\[
\hat{X}(j\omega_1, j\omega_2) = \frac{2\pi \delta(\omega_2 - 2\omega_1)}{4 + j\omega_4}
\]

\[
x(t_1, t_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2 \frac{2\pi \delta(\omega_2 - 2\omega_1)}{4 + j\omega_4} \hat{X}(j\omega_1, j\omega_2) e^{j\omega_1 t_1} e^{j\omega_2 t_2}
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega_1 \frac{e^{j\omega_1 t_1}}{4 + j\omega_4} \int_{-\infty}^{\infty} d\omega_2 \delta(\omega_2 - 2\omega_1) e^{j\omega_2 t_2} e^{j2\omega_1 t_2}
\]

\[
\quad = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega_1 \frac{e^{j\omega_1 (t_1 + 2t_2)}}{4 + j\omega_4} \quad \delta(t_1 + 2t_2) = e^{-4(t_1 + 2t_2)} u(t_1 + 2t_2)
\]

Table 4.2:
\[
ex^{-at} u(t) \quad \leftrightarrow \quad \frac{1}{a + j\omega}
\]

\[
\hat{X}(j\omega) = \frac{1}{a + j\omega} = \int_{-\infty}^{\infty} dt \quad e^{-at} u(t) e^{j\omega t}
\]

\[
x(t) = e^{-at} u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \quad \frac{1}{a + j\omega} \quad e^{j\omega t}
\]
Chapter 4:
\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} \, d\omega
\]
(4.8)
\[
\hat{X}(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt
\]
(4.9)

Chapter 5:
\[
x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega
\]
(5.8)
\[
\hat{x}[n] = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}
\]
(5.9)

Difference:
\[
\hat{X}(e^{j\omega+n\Omega}) = \hat{X}(e^{j\omega})
\]
Discrete-time Fourier Transform \(\hat{X}(e^{j\omega})\)
DTFT is always periodic in \(\omega\), period \(2\pi\).

Chapter 6: Time & Frequency Characterization of Signals and Systems

In connection with pg 327:

HW 6:
6.5; 6.18; 6.19; 6.23; 6.27; 6.30; 6.31; 6.33; 6.40; 6.51

6.5:

![Diagram of signal processing](image)

\[x(t) \rightarrow h(t) \rightarrow y(t)\]
\[h(t) = \frac{\sin \omega_c t}{\pi t} \rightarrow q(t)\]

\[H(j\omega) = \begin{cases} 
1 & 0 < \omega_c \leq |\omega| < 3\omega_c \\
0 & \text{elsewhere}
\end{cases}\]

Table 4.2:
\[
\frac{\sin \omega_c t}{\pi t} \rightarrow \hat{X}(j\omega) = \begin{cases} 
1 & |\omega| < \omega_c \\
0 & \text{elsewhere}
\end{cases}
\]

\[
H(j\omega) = \frac{\hat{X}(j\omega-2\omega_c) + \hat{X}(j(\omega+2\omega_c))}{\omega_c + \omega} = x(t) \frac{\sin \omega_c t}{\pi t} \left( e^{-j\omega t} + e^{j\omega t} \right) \]

\[
\frac{\sin \omega_c t}{\pi t} = e^{-j\omega t} \sin \omega_c t + e^{j\omega t} \sin \omega_c t
\]
Can step response show oscillatory behavior?

\[ u(t) \rightarrow \text{oscillatory behavior?} \]

1) Frequency domain:
\[
\hat{H} = \frac{\hat{Y}}{\hat{X}} = \frac{1}{\frac{1}{j\omega C} + R} = \frac{1}{1 + j\omega RC} = \hat{H}(j\omega)
\]

\[
\hat{X}(j\omega) = \hat{Y}(j\omega) = \hat{X}\hat{H} = \frac{1}{j\omega + R(j\omega)} \quad \rightarrow \text{oscillation in time?}
\]

If \( \frac{1}{j\omega + R(j\omega)} \)

Table: \( \frac{1}{j\omega + R(j\omega)} \)

No. (denominator is of first order \( \rightarrow \) in time domain it is an exponential)

Can step response have \( \cdots \), how to eliminate oscillation?

\[
\hat{H} = \frac{\hat{Y}}{\hat{X}} = \frac{1}{\frac{1}{j\omega C}} = \frac{1}{R + j\omega L + \frac{1}{j\omega C}} = \frac{1}{j\omega RC + (j\omega)^2 C + 1}
\]

second order denominator \( \rightarrow \) ripple in time.

How to eliminate ripple?
Quadratic pole: \[ \frac{1}{1 + 2(j\omega C) + (j\omega)^2} \]

To eliminate one (ripple) \( \Rightarrow \Re > 1 \)

\[ 1 + j\omega RC + (j\omega LC)^2 = 1 + 2(j\omega C)^2 + (j\omega)^2 \]

\[ z = \frac{RC}{2\sqrt{LC}} \]

\[ RC = 2\sqrt{LC} z \]

\[ \Re \Rightarrow \frac{R}{2\sqrt{LC}} \geq 1 \]

\[ R > 2\sqrt{\frac{L}{C}} \]

Low-pass filter:

\[ |\hat{H}(j\omega)| \]

Sketch impulse response for:

1. \( \operatorname{Phase}(\hat{H}(j\omega)) = 0 \)
2. \( \operatorname{Phase}(\hat{H}(j\omega)) = \omega T \)
3. \( \operatorname{Phase}(\hat{H}(j\omega)) = \begin{cases} \frac{\pi}{2} & \omega > 0 \\ -\frac{\pi}{2} & \omega < 0 \end{cases} \)

\[ h(t) = F^{-1}[\hat{H}(j\omega)] = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases} \]

\[ h(t) = \frac{\sin \omega_c t}{\pi t} \]

\[ \hat{h}(j\omega) = j\omega T \Rightarrow h(t) = \frac{\sin \omega_c (t + T)}{\pi (t + T)} \]

\[ \hat{z}(\omega) = |\hat{z}| e^{j \text{Phase}(\hat{z})} = (|\hat{z}| \cos [\text{Phase}(\hat{z})], \hat{z} \sin [\text{Phase}(\hat{z})]) \]
\[ x_{x}(t) = x(t) * y(t) \]

\[ x_{x}(t) = \sum_{k=-\infty}^{\infty} c_{k} e^{-jkw_{0}t} \]

\[ w_{0} = \frac{2\pi}{T} \]

\[ c_{k} = a_{k} * b_{k} = \sum_{n=-\infty}^{\infty} a_{n} b_{k-n} \]

\[ x(t) = \cos 2\pi nt = \frac{e^{j2\pi nt} + e^{-j2\pi nt}}{2} = \frac{1}{2} e^{j2\pi nt} + \frac{1}{2} e^{-j2\pi nt} \]

\[ b_{k} = \frac{1}{2} \delta(k-30) + \frac{1}{2} \delta(k+30) \]

\[ a_{k} = \begin{cases} \frac{1}{2} & \text{if } T_{1} = 1 \\ \frac{\sin k \omega_{0}}{\pi k} & \text{if } T_{1} = 3 \end{cases} \]

\[ c_{k} = a_{k} * b_{k} = \left[ \frac{1}{2} \delta(k-30) + \frac{1}{2} \delta(k+30) \right] * \frac{\sin \frac{2\pi k}{3}}{\pi k} \]

\[ \text{property of convolution} \]
3.46 (iii)

\[ x_3(t) \] \quad T_0 = 4 \quad \rightarrow \quad \omega_c = \frac{2\pi}{4} = \frac{\pi}{2}

\[ e^{-|t|} \text{ rect} \]

Can't correlate \( c_k \) in part i) with \( d_k \) in \( e^{-|t|} \sum_{k=-\infty}^{\infty} d_k e^{-j\omega_k t} \).

b/c \( e^{-|t|} \) only applies within each rectangular pulse.

\[ x_3(t) = x(t) y(t) \quad \text{ convolve} \quad \rightarrow \quad b_k = \frac{1}{2} \left[ d(k - \omega_0) + s(k + \omega_0) \right] \]

\[ T_0 = 4, \quad T_1 = 1 \]

\[ 1 \quad 1 \]

\[ 0, 1, 2, 3, 4, 5 \]

\[ a_k = ? \]

Similar to page 26:

\[ a_k = \frac{1}{T_0} \int_0^{T_0} dt \quad e^{-j\omega_k t} = \frac{1}{4} \int_{-1}^{0} dt \quad e^{-j\frac{\pi}{2} t} + \frac{1}{4} \int_0^{1} dt \quad e^{-j\frac{\pi}{2} t} \]

\[ + \frac{1}{4} \int_0^1 dt \quad e^{-j\frac{\pi}{2} t} = \frac{1}{4} \int_0^1 dt \quad e^{-j\frac{\pi}{2} t} \]

\[ = \frac{1}{4} \left[ \frac{e^{-j\frac{\pi}{2} t} - 1}{-1} \right] = \frac{1}{4} \left[ \frac{-j\frac{\pi}{2} t}{-1} \right] \]

\[ = \frac{1}{4} \left[ \frac{e^{-(1+j\frac{\pi}{2}) t}}{e^{-(1+j\frac{\pi}{2}) t}} - \frac{e^{-(1-j\frac{\pi}{2}) t}}{e^{-(1-j\frac{\pi}{2}) t}} \right] \]

\[ = \frac{1}{4} \left[ \frac{1 - e^{-(1+j\frac{\pi}{2}) t}}{1 + e^{-(1+j\frac{\pi}{2}) t}} - \frac{1 - e^{-(1-j\frac{\pi}{2}) t}}{1 + e^{-(1-j\frac{\pi}{2}) t}} \right] \]

\[ = \frac{1}{4} \left[ \frac{1 - e^{-j\frac{\pi}{2} t}}{1 + e^{-j\frac{\pi}{2} t}} - \frac{1 - e^{j\frac{\pi}{2} t}}{1 + e^{j\frac{\pi}{2} t}} \right] \]

\[ = \frac{1}{4} \left[ \frac{1 - e^{-j\frac{\pi}{2} t}}{1 + e^{-j\frac{\pi}{2} t}} - \frac{1 + e^{j\frac{\pi}{2} t}}{1 - e^{j\frac{\pi}{2} t}} \right] \]
\[
\frac{1}{4} \left[ \frac{1}{1 + \frac{k^2\pi^2}{\alpha^2}} \left( \frac{1 + j\frac{k^2\pi}{\alpha}}{e^\frac{j\frac{k^2\pi}{\alpha}}{\alpha}} - (1 - j\frac{k^2\pi}{\alpha}) \right) \right]
\]

\[
\frac{1}{4} \left[ \frac{1}{1 + \frac{k^2\pi^2}{\alpha^2}} \left( \frac{2 - \left( e^{-\frac{j\frac{k^2\pi}{\alpha}}{\alpha}} + e^{\frac{j\frac{k^2\pi}{\alpha}}{\alpha}} \right) e^{-1} + j\frac{k^2\pi}{\alpha} \left( -e^{-\frac{j\frac{k^2\pi}{\alpha}}{\alpha}} + e^{\frac{j\frac{k^2\pi}{\alpha}}{\alpha}} \right) e^{-1} }{2\sin\frac{k\pi}{\alpha}} \right] \right]
\]

\[
\eta_k = \frac{1}{4} \left[ \frac{1}{1 + \frac{k^2\pi^2}{\alpha^2}} \left( 2 - e^{-1} \left( 2\omega \frac{k\pi}{\alpha} + k\pi \sin \frac{k\pi}{\alpha} \right) \right) \right]
\]