4.34) Causal & Stable LTI $\hat{H}(j\omega) = \dfrac{j\omega + 4}{6 - \omega^2 + 5j\omega}$

a) Determine a D.E relating $x(t)$ to $y(t)$:

$$\hat{H}(j\omega) = \frac{\hat{Y}(j\omega)}{\hat{X}(j\omega)} = \frac{j\omega + 4}{6 + (j\omega)^2 + 5j\omega}$$

Frequency $\quad 6\hat{Y} + (j\omega)^2\hat{Y} + 5j\omega\hat{Y} = j\omega\hat{X} + 4\hat{X}$

Time $\quad 6y(t) + \dfrac{d^2y}{dt^2} + 5\dfrac{dy}{dt} = \dfrac{dx}{dt} + 4x(t)$

b) determine $h(t)$:
- or $F^{-1}[\hat{H}(j\omega)]$
- or solve this DE and identify $h(t)$ inside the integral.

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} d\tau\, x(\tau)\, h(t-\tau)$$

Laplace Transform $(s \to j\omega)$ $\quad \hat{Y}(s) = \int_0^{\infty} dt\, e^{-st} x(t)$

$$6\hat{Y}(s) + s^2\hat{Y} + 5s\hat{Y} = +s\hat{X} + 4\hat{X}$$ (system at rest initially)

$$(6 + s^2 + 5s)\hat{Y} = (+s+4)\hat{X}$$

$$\hat{Y} = \underbrace{\left[\frac{s+4}{s^2+5s+6}\right]}_{\hat{H}(s)}\hat{X} \quad \rightarrow \quad h(t) = L^{-1}\left[\frac{s+4}{s^2+5s+6}\right]$$

$$s^2 + 5s + 6 = 0 \rightarrow s = \frac{-5 \pm \sqrt{\;}}{} \quad \begin{cases} -3 \\ -2 \end{cases}$$

$$\| \quad (s+3)(s+2)$$

$$h(t) = L^{-1}\left\{\frac{A}{s+3} + \frac{B}{s+2}\right\} \quad \begin{cases} (A+B)s + 2A+3B = s+4 \\ A+B = 1 \\ 2A+3B = 4 \end{cases} \quad \begin{matrix} B = 2 \\ A = -1 \end{matrix}$$

$$= L^{-1}\left\{\frac{-1}{s+3} + \frac{2}{s+2}\right\} = \left[-e^{-3t} + 2e^{-2t}\right]u(t)$$

i) what is $y(t)$ when $x(t) = e^{-4t}u(t) - te^{-4t}u(t)$ ?

$$\hat{X}(j\omega) = \frac{1}{4+j\omega} - \frac{1}{(4+j\omega)^2} = \frac{1}{4+j\omega}\left(1 - \frac{1}{4+j\omega}\right) = \frac{3+j\omega}{(4+j\omega)^2}$$

$$\hat{Y}(j\omega) = \hat{H}(j\omega)\hat{X}(j\omega) = \frac{\cancel{4+j\omega}}{\cancel{(3+j\omega)}(2+j\omega)} \frac{\cancel{(3+j\omega)}}{(4+j\omega)^{\cancel{2}}} = \frac{1}{(2+j\omega)(4+j\omega)}$$

$$y(t) = F^{-1}\left\{\frac{A}{2+j\omega} + \frac{B}{4+j\omega}\right\} = \frac{1}{2}\left(e^{-2t} - e^{-4t}\right)u(t)$$

$$\left.\begin{array}{l} A+B=0 \\ 4A+2B=1 \end{array}\right\} A=\tfrac{1}{2};\ B=-\tfrac{1}{2}$$

$$A+B=0$$
$$2A+B=1$$

# Ch5: Discrete-Time Fourier Transform.

**HW5:** 5.8; 5.13; 5.19; 5.22 (a→e); 5.26; 5.27; 5.33; 5.36 c); 5.41 b); 5.51

$$\boxed{\hat{X}(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn}}$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} \hat{X}(e^{jw})\, e^{jwn}\, dw \quad \text{(inverse discrete-time Fourier Transform)}$$

Properties: Table 5.1 ( Linearity, time-shift, convolution, etc. )

Pairs: Table 5.2

Difference versus continuous-time Fourier Transform:

From Chapter 1:

| | |
|---|---|
| $e^{jwt}$ | $e^{jwn}$ |
| $e^{j(w+2\pi)t} \neq e^{jwt}$ | $e^{j(w+2\pi)n} = e^{jwn}$ |

$$\downarrow$$

$\hat{X}(e^{j(w+2\pi)})$ → Conclusion:

Discrete-time F.T. is ALWAYS periodic in frequency.

**5.8** Use Tables 5.1 & 5.2 to get $x[n]$ if

$$\hat{X}(e^{j\omega}) = \frac{1}{1-e^{-j\omega}} \boxed{\frac{\sin\frac{3}{2}\omega}{\sin\frac{1}{2}\omega}} + \underbrace{5\pi\,\delta(\omega)} \;; \quad -\pi < \omega < \pi$$

$$\boxed{5.1.\ \text{Accumulation}: \quad \sum_{k=-\infty}^{n} x[k] \longrightarrow \frac{1}{1-e^{-j\omega}}\underbrace{\hat{X}(e^{j\omega})} + \pi\,\hat{X}(e^{j0})\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)}$$

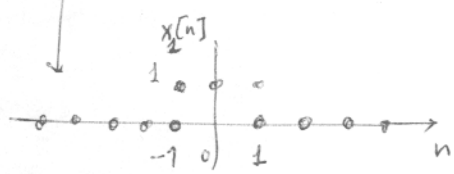
$$\frac{\sin\frac{3\omega}{2}}{\sin\frac{\omega}{2}}$$

$\hat{X}_1(e^{j\omega})$

?

$$\boxed{5.2: \quad x_1[n] = \begin{cases} 1 & |n| \le N_1 \\ 0 & |n| > N_1 \end{cases} \longleftarrow \frac{\sin\omega\left(N_1+\frac{1}{2}\right)}{\sin\frac{\omega}{2}}}$$

$$N_1 + \frac{1}{2} = \frac{3}{2} \rightarrow N_1 = 1$$




$$\sum_{k=-\infty}^{n} x_1[n] \longrightarrow \frac{1}{1-e^{-j\omega}}\,\frac{\sin\frac{3\omega}{2}}{\sin\frac{\omega}{2}} + \pi\,\underbrace{3}\,\underbrace{\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)}$$

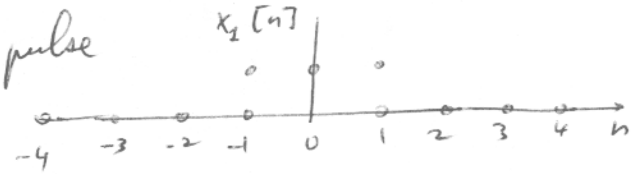
$$\lim_{\omega\to 0}\frac{\sin\frac{3\omega}{2}}{\sin\frac{\omega}{2}} = \lim_{\omega\to 0}\frac{\frac{3}{\cancel{2}}\cos\frac{3\omega}{2}}{\frac{1}{\cancel{2}}\cos\frac{\omega}{2}} = 3$$

L' Hôpital Rule

But DTFT $x_1[n] \rightarrow \hat{X}_1(e^{j\omega})$ is periodic in $\omega$ with period $2\pi$
$\rightarrow$ we just need to look at interval $-\omega \rightarrow \omega$,
i.e $k=0$ in this sum
or just $\delta(\omega)$ here

Summarizing:

with accumulation of the rectangular pulse



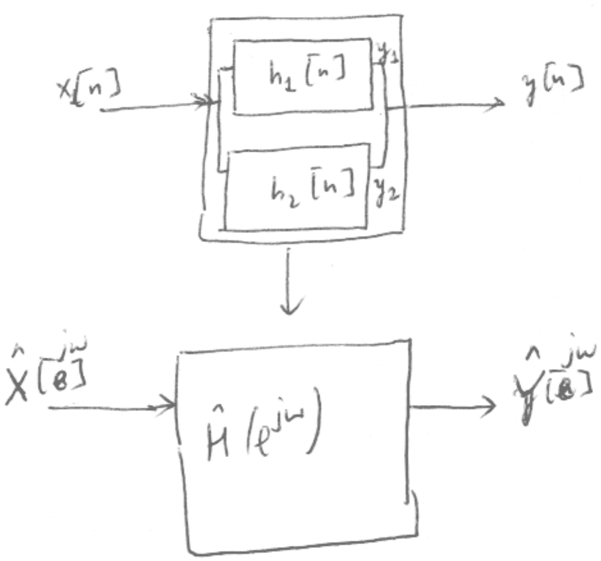

we have as DTFT: $\dfrac{1}{1-e^{-j\omega}} \dfrac{\sin \frac{3\omega}{2}}{\sin \frac{\omega}{2}} + 3\pi\,\delta(\omega)$

Now we need to add to the accumulation a signal that has as DTFT $2\pi\delta(\omega)$. From Table 5.2: $x[n] = 1 \longleftrightarrow 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l)$

again $l = 0$ for $-\omega \rightarrow \omega$ interval. or $2\pi\delta(\omega)$

$\rightarrow$ Finally: $\boxed{1 + \sum_{k=-\infty}^{n} x_1[n]} \longrightarrow \dfrac{1}{1-e^{-j\omega}} \dfrac{\sin\frac{3\omega}{2}}{\sin\frac{\omega}{2}} + 5\pi\,\delta(\omega)$

5.13/ LTI with $h_1[n] = \left(\frac{1}{3}\right)^n u[n]$ in parallel with $h_2[n]$




$h_2[n]$ ?

$$\boxed{\begin{aligned} y &= y_1 + y_2 \\ &= x * h_1 + x * h_2 \\ &= x * [h_1 + h_2] \\ \hat{Y}(e^{j\omega}) &= \hat{X}(e^{j\omega}) \cdot \underbrace{[\hat{H}_1(e^{j\omega}) + \hat{H}_2(e^{j\omega})]}_{\hat{H}(e^{j\omega})} \end{aligned}}$$

$$\hat{H}(e^{j\omega}) = \frac{-12 + 5e^{-j\omega}}{12 - 7e^{-j\omega} + e^{-j2\omega}}$$

So: $\hat{H} = \hat{H}_1 + \hat{H}_2 \rightarrow \hat{H}_2 = \hat{H} - \hat{H}_1$

$h_1[n] = \left(\frac{1}{3}\right)^n u[n] \rightarrow \dfrac{1}{1 - \frac{1}{3}e^{-j\omega}} = \dfrac{3}{3 - e^{-j\omega}}$

$$\rightarrow \hat{H}_2 = \frac{-12 + 5e^{-j\omega}}{12 - 7e^{-j\omega} + e^{-j2\omega}} - \frac{3}{3 - e^{-j\omega}}$$

$$= \frac{-36 + 27e^{-j\omega} - 5e^{-j2\omega} - 36 + 21e^{-j\omega} - 3e^{-j2\omega}}{(12 - 7e^{-j\omega} + e^{-j2\omega})(3 - e^{-j\omega})}$$