# Ch3 (cont.)

- <u>Fourier Series and Linear Systems</u>:

if $a_k$ are F.S. for $x(t)$, what are the F.S. coefficients for $y(t)$?

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y = x * h$$

↓

we should be able to get $b_k$ from $a_k$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad ; \quad y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$$

What is $b_k$ in term of $a_k$?

1) What is the output for $e^{jk\omega_0 t}$?

$$e^{jk\omega_0 t} * h(t) = \int_{-\infty}^{\infty} d\tau\, h(\tau)\, e^{jk\omega_0 (t-\tau)}$$

↓ def. of a convolution

$$= e^{jk\omega_0 t} \underbrace{\int_{-\infty}^{\infty} d\tau\, h(\tau)\, e^{-jk\omega_0 \tau}}_{\text{Fourier Transform of } h(t)\ \hat{H}(jk\omega_0)}$$

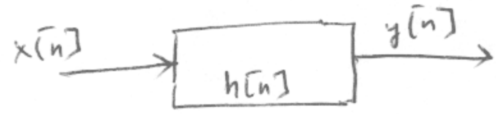
$\Rightarrow$ output for $e^{jk\omega_0 t}$ is $e^{jk\omega_0 t} \cdot \hat{H}(jk\omega_0)$

2) What is then output for $x(t)$?

$$y(t) = \sum_{k=-\infty}^{\infty} \underbrace{a_k \hat{H}(jk\omega_0)}_{b_k} e^{jk\omega_0 t} \rightarrow \boxed{b_k = a_k \hat{H}(jk\omega_0)}$$

## F.S. & linear discrete-time systems:

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n} \quad ; \quad y[n] = \sum_{k=\langle N\rangle} b_k e^{jk\frac{2\pi}{N}n}$$




Then $b_k = a_k \hat{H}(e^{j\omega})$ where $\hat{H}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$

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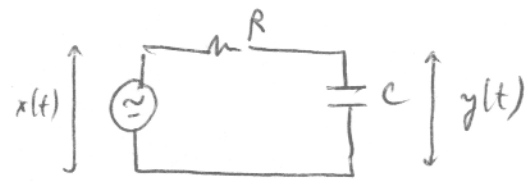
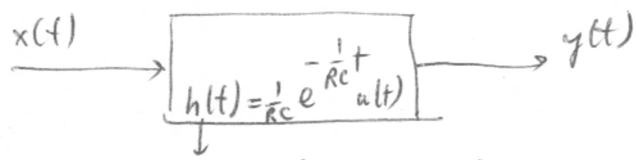
**Filtering**: Low-pass:

**Example:** RC circuit:

**Continuous-time**



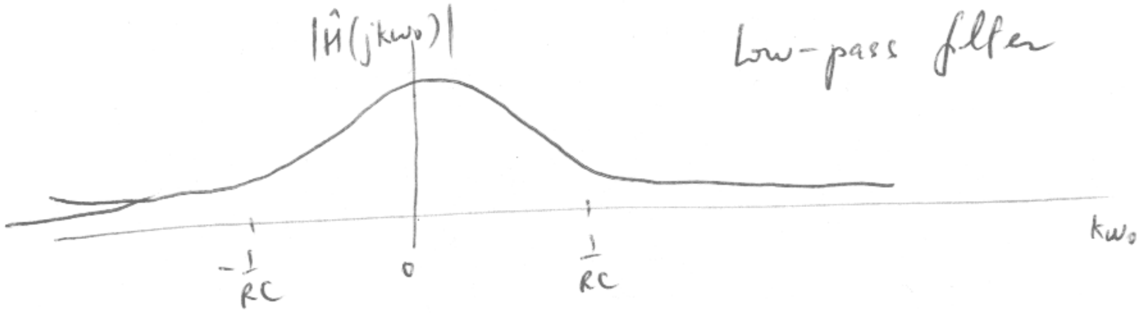




From solution to D.E.

$$\boxed{RC\frac{dy}{dt} + y = x(t)}$$

$$\hat{H}(jk\omega_0) = \int_{-\infty}^{\infty} dt\, h(t)\, e^{-jk\omega_0 t}$$

$$= \frac{1}{1 + RCjk\omega_0}$$




Low-pass filter

Discrete-time filter:

Would be represented by the discrete-time version of

$$RC \frac{dy}{dt} + y = x$$

$\underbrace{\text{Finite}}$-difference replacement for time derivative $\frac{dy}{dt}$:

$$\frac{dy}{dt} = \lim_{\Delta \to 0} \frac{y(t) - y(t-\Delta)}{\Delta}$$

$$\frac{RC}{\Delta}\left(y[n] - y[n-1]\right) + y[n] = x[n]$$

$$\left(\frac{RC}{\Delta} + 1\right) y[n] - \frac{RC}{\Delta} y[n-1] = x[n]$$

In general: $\boxed{\beta\, y[n] + \alpha\, y[n-1] = x[n]}$

Represents a discrete-time low-pass filter.

XC1 (= 1 HW set)
- a) Derive $\hat{H}(e^{j\omega})$ for this discrete-time filter
- b) Derive $h[n]$

3.5

$x_1(t)$ with fundamental frequency $w_1$ ; F.S. $a_k$

$x_2(t) = x_1(1-t) + x_1(t-1)$ $\begin{cases} w_2 \overset{?}{=} w_1 \\ \text{F.S. } x_2(t) \overset{?}{=} \end{cases}$

time shift : same freq.
reflexion : " "
sum : " "

3.12

$x_1[n]$ ; $x_2[n]$ both with $N=4$

$a_k$ $b_k$ Find $c_k$ for $g[n] = x_1[n] . x_2[n]$

3.14

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n-4k] \quad ; \quad y[n] = \cos\left[\frac{5\pi}{2}n + \frac{\pi}{4}\right]$$

$\hat{H}(e^{jk\frac{\pi}{2}})$ for $k = 0, 1, 2, 3$ ?

just use $b_k = a_k \hat{H}(e^{jw})$

$b_k \downarrow$ F.S. of $y[n]$ ; $a_k \downarrow$ F.S. of $x[n]$

$\hookrightarrow \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\frac{2\pi}{N}n}$

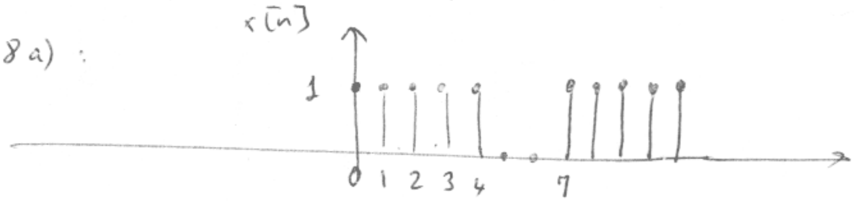
$$\cos\left(\frac{5\pi}{2}n + \frac{\pi}{4}\right) = \frac{1}{2}\left(e^{j\left(\frac{5\pi n}{2} + \frac{\pi}{4}\right)} + e^{-j\left(\frac{5\pi n}{2} + \frac{\pi}{4}\right)}\right)$$

$$\frac{2k}{N} = \frac{5}{2} \rightarrow k = \frac{5}{4}N = \frac{5 \cdot 4}{4}$$

$\hookrightarrow b_5 = \frac{1}{2} e^{j\frac{\pi}{4}}$ ; $b_{-5} = \frac{1}{2} e^{-j\frac{\pi}{4}}$

3.22

3.28 a) Figure P3.28 a) :




$N = 7$

$N_1 = 4$

Notes pg#30 :

$$\begin{cases} a_k = \dfrac{1}{7} \dfrac{\sin \frac{2\pi k}{7}\left(4+\frac{1}{2}\right)}{\sin \frac{2\pi k}{14}} = \dfrac{1}{7} \dfrac{\sin\left(\frac{9\pi k}{7}\right)}{\sin\left(\frac{\pi k}{7}\right)} \\ a_k = \dfrac{1}{7}(9) = \dfrac{9}{7} \quad (k = 0, \pm 7, \pm 14, \ldots) \end{cases}$$

Figure P3.28 b) Similar.

Figure P3.28 c):

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n]\, e^{-jk\frac{2\pi}{N}n}$$

$N = 6$ $\quad x[0] = 1 \; ; \; x[1] = 2 \; ; \; x[2] = -1 \; ; \; x[3] = 0 \; ; \; x[4] = -1 \; ; \; x[5] = 2$

$$a_k = \frac{1}{6}\Big[ \underbrace{1\, e^{-jk\frac{2\pi}{6}\cdot 0}}_{1} + 2\, e^{-jk\frac{2\pi}{6}\cdot 1} - 1\, e^{-jk\frac{2\pi}{6} 2} + 0 - 1 e^{-jk\frac{2\pi}{6}4} + 2\, e^{-jk\frac{2\pi}{6}5} \Big]$$

$$= \frac{1}{6}\left[1 + 2\, e^{-jk\frac{\pi}{3}} - 1\, e^{-jk\frac{2\pi}{3}} - 1 e^{-jk\frac{4\pi}{3}} + 2 e^{-jk\frac{5\pi}{3}}\right]$$

$$\cos A = \frac{e^{-jA} + e^{jA}}{2}$$

Note: $e^{-jC} = e^{j(-C+2\pi k)}$

$$e^{-jk\frac{5\pi}{3}} = e^{jk\left(-\frac{5\pi}{3}+2\pi\right)} = e^{+jk\frac{\pi}{3}}$$

$$e^{-jk\frac{4\pi}{3}} = e^{jk\left(-\frac{4\pi}{3}+2\pi\right)} = e^{jk\frac{2\pi}{3}}$$

$$= \frac{1}{6}\left[1 + 2\underbrace{\left(e^{-jk\frac{\pi}{3}} + e^{+jk\frac{\pi}{3}}\right)} - 1\left(e^{-jk\frac{2\pi}{3}} + e^{jk\frac{2\pi}{3}}\right)\right]$$

$$= \frac{1}{6}\left[1 + 4 \cos \frac{k\pi}{3} - 2 \cos \frac{2\pi k}{3}\right]$$

3.28b Find $a_k$ for $x[n] = \sin\left(\frac{2\pi n}{3}\right)\cos\left(\frac{\pi n}{2}\right)$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} \sin\frac{2\pi n}{3}\cos\frac{\pi n}{2}\, e^{-jk\frac{2\pi}{N}n}$$ , What's $N$? $N = 6$

$$= \frac{1}{6}\sum_{n=0,1,2,3,4,5}\left( \qquad \right)$$

$$= \frac{1}{6}\Bigg( \sin\frac{2\pi}{3}\cos\frac{\pi}{3}\, e^{-jk\frac{\pi}{3}} + \sin\frac{4\pi}{3}\underbrace{\cos\pi}_{-1}\, e^{-jk\frac{2\pi}{3}} + \underbrace{\sin 2\pi}_{0}\,.$$

$$+ \sin\frac{8\pi}{3}\underbrace{\cos 2\pi}_{1}\, e^{-jk\frac{4\pi}{3}} + \sin\frac{10\pi}{3}\underbrace{\cos\frac{5\pi}{2}}_{=\cos\frac{\pi}{2}=0} \Bigg)$$

$$= \frac{1}{6}\left( \sin\frac{2\pi}{3}\cos\frac{\pi}{3}\, e^{-jk\frac{\pi}{3}} - \sin\frac{4\pi}{3}\, e^{-jk\frac{2\pi}{3}} + \sin\frac{8\pi}{3}\, e^{-jk\frac{4\pi}{3}} \right)$$

regrouping ...