Ch5: Discrete-time Fourier Transform

HW5: 5-8; 5.13; 5.19; 5.22 (a→e); 5-26; 5.27; 5.33; 5.36 c); 5.41 b); 5.51

Continuous-time Fourier Transform	Discrete-time Fourier Transform
$\hat{X}(j\omega) = \int_{-\infty}^{\infty} dt\, x(t)\, e^{-j\omega t}$	$\hat{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$
$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \hat{X}(j\omega)\, e^{j\omega t}$	$x[n] = \frac{1}{2\pi}\int_{2\pi} d\omega\, \hat{X}(e^{j\omega})\, e^{j\omega n}$
	• Time is discrete while frequency is continuous
• $e^{j\omega t}$	• $e^{j\omega n}$
$e^{j(\omega+2\pi)t} \neq e^{j\omega t}$	$e^{j(\omega+2\pi)n} = e^{j\omega n}$
	$\downarrow$
	$\hat{X}\left(e^{j(\omega+2\pi)}\right) = \hat{X}(e^{j\omega})$
	The discrete-time Fourier Transform is $\boxed{\text{always periodic in frequency}}$
	→ This is related to pg 69 (notes) "Discreteness in time → repetition in frequency"
	→ This is why we write $(e^{j\omega})$ as the argument

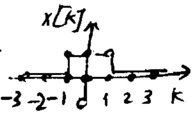
Properties: Table 5.1 (linearity, time-shift, convolution, etc.)
Table 5.2 (pairs of DT-FT)

5.8 | Use Tables 5.1 & 5.2 to determine $x[n]$ for:

$$\hat{X}(e^{j\omega}) = \frac{1}{1-e^{-j\omega}}\left(\frac{\sin\frac{3}{2}\omega}{\sin\frac{\omega}{2}}\right) + 5\pi\,\delta(\omega) \qquad \boxed{-\pi<\omega<\pi}$$

Table 5.1: # 5.3.5 Accumulation: $\sum_{k=-\infty}^{n} x[k] \longleftrightarrow \frac{1}{1-e^{-j\omega}}\hat{X}(e^{j\omega}) + \pi\hat{X}(e^{j0})\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$

What is $x[k]$? $\rightarrow$ $x_1[k] = DTFT^{-1}\left\{\frac{\sin\frac{3\omega}{2}}{\sin\frac{\omega}{2}}\right\} = \begin{cases} 1 & |k|\le 1 \\ 0 & |k|>1 \end{cases}$



In Table 5.2: $x[n] = \begin{cases} 1 & |n|\le N_1 \\ 0 & |n|>N_1 \end{cases} \longleftrightarrow \frac{\sin\left(\omega\left(N_1+\frac{1}{2}\right)\right)}{\sin\frac{\omega}{2}}$

$$\sum_{k=-\infty}^{n} x_1[k] \longrightarrow \frac{1}{1-e^{-j\omega}}\frac{\sin\frac{3\omega}{2}}{\sin\frac{\omega}{2}} + \pi\underbrace{\left[\left.\frac{\sin\frac{3\omega}{2}}{\sin\frac{\omega}{2}}\right|_{\omega\to 0}\right]}_{\text{L'Hôpital rule:}}\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$$

L'Hôpital rule:

$$\left[\left.\frac{\frac{3}{2}\cos\left(\frac{3\omega}{2}\right)}{\frac{1}{2}\cos\left(\frac{\omega}{2}\right)}\right|_{\omega\to 0}\right] = 3$$

$$= \frac{1}{1-e^{-j\omega}}\frac{\sin\frac{3\omega}{2}}{\sin\frac{\omega}{2}} + 3\pi\underbrace{\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)}$$

DTFT is periodic with period 2π $\rightarrow$ just b/w $-\pi$ & π is enough.
$\hookrightarrow$ $k=0$

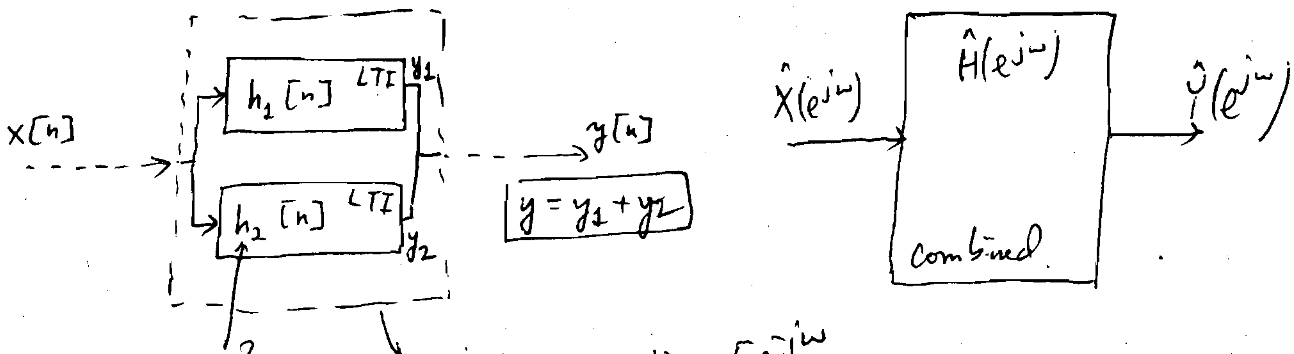
$$= \frac{1}{1-e^{-j\omega}}\frac{\sin\frac{3\omega}{2}}{\sin\frac{\omega}{2}} + 3\pi\,\delta(\omega-0)$$

To get an extra $2\pi\,\delta(\omega)$: Table 5.2: $x[n]=1 \leftrightarrow 2\pi\sum_{l=-\infty}^{\infty}\delta(\omega-2\pi l)$

or $2\pi\,\delta(\omega)$ for $-\pi<\omega<\pi$
$(l=0)$

$$\boxed{x[n] = 1 + \sum_{k=-\infty}^{n} x_1[k] \quad \text{where } x_1[k] = \begin{cases} 1 & |k|\le 1 \\ 0 & |k|>1 \end{cases}}$$

5.13 LTI with $h_1[n] = \left(\frac{1}{3}\right)^n u[n]$ in parallel with a second LTI with $h_2[n]$



$$\hat{H}(e^{j\omega}) = \frac{-12 + 5e^{-j\omega}}{12 - 7e^{-j\omega} + e^{-j2\omega}}$$

Parallel connection: both systems receive a same input.
Series connection: input to 2nd system is the output to the 1st system

Need to find $h_2[n]$:

$$y = y_1 + y_2 = x * h_1 + x * h_2 = x * (h_1 + h_2)$$

So: $y = x * (h_1 + h_2)$

Apply DT-FT on both sides: $\hat{Y}(e^{j\omega}) = \hat{X}(e^{j\omega}) \cdot \underbrace{(\hat{H}_1 + \hat{H}_2)}_{\hat{H}}$

Table 5.1 (#5.4)

$$\hat{H}_2 = \hat{H} - \hat{H}_1 = \frac{-12 + 5e^{-j\omega}}{12 - 7e^{-j\omega} + e^{-j2\omega}} - \frac{3}{3 - e^{-j\omega}}$$

$$h_1[n] = \left(\frac{1}{3}\right)^n u[n] \rightarrow \hat{H}_1(e^{j\omega}) = \frac{1}{1 - \frac{1}{3}e^{j\omega}} = \frac{3}{3 - e^{-j\omega}}$$

Table 5.2: $a^n u[n]$, $|a| < 1 \rightarrow \frac{1}{1 - ae^{-j\omega}}$

$$= \frac{(-12 + 5e^{-j\omega})(3 - e^{-j\omega}) - 3(12 - 7e^{-j\omega} + e^{-j2\omega})}{(12 - 7e^{-j\omega} + e^{-j2\omega})(3 - e^{-j\omega})}$$

$$= \frac{-36 + 12e^{-j\omega} + 15e^{-j\omega} - 5e^{-j2\omega} - 36 + 21e^{-j\omega} - 3e^{-j2\omega}}{36 - 21e^{-j\omega} + 3e^{-j2\omega} - 12e^{-j\omega} + 7e^{-j2\omega} - e^{-j3\omega}}$$

$$\hat{H}_2 = \frac{-72 + 48 e^{-j\omega} - 8 e^{-j2\omega}}{36 - 33 e^{-j\omega} + 10 e^{-j2\omega} - e^{-j3\omega}} = \frac{-2\cancel{(36 - 24e^{-j\omega} + 4e^{-j2\omega})}}{\cancel{(36 - 24e^{-j\omega} + 4e^{-j2\omega})}(\frac{1}{4}e^{-j\omega} - 1)}$$

Division:

$$\begin{array}{rrrr|l}
36 & -33e^{-j\omega} & +10e^{-j2\omega} & -e^{-j3\omega} & 36 - 24e^{-j\omega} + 4e^{-j2\omega} \\
\hline
0 & 9e^{-j\omega} & -6e^{-j2\omega} & +e^{-j3\omega} & \frac{1}{4}e^{-j\omega} - 1 \\
\hline
36 & -24e^{-j\omega} & 4e^{-j2\omega} & 0 & \\
-36 & +24e^{-j\omega} & -4e^{-j2\omega} & & \\
\hline
 & & 0 & &
\end{array}$$

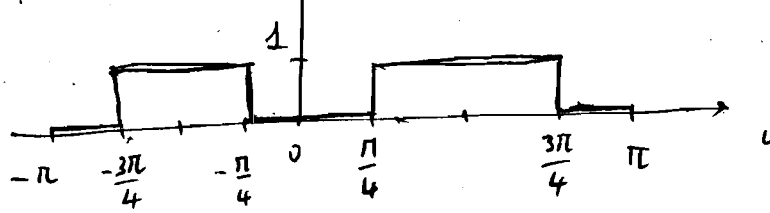
This says: $36 - 33e^{-j\omega} + 10e^{-j2\omega} - e^{-j3\omega} = (36 - 24e^{-j\omega} + 4e^{-j2\omega}) \cdot (\frac{1}{4}e^{-j\omega} - 1)$

$$\hat{H}_2 = \frac{2}{1 - \frac{1}{4}e^{j\omega}} \longrightarrow \boxed{h_2[n] = 2\left(\frac{1}{4}\right)^n u[n]}$$

Table 5.2

5.22 a) $\hat{X}(e^{j\omega}) = \begin{cases} 1 & \frac{\pi}{4} \le |\omega| \le \frac{3\pi}{4} \\ 0 & \frac{3\pi}{4} \le |\omega| \le \pi, \quad 0 \le |\omega| \le \frac{\pi}{4} \end{cases}$

$\hat{X}(e^{j\omega})$ → b/c it's periodic in frequency.



(repeating every multiple of 2π)

To find $x[n] = \frac{1}{2\pi}\int_{2\pi} d\omega \, \hat{X}(e^{j\omega}) e^{j\omega n} = \frac{2}{2\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\omega \, e^{j\omega n}$?

$$\overset{?}{=} \underbrace{\frac{1}{2\pi}\int_{-\frac{3\pi}{4}}^{-\frac{\pi}{4}} d\omega \, e^{j\omega n}} + \underbrace{\frac{1}{2\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\omega \, e^{j\omega n}}$$

$$= \underbrace{\frac{1}{2\pi n j}\left(e^{-j\frac{\pi}{4}n} - e^{-j\frac{3\pi}{4}n}\right)} + \underbrace{\frac{1}{2\pi n j}\left(e^{jn\frac{3\pi}{4}} - e^{jn\frac{\pi}{4}}\right)}$$

$$= \frac{e^{-jn\frac{2\pi}{4}}}{2\pi n j}\underbrace{\left(e^{j\frac{\pi n}{4}} - e^{-j\frac{\pi n}{4}}\right)}_{2j\sin\left(n\frac{\pi}{4}\right)} + \frac{e^{jn\frac{2\pi}{4}}}{2\pi n j}\underbrace{\left(e^{jn\frac{\pi}{4}} - e^{-jn\frac{\pi}{4}}\right)}_{2j\sin\left(n\frac{\pi}{4}\right)}$$

$$= \frac{1}{2\pi n \cancel{j}} \cancel{2j} \sin\left(n\frac{\pi}{4}\right)\left(2\cos n\frac{2\pi}{4}\right) = \frac{4}{2\pi n}\sin\left(n\frac{\pi}{4}\right)\cos\left(\frac{n 2\pi}{4}\right)$$