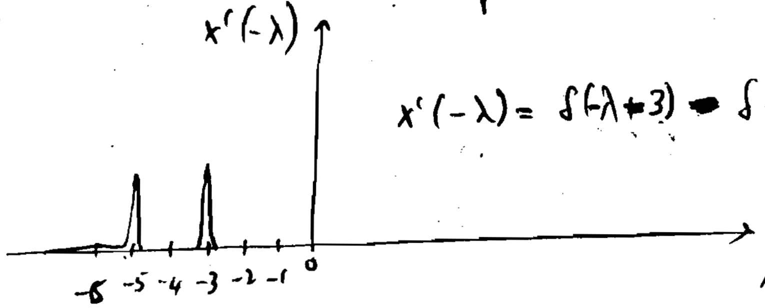
$5 < t$: overlapping interval $(t-5, t-3)$

$$y(t) = \begin{cases} t<0 : & 0 \\ 3 \leq t \leq 5 : & \displaystyle\int_0^{t-3} d\lambda\, e^{-3\lambda}\, 1 = \left[\frac{e^{-3\lambda}}{-3}\right]_0^{t-3} = \frac{1 - e^{-3(t-3)}}{3} \\ 5<t : & \displaystyle\int_{t-5}^{t-3} d\lambda\, e^{-3\lambda} \cdot 1 = \left[\frac{e^{-3\lambda}}{-3}\right]_{t-5}^{t-3} = \frac{e^{-3(t-5)} - e^{-3(t-3)}}{3} \end{cases}$$

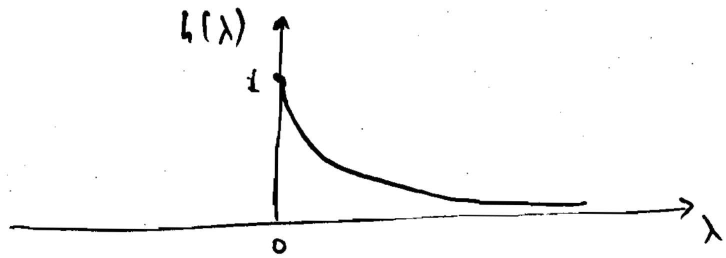
b) $g(t) = \frac{dx}{dt} * h(t) = x'(t) * h(t) = \int_{-\infty}^{\infty} d\lambda \, h(\lambda) \, x'(t-\lambda)$



$x'(\lambda) = \delta(\lambda-3) - \delta(\lambda-5)$



$x'(-\lambda) = \delta(-\lambda+3) - \delta(-\lambda+5)$



intervals for t:

$t < 3$: no overlap.

$3 \leq t < 5$: 1 overlap by $\delta(-\lambda+3)$

$5 \leq t$: 2 overlaps by $\delta(-\lambda+3)$ & $\delta(-\lambda+5)$

$$g(t) = \begin{cases} t<3 : & 0 \\ 3 \leq t < 5 : & \int_0^{\infty} d\lambda \, e^{-3\lambda} \, \delta(-\lambda-3+t) = e^{-3(t-3)} \checkmark \\ 5 \leq t : & \int_0^{\infty} d\lambda \, e^{-3\lambda} \underbrace{\delta(-\lambda-3+t)}_{\text{centered at } t-3} - \int_0^{\infty} d\lambda \, e^{-3\lambda} \underbrace{\delta(-\lambda-5+t)}_{\text{centered at } t-5} \\ & = e^{-3(t-3)} - e^{-3(t-5)} \checkmark \end{cases}$$

centered at $+(t-3)$

$-(\lambda+3-t)$

$$\boxed{\int_0^{\infty} f(\lambda) \, \delta(\lambda-a) = f(a)}$$

$a>0$

c)