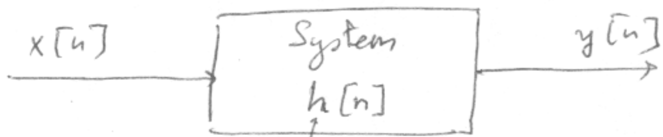
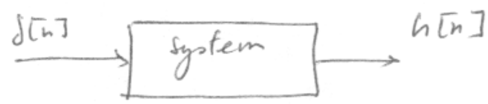
Ch 2: Linear - Time Invariant Systems.

HW 2: 2.7; 2.8; 2.11; 2.17; 2.22; 2.24; 2.31; 2.40; 2.47; 2.61

Convolution: allows to calculate $y[n]$ from $x[n]$ and $h[n]$



impulse response
of a discrete-time system.
(output to an impulse input)



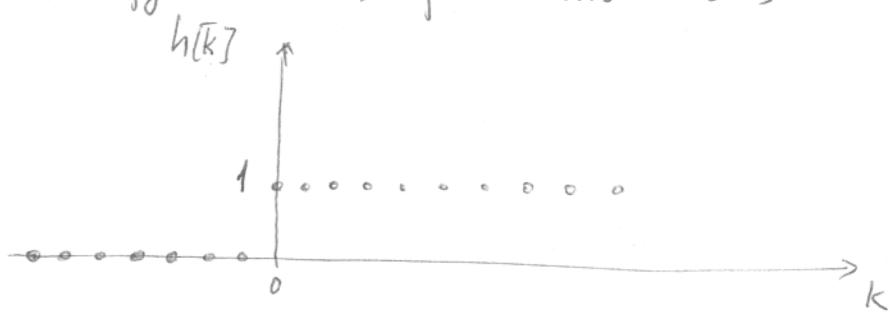
$$y[n] = x[n] * h[n] \equiv \sum_{k=-\infty}^{\infty} x[k] . h[n-k]$$

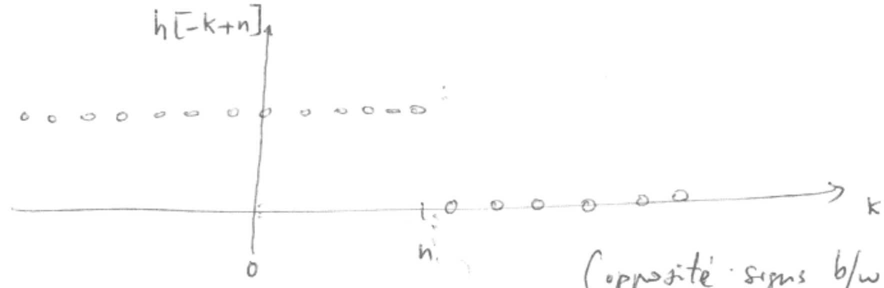
"convolution with"

Example: Find $y[n]$ from $x[n] = \alpha^n u[n]$ and $\boxed{h[n] = u[n]}$

step functions

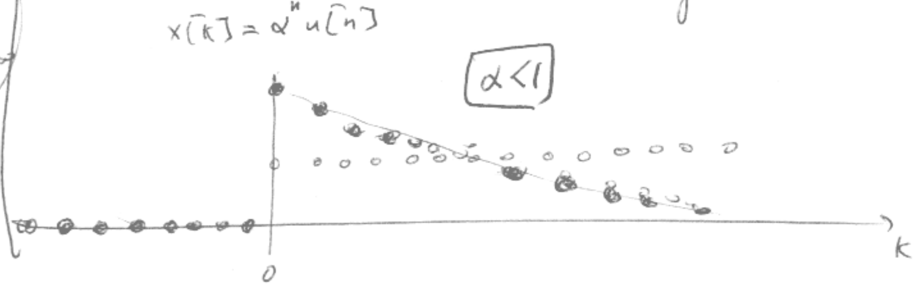
To do $x[n] * h[n] \equiv \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$ we need to be careful to get the right $h[n-k]$: This is $h[-k \underbrace{+n}_{\text{shift}}]$: we need to identify $h[-k+n]$ from $h[k] = u[k]$





(opposite signs b/w shift and original argument → shift to the right)

pt by pt product then $\sum_{k=-\infty}^{+\infty}$



By observation we only need $\sum_{k=0}^{n}$ here.

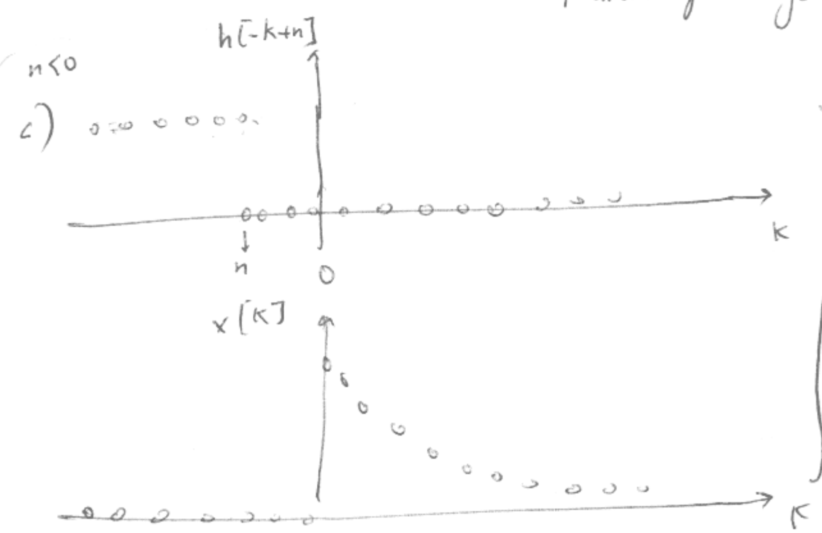
Distinguishing

a) $n < 0$: $h[-k+n]$ starts become 1 to the left of 0 → no overlap b/w $h[-k+n]$ & $x[k]$ → $\Sigma = 0$

b) $n = 0$ one overlap : $\Sigma = 1$

c) $n > 0$ $\sum_{k=0}^{n} \alpha^n$

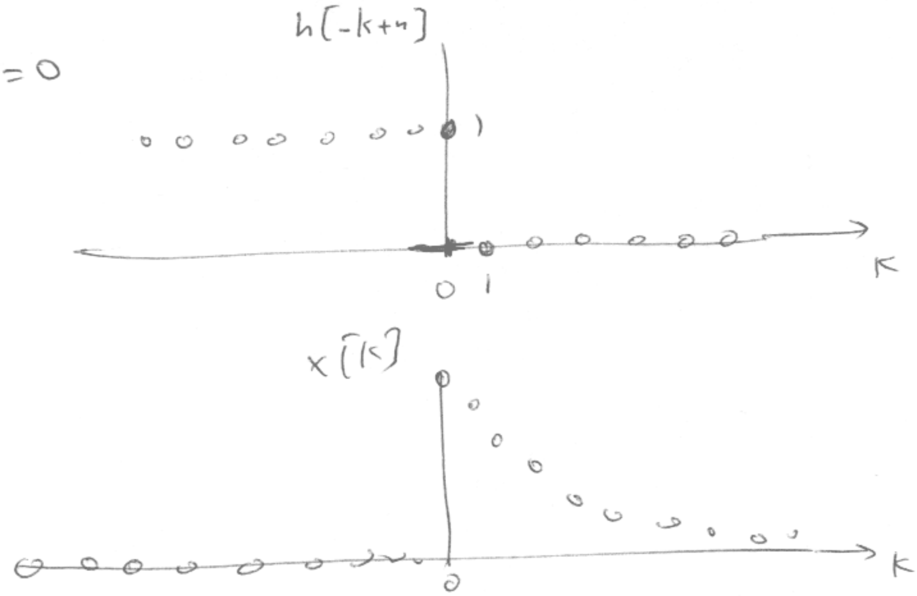
Formula for a geometric sum.



point-to-point products always are 0

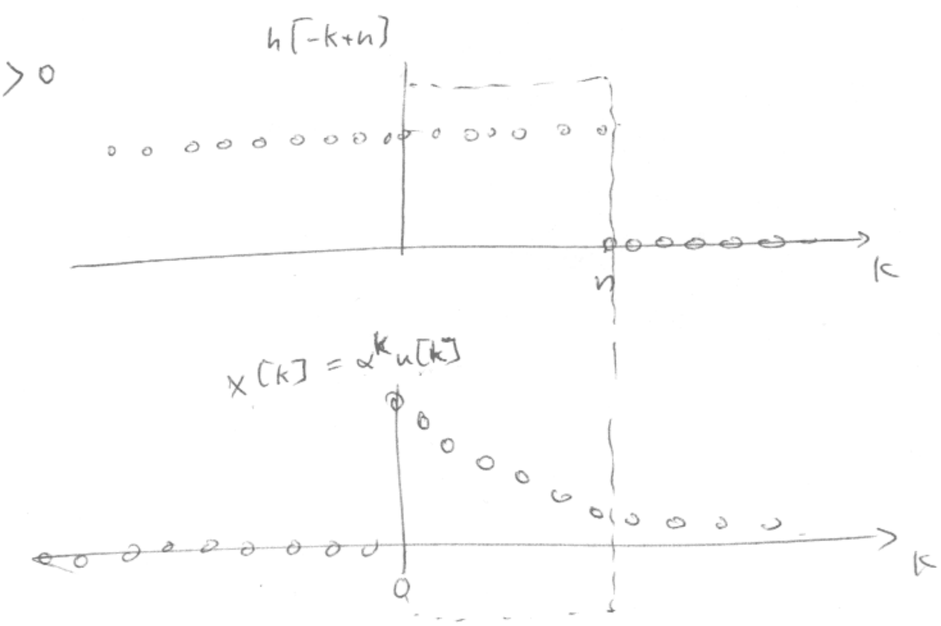
$y[n] = x[n] * h[n]$ is 0 for $n \leqslant 0$

b) $n=0$



$\rightarrow y[n] = 1$ for $n=0$

c) $n>0$



intervals with overlap

$$\boxed{y[n] = \sum_{k=0}^{n} \alpha^k} \stackrel{?}{=} \frac{1-\alpha^{n+1}}{1-\alpha} \quad (n>0)$$

$$\sum_{k=0}^{n} \alpha^k = 1 + \alpha + \alpha^2 + \cdot + \alpha^{n-1} + \alpha^n = S$$

$$\alpha + \alpha^2 + \dots + \alpha^n = S-1$$

$$\alpha \underbrace{\left(1 + \alpha + \alpha^2 + \dots \alpha^{n-1}\right)}_{S-\alpha^n} = S-1 \Rightarrow 1-\alpha^{n+1} = S(1-\alpha)$$

$$\Rightarrow \boxed{S = \frac{1-\alpha^{n+1}}{1-\alpha}}$$

, Convolution for continuous-time signal & system.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$h(t)$ ↓ impulse response $\left(\xrightarrow{\delta(t)} \boxed{\text{System}} \xrightarrow{h(t)} \right)$

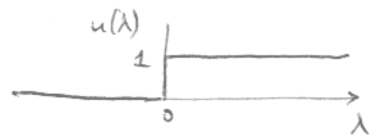
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} d\lambda \; \underbrace{x(\lambda)} \; \underbrace{h(t-\lambda)} = \int_{-\infty}^{\infty} d\lambda \; x(t-\lambda)\, h(\lambda)$$

$= h(t) * x(t)$

(convolution is commutative)

<u>Example</u>: $x(t) = e^{-at} u(t)$; $h(t) = u(t)$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} d\lambda \; \overbrace{e^{-a\lambda} \underbrace{u(\lambda)}} \; \overbrace{u(t-\lambda)}$$

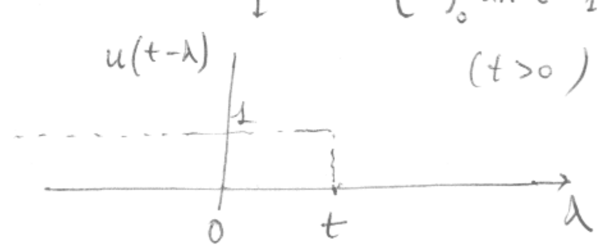


→ This makes $\int_{-\infty}^{\infty} \longrightarrow \int_{0}^{\infty}$

$$\int_{-\infty}^{\infty} = \int_{-\infty}^{0} + \int_{0}^{\infty}$$

$$= \int_0^{\infty} d\lambda \; e^{-a\lambda} \, 1 \; \underbrace{u(t-\lambda)}_{1} \begin{cases} = 0 & (t<0) \\ = 0 & (t=0) \\ = \int_0^t d\lambda \; e^{-a\lambda} \, 1 \cdot 1 = \left[\dfrac{e^{-a\lambda}}{-a} \right]_0^t = \dfrac{e^{-at} - 1}{-a} & (t>0) \end{cases}$$

$$= \frac{1 - e^{-at}}{a}$$



$(t=0)$

$(t<0)$