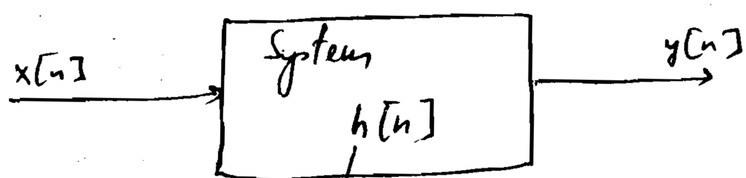
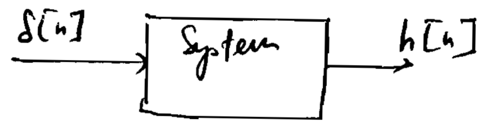
Ch 2: Linear Time-Invariant Systems

<u>HW 2:</u> 2.7; 2.8; 2.11; 2.19; 2.22; 2.24; 2.31; 2.40; 2.47; 2.61

<u>Convolution</u>: it is a mathematical operation that allows us to
"$*$" calculate $y[n]$ from $x[n]$ and $h[n]$



impulse-response
for a discrete-time system
($h[n]$ is the output for an impulse input by the system). It is used to characterize the system.



$$y[n] = x[n] * h[n] \quad (y[n] \text{ is } x[n] \text{ convolution with } h[n])$$
$$= \sum_{k=-\infty}^{\infty} x[k] \cdot \underbrace{h[n-k]}$$

<u>Example</u>: Find $y[n]$ from $x[n] = \alpha^n u[n]$ & $h[n] = u[n]$

step function; the power of constant elevated to the time index

(We work with a system whose response to an impulse input is the step function) $\longrightarrow$ e.g. the light switch. We would like to know what is the output $y[n]$ produced by the light switch by an input that starts at $n=0$ and increases as α^n. We will find out by applying the convolution b/w $\alpha^n u[n]$ & $u[n]$

$$\Rightarrow \; y[n] = x[n] * h[n] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha} \qquad (n>0)$$

Derivation of the formula for the sum of the geometric series

$$\sum_{k=0}^{n} \alpha^k = \underbrace{\alpha^0}_{1} + \alpha^1 + \alpha^2 + \alpha^3 + \cdots + \alpha^{n-1} + \alpha^n = S$$

$$\alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^{n-1} + \alpha^n = S-1$$

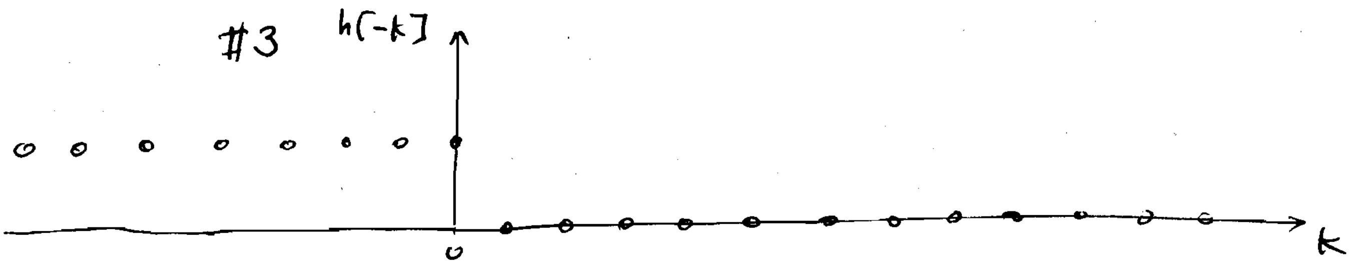
$$\alpha\underbrace{\left(1 + \alpha + \alpha^2 + \cdots + \alpha^{n-2} + \alpha^{n-1}\right)}_{S-\alpha^n} = S-1$$

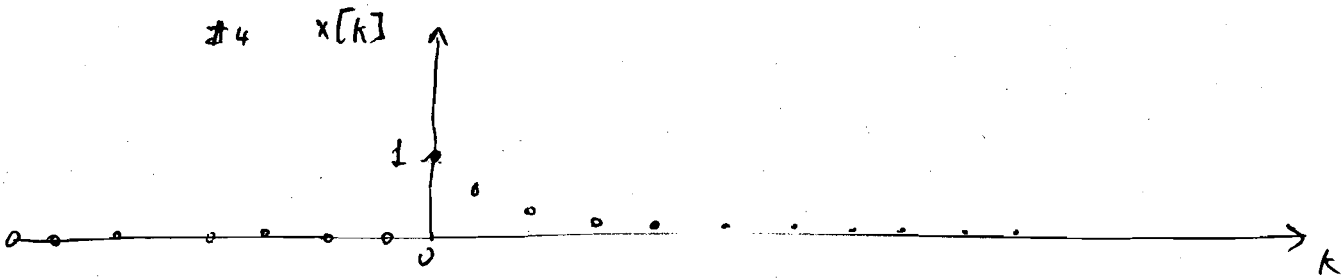
$$\alpha(S-\alpha^n) = S-1$$

$$1-\alpha^{n+1} = S - \alpha S \rightarrow S = \frac{1-\alpha^{n+1}}{1-\alpha}$$

Now $y[n] = \frac{1-\alpha^{n+1}}{1-\alpha}$ for $n>0$ as we assume a positive n from graph #2 to graph #3. What about $n \leq 0$?

1) $n=0$: what is $y[0] = x[n] * h[n] \big/_{n=0}$?



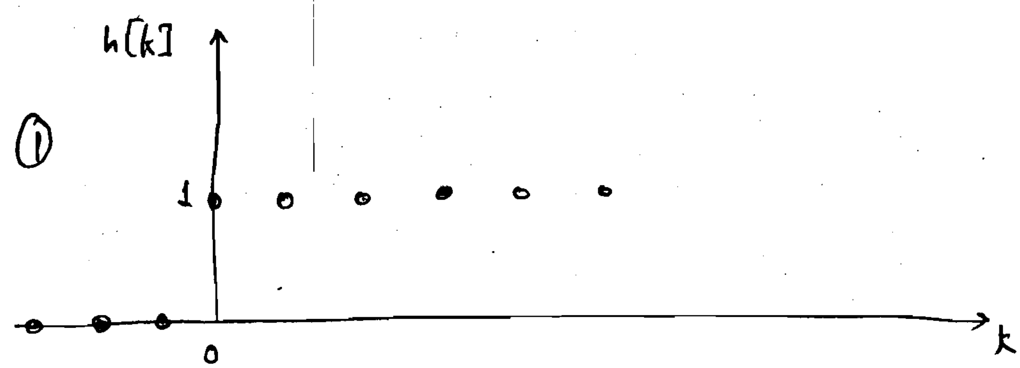


$\rightarrow y[0] = 1 + 0 + 0 + \cdots = 1$ just one non-zero ~~[illegible]~~ overlapping point b/w graph #3 & graph #4.

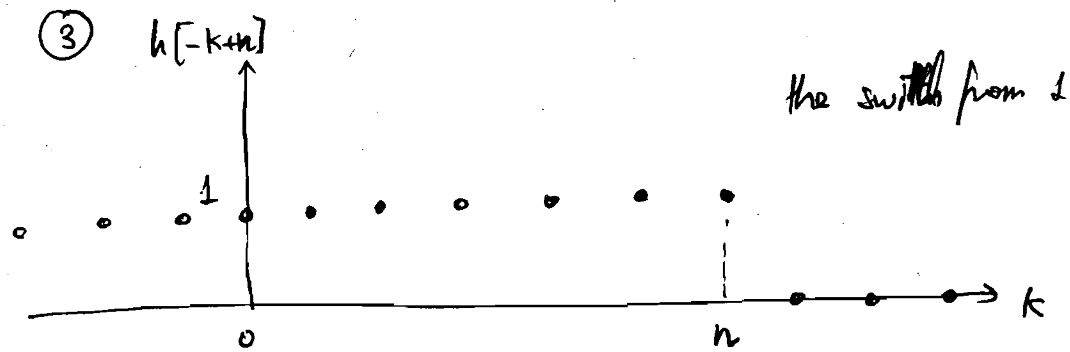
looking at the definition of the convolution we need to multiply

$x[k]$ to $h[n-k]$

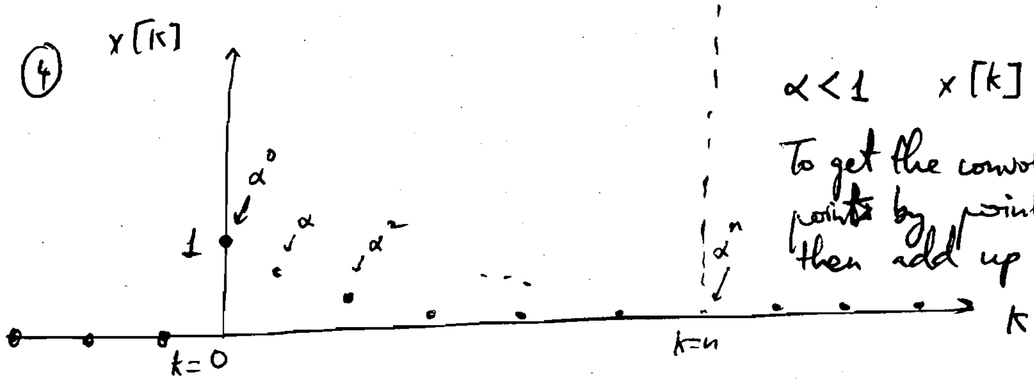
$\underbrace{\qquad\qquad}$ can be obtained from $h[k]$ by $\begin{cases} \to \text{time reversal} \\ \& \\ \to \text{time shift of } n \end{cases}$



time-reversal is a graphical reflection wrt the vertical axis.



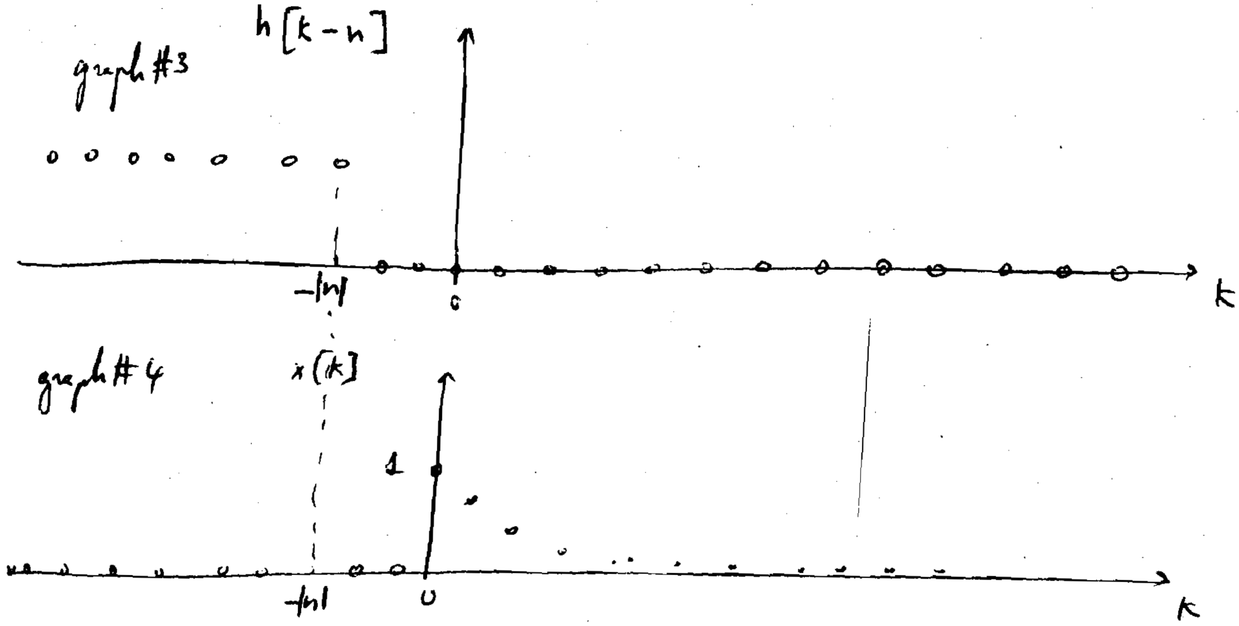
the switch from 1 to 0 happens at n



$\alpha < 1 \quad x[k] = \alpha^k u[k]$

To get the convolution we multiply points by point graph #3 & graph #4 then add up all these products

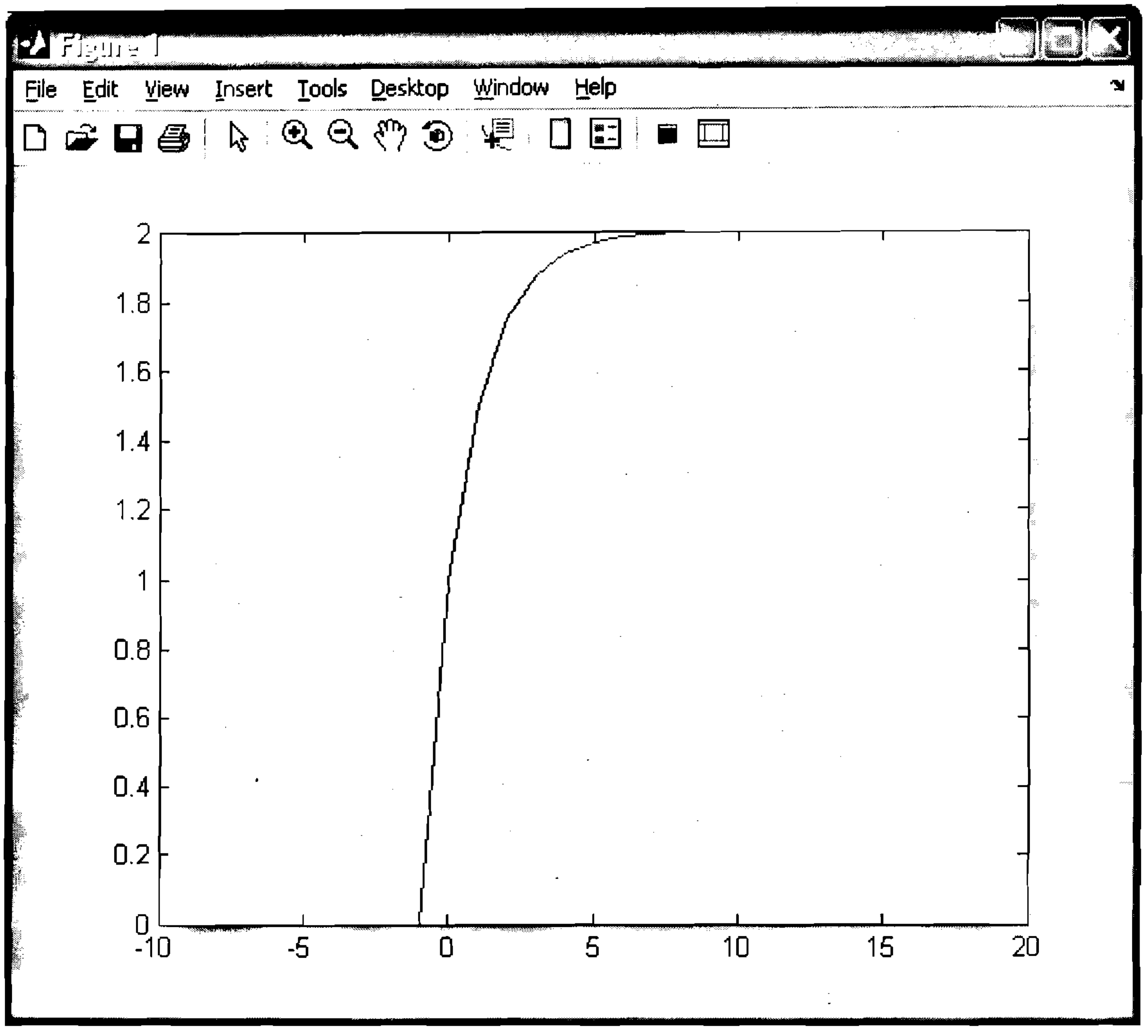
2) For $n < 0$:



$y[n] = x[n] * h[n] \Big|_{n<0} = 0$ | There is no non-zero overlapping points b/w graph#3 & graph #4.

The result from applying $x[n] = \alpha^n u[n]$ to a light-switch-like system is

$$y[n] = x[n] * h[n] = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ \dfrac{1-\alpha^{n+1}}{1-\alpha} & n > 0 \end{cases} = \frac{1-\alpha^{n+1}}{1-\alpha} u[n]$$




```
%Output by a light switch like system (h[n]=u[n]);
%plotting from -10 to 20
a=zeros(1,10);

alpha=0.5;
for n=1:20;
y(n)= (1-alpha^(n+1))/(1-alpha);
end
y1=[a 1 y]; %we insert y[0]=1 in between since Matlab does not like
zero as an index in y(n)
n1=-10:20;
plot(n1, y1)
```

c) Is this system LTI? - Linearity clear from input/output equation

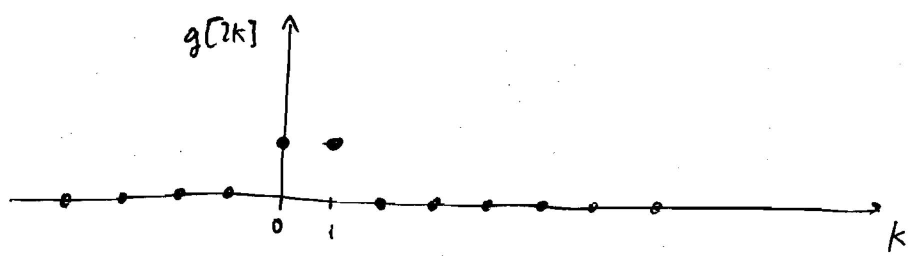
Based on a) & b)

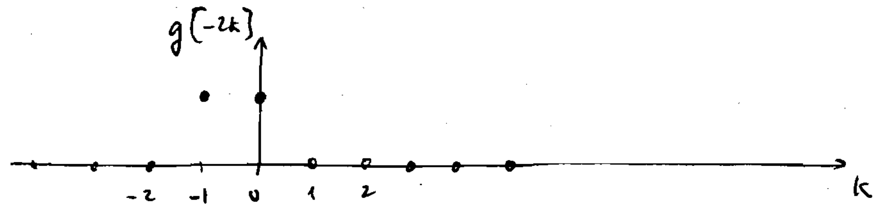
when input gets a time shift of 1, the output is time shifted by 2 → This indicates the system is not time-invariant.
(It would be if the output is time-shift by 1 as well)

d) $y[n]$? when $x[n] = u[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} u[k]\, g[n-2k]$$

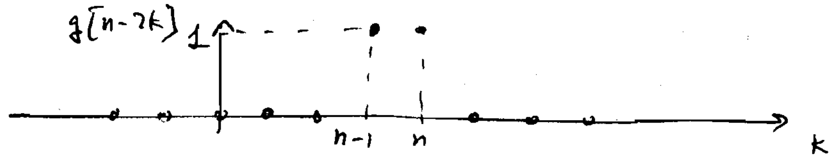
$$g[2k] = u[2k] - u[2k-4]$$





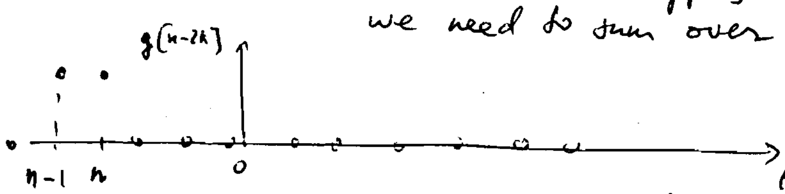
$n = 0$ $\quad y[n] = 1$ (only one overlapping term)

$n > 0$



$y[n] = 2$ (there are 2 overlapping terms : $k = n-1$ & $k = n$ we need to sum over)

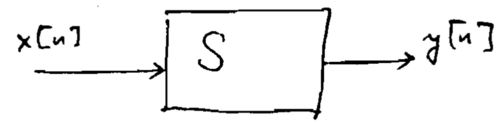
$n < 0$



$y[n] = 0$ (there is no overlapping terms with $u[k]$)

$$\rightarrow y[n] = 2u[n] - \delta[n]$$

2.7

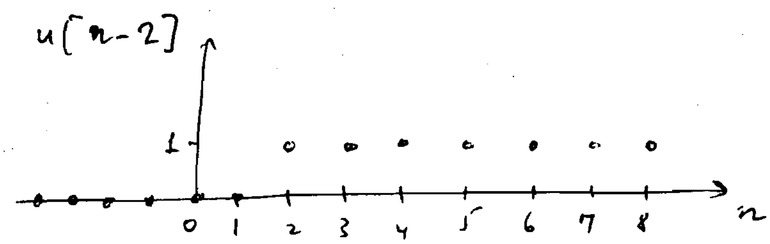


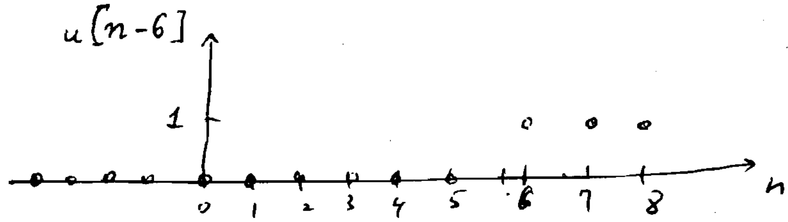
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, g[n-2k] \quad ; \quad g[n] = u[n] - u[n-4]$$

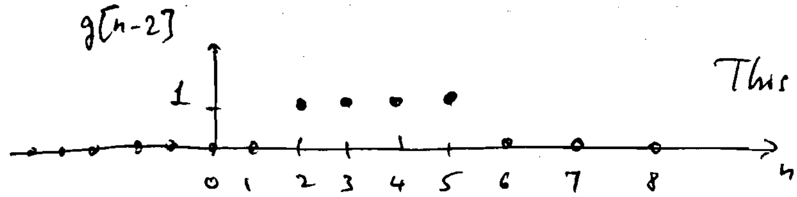
a) $y[n]$? when $x[n] = \delta[n-1]$.

$$y[n] = \sum_{k=-\infty}^{\infty} \delta[k-1] \cdot g[n-2k] \stackrel{?}{=} g[n-2] = u[n-2] - u[n-6]$$

This is a convolution: from the example at beginning of chapter we need to see the overlap b/w $\delta(k-1)$ & $g[n-2k]$. Since $\delta[k-1]$ just has 1 point at $k=1$. This summation just has 1 term at $k=1$



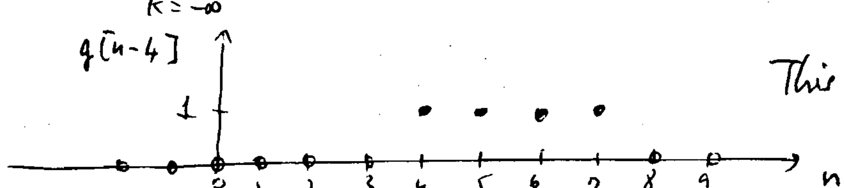




This is $y[n]$
(just 1 b/w 2 and 5)

b) $y[n]$? when $x[n] = \delta[n-2]$

$$y[n] = \sum_{k=-\infty}^{\infty} \delta[k-2]\, g[n-2k] = g[n-4] = u[n-4] - u[n-8]$$



This is $y[n]$.

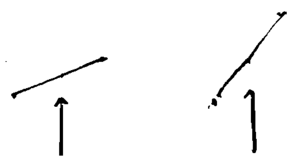
$x(t) \rightarrow \boxed{S} \rightarrow y(t)$ → S is characterized by this input/output differential equation

1.27 g) $y(t) = \underbrace{\dfrac{dx}{dt}}_{\text{slope}}$

$$\boxed{\begin{array}{l} \text{Memoryless} \rightarrow \text{No} \\ \text{Time invariant} \rightarrow \text{Yes.} \\ \text{Linear} \rightarrow \text{Yes (derivative is linear)} \\ \text{Causal} \rightarrow \text{Yes.} \\ \text{Stable} \rightarrow \text{NO} \end{array}}$$

Memory:

- $\dfrac{dx}{dt} = \lim_{\varepsilon \to 0} \dfrac{x(t+\frac{\varepsilon}{2}) - x(t-\frac{\varepsilon}{2})}{\varepsilon} = \xrightarrow[\alpha = -\frac{\varepsilon}{2}]{} = \lim_{\varepsilon \to 0} \dfrac{x(t) - x(t-\varepsilon)}{\varepsilon}$
- existence requires continuity of $x(t)$
- graphically:



Time inv: if time is shifted by α in $x(t)$ would it be shifted by the same amount in $y(t)$

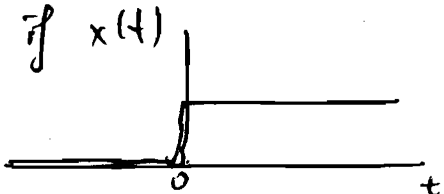
$x(t) \rightarrow x(t+\alpha)$

$$\lim_{\varepsilon \to 0} \frac{x(t+\alpha+\frac{\varepsilon}{2}) - x(t+\alpha-\frac{\varepsilon}{2})}{\varepsilon} = \frac{dx(t+\alpha)}{dt} = y(t+\alpha) \checkmark$$

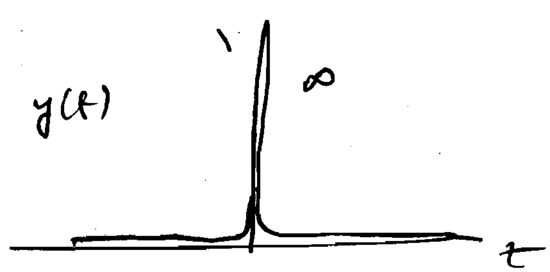
Causality: can refer to the discrete-time version:

$$y(t) = \lim_{\varepsilon \to 0} \frac{x(t) - x(t-\varepsilon)}{\varepsilon} \rightarrow y[n] = \frac{x[n] - x[n-1]}{\Delta}$$

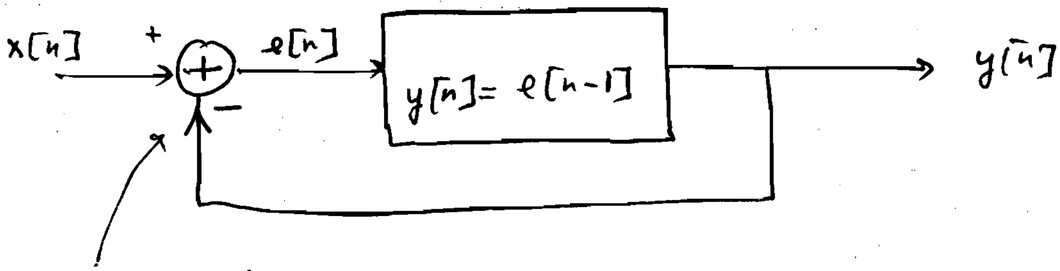
Stable:
↳ No: if $x(t)$



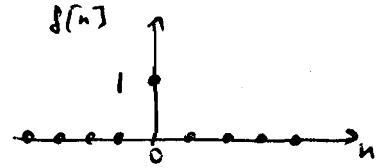
$y(t)$



1.46



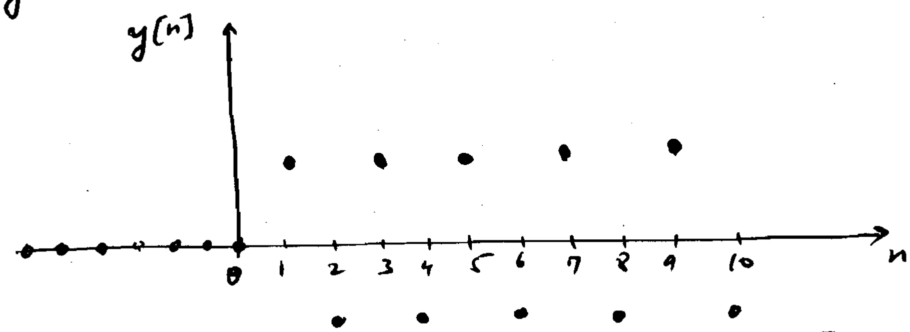
- Feedback system
- System defined by input/output difference equation: $y[n] = e[n-1]$
- Assume $y[n]=0$ for $n<0$

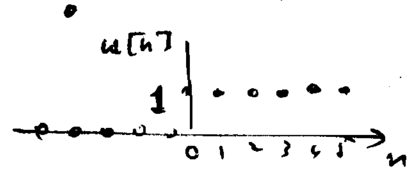


a) Sketch $y[n]$ when $x[n] = \delta[n]$ →

$$
\left.\begin{aligned}
y[0] &= e[-1] = x[-1] - y[-1] = 0-0 = 0 \\
y[1] &= e[0] = x[0] - y[0] = 1-0 = 1 \\
y[2] &= e[1] = x[1] - y[1] = 0-1 = -1 \\
y[3] &= e[2] = x[2] - y[2] = 0-(-1) = +1 \\
y[4] &= e[3] = x[3] - y[3] = 0-1 = -1
\end{aligned}\right\}
$$

using iterations: (use previously calculated values)





b) Sketch $y[n]$ when $x[n] = u[n]$

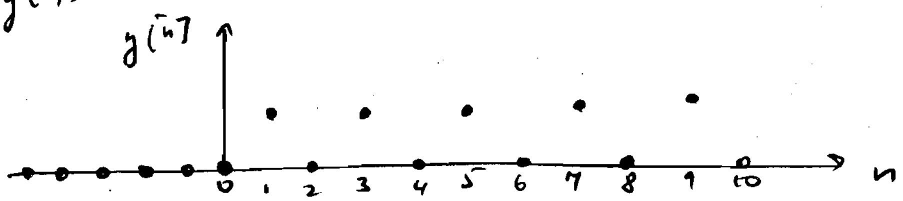
$y[0] = e[-1] = x[-1] - y[-1] = 0-0 = 0$

$y[1] = 1$

$y[2] = e[1] = x[1] - y[1] = 1-1 = 0$

$y[3] = 1-0 = 1$

$y[4] = 0$



- **Convolution**: application to a ~~conta~~ continuous-time signal & system: to find the output from the input & the impulse-response characterizing the system involved.

$$x(t) \longrightarrow \boxed{h(t) \quad \text{System}} \longrightarrow y(t)$$

$h(t)$: impulse-response function ($\delta(t) \longrightarrow \boxed{\text{System}} \longrightarrow h(t)$)

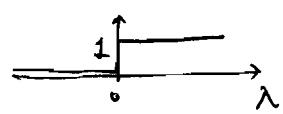
$$\boxed{y(t) = x(t) * h(t)} = \int_{-\infty}^{\infty} d\lambda\; x(\lambda)\, h(t-\lambda) = \int_{-\infty}^{\infty} d\lambda\; x(t-\lambda)\, h(\lambda)$$

by a change of variable

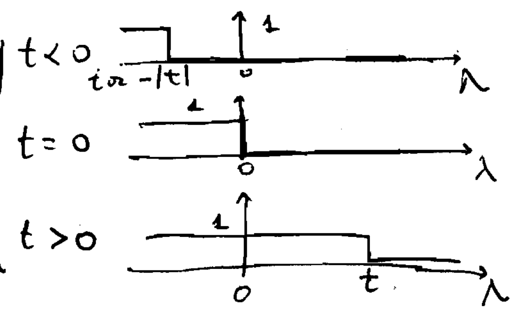
- Here λ plays a similar role as the index k in discrete-time convolution.
- Time-reversal & shift can go with h or with x giving the same result. $\rightarrow$ $y(t) = x(t) * h(t) = h(t) * x(t)$.
("Convolution is commutative")

- Example: find $y(t)$ for $x(t) = e^{-at}u(t)$ & $h(t) = u(t)$

$$y(t) = \int_{-\infty}^{\infty} d\lambda\; e^{-a\lambda}\, u(\lambda)\; u(t-\lambda) = \int_{0}^{\infty} d\lambda\; e^{-a\lambda}\, 1\; u(t-\lambda)$$



Now $\boxed{u(t-\lambda)}$ depends on t:



$$\rightarrow y(t) = \begin{cases} t<0 & 0 \\ t=0 & 0 \\ t>0 : & \int_0^t d\lambda\, e^{-a\lambda}\, 1.1 = \left[\dfrac{e^{-a\lambda}}{-a}\right]_0^t = \dfrac{1-e^{-at}}{a} \end{cases}$$

$$\rightarrow \boxed{y(t) = \frac{1-e^{-at}}{a}\, u(t)}$$

1.36

$$x(t) = e^{jw_0 t}$$

Fund. freq w_0 and $T_0 = \frac{2\pi}{w_0}$

$$\rightarrow x[n] = x(nT) = e^{jw_0 nT}$$

a) $x[n]$ is periodic IFF $\frac{T}{T_0}$ is rational (some multiple of the $\boxed{\text{sampling interval}}$ exactly equals a multiple of the $\boxed{\text{period of } x(t)}$

$\rightarrow T$ (sampling interval), $\rightarrow T_0$ (period of $x(t)$)

Produce this discrete-time signal:

$$\left.\begin{aligned} x[0] &= x(0) \\ x[1] &= x(T) \\ x[2] &= x(2T) \\ x[3] &= x(3T) \end{aligned}\right\} \text{T is the sampling interval.}$$

Need to prove: $\frac{T}{T_0}$ is rational.