0	0	0

for 'OR' is:

Ch3 Boolean Algebra & Combinational Networks:
or Logic Circuits.

Boolean Algebra: $\boxed{+ \leftrightarrow \text{OR}}$ & $\boxed{\cdot \leftrightarrow \text{AND}}$

a) Commutative:

$A + B = B + A$ $\qquad$ $A \cdot B = B \cdot A$

b) Associative,

$(A+B)+C = A+(B+C)$ $\qquad$ $(A \cdot B) \cdot C = A \cdot (B \cdot C)$

c) Neutral element.

$A + 0 = A$ $\qquad$ $A \cdot 1 = A$

d) Distributive:

$A \cdot (B+C) = A \cdot B + A \cdot C$ ① / $\boxed{A + B \cdot C = (A+B) \cdot (A+C)}$ ②

(Not in Number Algebra)

$A + \underbrace{1 \cdot 1}_{1} = (A+1) \cdot (A+1)$

$(A+1) = (A+1) \cdot (A+1)$

$\Rightarrow \boxed{A + 1 = 1}$

"A or 1 is 1"

(Use set theory & Venn Diagrams to visualize)

e) Complement of A is $\overline{A}$ ('A bar' or 'A not'):

$A + \overline{A} = 1$ $\qquad$ $A \cdot \overline{A} = 0$

We can write the truth table for f from this MAX TERM

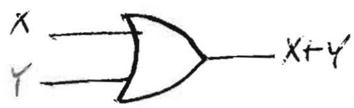
Truth Tables:

Operations (Boolean) b/w X & Y

X	Y	X+Y ('X or Y')
0	0	0
1	0	1
0	1	1
1	1	1

$\boxed{\text{OR}}$

Logic circuit symbol for 'OR' is:



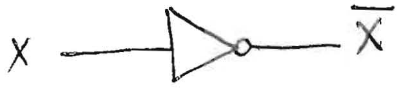
X	Y	X·Y ('X AND Y')
0	0	0
1	0	0
0	1	0
1	1	1

$\boxed{\text{AND}}$

Logic circuit symbol for 'AND' is

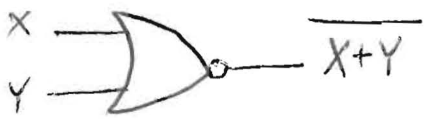
X	$\overline{X}$
0	1
1	0

Logic circuit symbol for 'Complement Op." is



X	Y	$\overline{X+Y}$ ('X NOR Y')
0	0	1
1	0	0
0	1	0
1	1	0

$\boxed{\text{NOR}}$



X	Y	$\overline{X \cdot Y}$ ('X NAND Y')
0	0	1
1	0	1
0	1	1
1	1	0

$\boxed{\text{NAND}}$

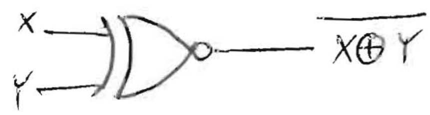
Exclusive OR 'XOR' one or the other (X or Y) but not both!

X	Y	$X \oplus Y$ ('X XOR Y')
0	0	0
1	0	1
0	1	1
1	1	0

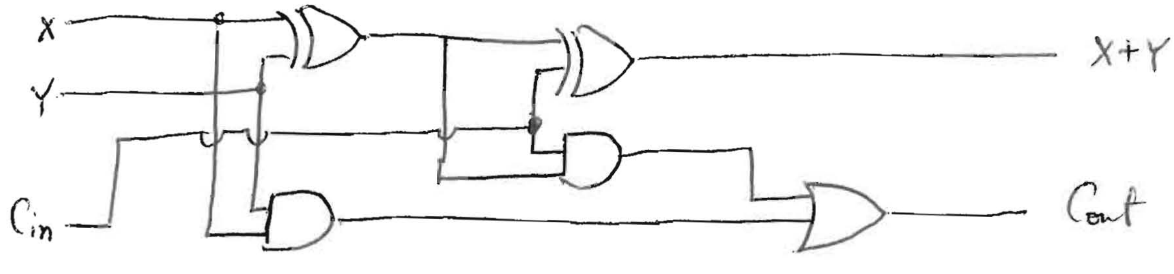
$\boxed{\text{XOR}}$

X	Y	$\overline{X \oplus Y}$ ('X XNOR Y')
0	0	1
1	0	0
0	1	0
1	1	1

$\boxed{\text{XNOR}}$



A one-bit adder can be implemented with 2 XOR's, 2 AND's and one OR: (Lab #2)



More complex operations → larger combinations of logic gates: It is possible to use properties of Boolean Algebra (including their corollaries) to find the simplest combination of logic gates to achieve certain task or operation (cost effective)

Corollaries of Boolean Algebra Properties:

a) $\boxed{\overline{\overline{A}} = A}$

↳ Proof: $A + \overline{A} = 1 \rightarrow$ extract A from both sides $\overline{A} = 1 - A \rightarrow$ 'Universe except A'

$$\boxed{\overline{\overline{A}}} = \overline{\overline{A}} + 0$$

$$= \overline{\overline{A}} + A \cdot \overline{A}$$

'Distributive' ② $= (\overline{\overline{A}} + A) \cdot (\overline{\overline{A}} + \overline{A})$

'Commutative' $= (A + \overline{\overline{A}}) \cdot \underbrace{(\overline{A} + \overline{\overline{A}})}_{1 \text{ 'Complement'}}$

$$= (A + \overline{\overline{A}}) \cdot \overbrace{(A + \overline{A})}^{1}$$

'Distributive' ② $= A + \underbrace{\overline{\overline{A}} \cdot \overline{A}}$

if $\overline{A} \equiv B$

$\overline{B} \cdot B = 0$

$$= \boxed{A}$$

b) De Morgan's Law #1 $\qquad \overline{A + B} = \overline{A} \cdot \overline{B}$

(Complement of A or B is the complement of A & the complement of B)

Simple proof using truth table (in Boolean algebra, $A = 0, 1$; $B = 0, 1$)

A	B	$\overline{A+B}$
0	0	1
0	1	0
1	0	0
1	1	0

A	B	$\bar{A}$	$\bar{B}$	$\bar{A}\cdot\bar{B}$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	0	0	0

A more general proof:

If De Morgan's Law #1 is true:

$$P + \bar{P} = 1$$

$P \equiv A+B \rightarrow$ $A+B + \overline{A+B} = 1 \longrightarrow A+B+\bar{A}\cdot\bar{B} = 1$

Using OR to AND distribution: $(\underbrace{A+B+\bar{A}}_{1})\cdot(\underbrace{A+B+\bar{B}}_{1})$

$$(\underbrace{B+1}_{1}) \cdot (\underbrace{A+1}_{1})$$

QED.

c) DeMorgan's Law #2: $\overline{A\cdot B} = \bar{A} + \bar{B}$

Proof: we know $A + \bar{A} = 1$

$A \cdot \bar{A} = 0$

$\rightarrow A\cdot B \cdot \overline{A\cdot B} = 0$

If De Morgan's Law # is true: $A\cdot B\cdot(\bar{A}+\bar{B}) = 0$

Using AND to OR distribution: $\underbrace{A\cdot B\cdot\bar{A}}_{0} + \underbrace{A\cdot B\cdot\bar{B}}_{0} = 0$ ✓

QED.

d) Absorption Laws:

1) $A + A \cdot B = A$

Proof: $A = A \cdot 1$

$A \cdot 1 + A \cdot B = A \cdot \underbrace{(1+B)}_{1} = A$ ✓

2) $A \cdot (A+B) = A$

Proof: $A \cdot (A+B) = A \cdot A + A \cdot B = A + A \cdot B \overset{1)}{=} A$

Consequence: $B \equiv 1 \rightarrow$ $\boxed{A + A = A}$

How to write Boolean Formulas & Functions:

Formula: Boolean combination of literals.
(Literal is a Boolean variable or its complement: x or $\bar{x}$)

$xyz + x\bar{y}z + \bar{x}yz$ (for example)

Function: a map of a set of Boolean variables (x, y, z) to a unique formula.

$f(x,y,z) = (\bar{x} + y) \cdot z$

$f(w,x,y,z) = z \cdot (x + \bar{y}) \cdot (w + \bar{x} + \bar{y})$

$\updownarrow$

Truth Table

→ Given a True Table, a Boolean function can be written in standard or in m-notation, using a MINTERM or a MAXTERM canonical formulas.

<u>MINTERM Canonical</u>: add terms which output a one on the truth table:

An m is assigned to each combination of x, y, z

	x	y	z	f(x,y,z)		
m=0	0	0	0	0		
m=1	0	0	1	1	← $\bar{x}\cdot\bar{y}\cdot z$	(since f gives a 1 when $x=0$, $y=0$, & $z=1$)
m=2	0	1	0	0		
m=3	0	1	1	1	← $\bar{x}\cdot y\cdot z$	(" " " a 1 when $x=0$, $y=1=z$)
m=4	1	0	0	1	← $x\cdot\bar{y}\cdot\bar{z}$	
⋮	1	0	1	0		
⋮	1	1	0	0		
m=7	1	1	1	0		

$f(x,y,z) = \bar{x}\bar{y}z + \bar{x}yz + x\bar{y}\bar{z}$ (Standard)

$f(x,y,z) = \sum m(1,3,4)$ (m-notation)

<u>MAXTERM Canonical</u>: add literals which output a zero on the truth table:

	x	y	z	f(x,y,z)		
M=0	0	0	0	0	→ $x+y+z$	(since this gives 0 when $x=y=z=0$)
M=1	0	0	1	1		
M=2	0	1	0	0	→ $x+\bar{y}+z$	
M=3	0	1	1	1		
M=4	1	0	0	1		
⋮	1	0	1	0	→ $\bar{x}+y+\bar{z}$	
⋮	1	1	0	0	→ $\bar{x}+\bar{y}+z$	
M=7	1	1	1	0	→ $\bar{x}+\bar{y}+\bar{z}$	

$f(x,y,z) = (x+y+z)\cdot(x+\bar{y}+z)\cdot(\bar{x}+y+\bar{z})\cdot(\bar{x}+\bar{y}+z)\cdot(\bar{x}+\bar{y}+\bar{z})$ Standard

$f(x,y,z) = \prod M(0,2,5,6,7)$ (M-notation)

Formula Manipulation: Converting to a Standard MINTERM Canonical

$$f(x,y,z) = \overline{(x+y)} + (y + x\cdot z)\cdot(x+\bar{y})$$

Let's convert it into a MINTERM Canonical:
(use properties & corollaries of Boolean algebra: (Table 3-1 page 70)

De Morgan #1: $\overline{(x+y)} = \bar{x}\cdot\bar{y}$

Distributive property: AND to OR:

$$(y+x\cdot z)\cdot(x+\bar{y}) = y\cdot x + \underbrace{y\cdot\bar{y}}_{0} + \underbrace{x\cdot z\cdot x}_{\substack{x\cdot z \\ (x\cdot x = x)}} + x\cdot z\cdot\bar{y}$$

$$f(x,y,z) = \bar{x}\cdot\bar{y} + x\cdot y + x\cdot z + x\cdot\bar{y}\cdot z$$

For a MINTERM Canonical all should have 3 literals!

using $z+\bar{z} = 1$ & $A\cdot 1 = A$

$$f(x,y,z) = \bar{x}\cdot\bar{y}\cdot(z+\bar{z}) + x\cdot y\cdot(z+\bar{z}) + x\cdot(y+\bar{y})\cdot z + x\cdot\bar{y}\cdot z$$

$$= \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z} + xyz + xy\bar{z} + xyz + \underbrace{x\bar{y}z + x\bar{y}z}$$

$$= \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z} + xyz + xy\bar{z} + x\bar{y}z$$ (dropped duplicats).

↓

f in MINTERM Canonical.

We can write truth table for $f(x,y,z)$ from it's MINTERM Canonical expression:

↓

five terms

→ five ones

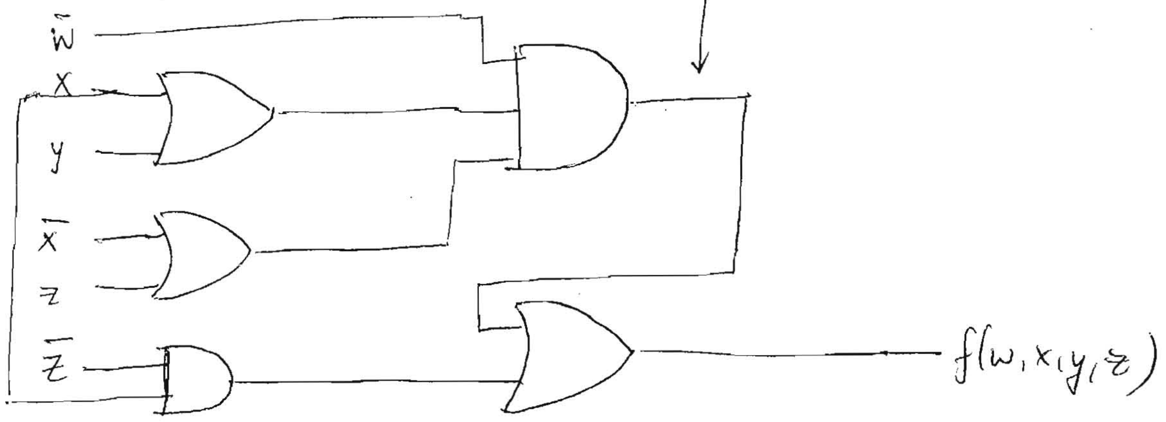
x	y	z	f(x,y,z)
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

3.19

a) Draw logic circuit (using AND $(\cdot)$ & OR $(+)$ gates)

for $f(w,x,y,z) = \underbrace{\bar{w}\cdot(x+y)\cdot(\bar{x}+z)} + x\cdot\bar{z}$

4 inputs : w, x, y, z :



(2 AND's & 3 OR's)

b) $f(v,w,x,y,z) = (x+y)\cdot\left\{\bar{v} + (\bar{w}+y)\cdot\left[v+(\bar{w}+z)\cdot(\bar{v}+x+z)\right]\right\}$

(3 AND's + 6 OR's)

0 0 1 | 1
0 1 0 | 0 (b) There are only
0 1 1 | 0 (c)

Boolean Formula Manipulation:

- Conversion to MIN TERM ✓ or MAX TERM Canonical ✓
- Expansion about a variable ✓
- Shannon's reduction theorems ✓
- Use universal gates: NAND's & NOR's ✓

– Example on converting a given Boolean function into MAX TERM canonical.
↳ properties of Boolean Algebra

$$f(x,y,z) = \overline{(x+y)} + (y + x\cdot z)\cdot(x+\bar{y})$$

. Distributive property: (OR to AND) $\quad y + x\cdot z = (y+x)\cdot(y+z)$

. De Morgan's Law #1: $\quad \overline{(x+y)} = \bar{x}\cdot\bar{y}$

$$f(x,y,z) = \bar{x}\cdot\bar{y} + (y+x)\cdot(y+z)\cdot(x+\bar{y})$$

. Distributive property (OR to AND's)

$$= (\bar{x}\cdot\bar{y} + y + x)\cdot(\bar{x}\cdot\bar{y} + y + z)\cdot(\bar{x}\cdot\bar{y} + x + \bar{y})$$

$$= \underbrace{(\bar{x} + y + x)}_{1}\cdot\underbrace{(\bar{y} + y + x)}_{1}\cdot(\bar{x} + y + z)\cdot\underbrace{(\bar{y}+y+z)}_{1}\cdot\underbrace{(\bar{x} + x + \bar{y})}_{1}\cdot(\bar{y} + x + \bar{y})$$

$$\begin{cases} x + \bar{x} = 1 \\ y + 1 = 1 \end{cases}$$

$$= (\bar{x} + y + z)\cdot(\bar{y} + x + \bar{y})$$

$\bar{y} + \bar{y} = \bar{y}$ $$= \underbrace{(\bar{x}+y+z)}_{\text{MAX TERM}}\cdot\underbrace{(\bar{y} + x)}_{\text{Missing } z!}$$

$z\cdot\bar{z} = 0$ $$= (\bar{x}+y+z)\cdot(x + \bar{y} + z\cdot\bar{z})$$

Distributive property $$= \underbrace{(\bar{x}+y+z)}_{a}\cdot\underbrace{(x+\bar{y}+z)}_{b}\cdot\underbrace{(x+\bar{y}+\bar{z})}_{c}$$

We can write the truth table for f from this MAX TERM canonical version of it:

x	y	z	$f(x,y,z)$	
0	0	0	1	
0	0	1	1	
0	1	0	0	(b)
0	1	1	0	(c)
1	0	0	0	(a)
1	0	1	1	
1	1	0	1	
1	1	1	1	

There are only three 0's (three factors in the MAX TERM Canonical)

Expansion about a Variable:

a) OR (+) expansion around x_i

$$f(x_1, x_2, \ldots, x_{n-1}, x_n) = x_i \cdot f(x_1, x_2, \ldots x_n)\Big/_{x_i=1} + \bar{x}_i \cdot f(x_1, x_2, \ldots x_n)\Big/_{x_i=0}$$

b) AND (·) expansion around x_i

$$f(x_1, x_2, \ldots x_n) = \left[x_i + f(x_1, x_2, \ldots x_n)\Big/_{x_i=0} \right] \cdot \left[\bar{x}_i + f(x_1, x_2, \ldots x_n)\Big/_{x_i=1} \right]$$

$$f(w,x,y,z) = \bar{w} \cdot \bar{z} + (w \cdot x + y) \cdot z$$

OR expansion of f around x:

$$f(w,x,y,z) = x \cdot [\bar{w} \cdot \bar{z} + (\underbrace{w \cdot 1}_{w} + y) \cdot z] + \bar{x} \cdot [\bar{w} \cdot \bar{z} + (\underbrace{w \cdot 0}_{0} + y) \cdot z]$$

$$= x \cdot [\bar{w} \cdot \bar{z} + (w + y) \cdot z] + \bar{x} \cdot [\bar{w} \cdot \bar{z} + y \cdot z]$$

a) Complements using NAND gate :

Shannon's Reduction Theorems :

a) $x_i \cdot f(x_1, x_2, \cdots x_i \cdots x_n)] = x_i \cdot f(x_1, x_2, \cdots 1, \cdots x_n)$

↑ x_i is replaced by 1

b) $x_i + f[x_1, x_2, \cdots x_i, \cdots x_n)] = x_i + f(x_1, x_2, \cdots 0, \cdots x_n)$

↑ x_i replaced by 0

c) $\overline{x_i} \cdot f(x_1, x_2, \cdots x_i, \cdots x_n) = \overline{x_i} \cdot f(x_1, x_2, \cdots 0, \cdots x_n)$

↑ x_i replaced by 0

d) $\overline{x_i} + f(x_1, x_2, \cdots x_i, \cdots x_n) = \overline{x_i} + f(x_1, x_2, \cdots, 1, \cdots x_n)$

↑ x_i replaced by 1

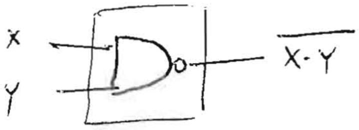
Universal Gates NAND's & NOR's :

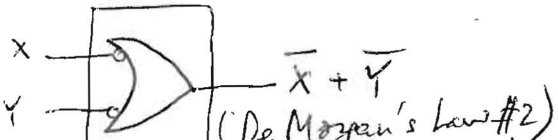
NAND & NOR are independent!

X	Y	$X \cdot Y$ (AND)	$\overline{X \cdot Y}$ NAND	$X+Y$ (OR)	$\overline{X+Y}$ (NOR)
0	0	0	1	0	1
0	1	0	1	1	0
1	0	0	1	1	0
1	1	1	0	1	0

NAND: $\quad NAND(x, y) = \overline{X \cdot Y}$

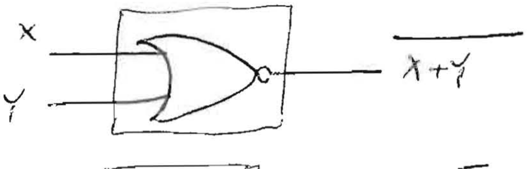
Logic Gate Symbol:

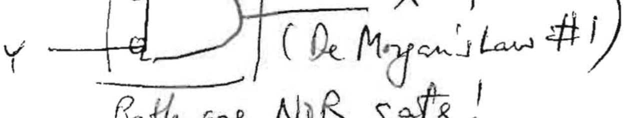




Both are NAND gates!

NOR: $\quad NOR(x, y) = \overline{X+Y}$

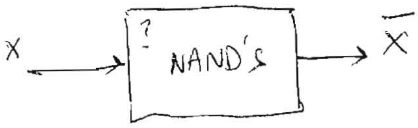




Both are NOR gates!

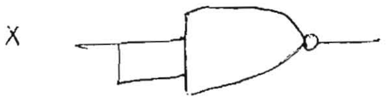
NAND's & NOR's are universal! ANY logic circuit can be implemented using <u>only</u> NAND's or NOR's!

a) <u>Complements using NAND gate</u>:



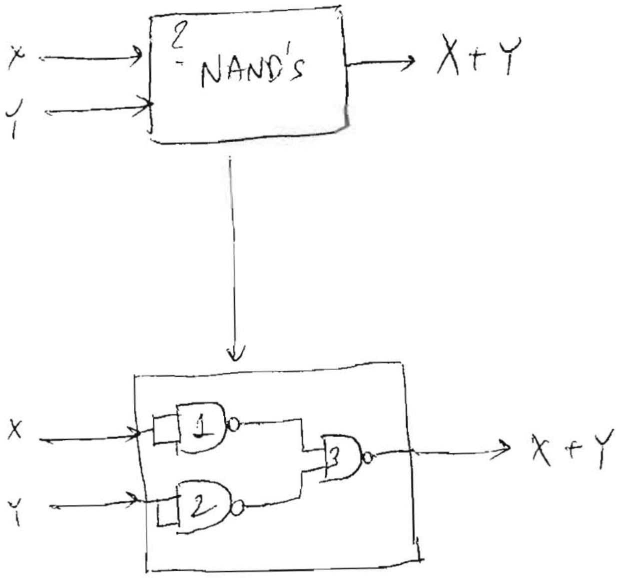
$$\begin{cases} X \cdot X = X \\ \overline{X \cdot X} = \overline{X} \end{cases}$$

$\downarrow$ NAND!



$\overline{X \cdot X} = \overline{X}$

b) <u>OR using NAND's</u>:



<u>De Morgan's Law</u>:

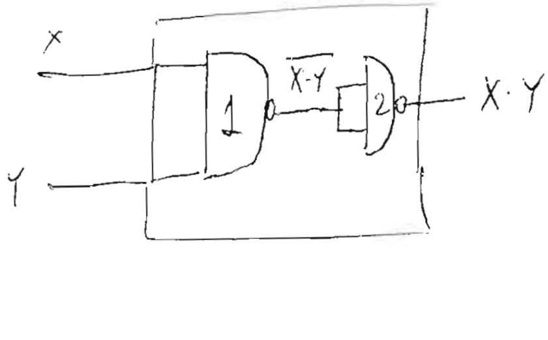
$$\overline{X \cdot Y} \overset{\#2}{=} \overline{X} + \overline{Y}$$

$$\overline{\overline{X} \cdot \overline{Y}} = \overline{\left(\overline{X} + \overline{Y}\right)} \overset{\#1}{=} \overline{\overline{X}} + \overline{\overline{Y}}$$

$$= X + Y$$

(can implement an OR using 3 NAND's)

c) <u>AND using 2 NAND's</u>



$$\overline{\overline{X \cdot Y} \cdot \overline{X \cdot Y}} = X \cdot Y$$

$$\hookrightarrow \overline{\overline{X \cdot Y}} + \overline{\overline{X \cdot Y}} = X \cdot Y + X \cdot Y$$

$$= X \cdot Y$$