Ch3: Fourier Series Representation of Periodic Signals

HW3: 3.5; 3.12; 3.14; 3.22; 3.25; 3.28; 3.32; 3.46; 3.48; 3.52 a) b) c) d)

→ Periodic Signals: (continuous-time)

$\cos wt$, $\quad Re\,[e^{jwt}]$

$2\cos(10t) + \sin(4t)$ $\quad$ (Prob. (1.10)) → $T = \pi$

Reverse process: any periodic signal can be expressed as an infinite sum of of $\cos(k w_0 t)$ or $e^{jkw_0 t}$

k → integer; w_0 → fundamental frequency

$$\boxed{x(t) = \sum_{k=-\infty}^{+\infty} \underbrace{a_k}_{\text{complex quantity, independent of time}} e^{jkw_0 t}} \quad (1)$$

Consequence: a periodic signal can be determined by knowing $\begin{cases} a_k \\ w_0 \end{cases}$

→ How to obtain a_k: from $x(t)$: $\displaystyle\int_0^T e^{j(k-n)w_0 t}\,dt = \begin{cases} T & \text{if } k=n \\ 0 & \text{if } k \neq n \end{cases}$

(Orthogonality of complex exponentials or plane waves)

Multiply both sides of (1) by $e^{-jnw_0 t}$ and $\int_0^T dt$

$$\boxed{\int_0^T dt\, e^{-jnw_0 t}\, x(t) = \sum_{k=-\infty}^{+\infty} a_k \underbrace{\int_0^T dt\, e^{j(k-n)w_0 t}}_{T\,\delta[k-n]}}$$

Then $\displaystyle\int_0^T dt\, e^{-jn\omega_0 t} x(t) = \underbrace{\sum_{k=-\infty}^{+\infty} \left(a_k\, T\, \delta[k-n]\right)}_{T a_n}$

$$\rightarrow \boxed{a_n = \frac{1}{T}\int_0^T dt\, e^{-jn\omega_0 t} x(t)}$$ ($T = \frac{2\pi}{\omega_0}$, is the fundamental period)

Properties of Fourier Series (Table 3.1 pg. 206)

Linearity: $\left\{\begin{array}{l} x(t) = \displaystyle\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \\ y(t) = \displaystyle\sum_{k=-\infty}^{+\infty} b_k e^{jk\omega_0 t} \end{array}\right\} \; A x(t) + B y(t) = \displaystyle\sum_{k=-\infty}^{\infty} (A a_k + B b_k) e^{jk\omega_0 t}$

Math. This says:

F.S. coeff. for $Ax(t) + By(t)$ are $Aa_k + Bb_k$

Time shift: $x(t) = \displaystyle\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$$\downarrow$$

$$x(t-t_0) = \sum_{k=-\infty}^{\infty} a_k \underbrace{e^{jk\omega_0 (t-t_0)}}_{e^{jk\omega_0 t}\, e^{-jk\omega_0 t_0}} = \sum_{k=-\infty}^{\infty} \left(a_k e^{-jk\omega_0 t_0}\right) e^{jk\omega_0 t}$$

group these two factors together

Math. This says

$\rightarrow$ F.S. coeff for $x(t-t_0)$ are $a_k e^{-jk\omega_0 t_0}$

Frequency Shift

Time reversal

Time scaling

Multiplication

Differentiation

Integration

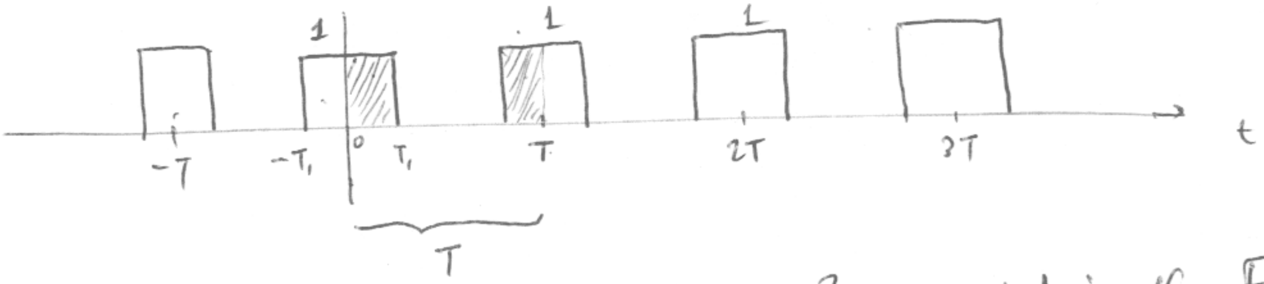
Parseval's Relation for Periodic Signals:

$$\underbrace{\frac{1}{T}\int_T |x(t)|^2\, dt}_{\text{signal}} = \underbrace{\sum_{k=-\infty}^{\infty} |a_k|^2}_{\text{F.S. coeffs.}}$$

Example 3.5:

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < \frac{T}{2} \end{cases}$$

(T_1 : half width of pulses ; T period of the train of pulses)



This is periodic → what are the a_k's? or what is the Fourier Series Representation of this signal?

$$a_k = \frac{1}{T}\int_0^T dt\, e^{jk\omega_0 t} x(t) = \frac{1}{T}\int_{-T_1}^{T_1} dt\, e^{jk\omega_0 t}\cdot 1 = \frac{1}{T}\left[\frac{e^{jk\omega_0 t}}{jk\omega_0}\right]_{-T_1}^{T_1}$$

Pick an interval of width T_1 could be $(-\frac{T}{2}, \frac{T}{2})$ since $\frac{T}{2} > T_1$

$$= \frac{1}{jk\omega_0 T}\underbrace{\left(e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}\right)}_{2j\sin(k\omega_0 T_1)} = \frac{2\sin k\omega_0 T_1}{k\underbrace{\omega_0 T}_{\to 2\pi}} = \boxed{\frac{\sin k\omega_0 T_1}{\pi k}}$$

$$e^{jk\omega_0 T_1} = \cos k\omega_0 T_1 + j\sin k\omega_0 T_1$$ (de Moivre's formula)

$$e^{-jk\omega_0 T_1} = \cos k\omega_0 T_1 - j\sin k\omega_0 T_1$$

$$e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1} = 2j \sin k\omega_0 T_1$$

or $$e^{jk\omega_0 T_1} + e^{-jk\omega_0 T_1} = 2\cos k\omega_0 T_1$$

what is a_0? do separately:

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt\, \underbrace{e^{j0\omega_0 t}}_{1} \cdot 1 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{1}{T}\left[t\right]_{-T_1}^{T_1} = \frac{2T_1}{T}$$