Boolean expressions $\longrightarrow$ simplify using { - Properties of Boolean Algebra; - with visual aid: Karnaugh maps. }

Karnaugh map: 4-variable Boolean function: $\boxed{\text{2 ones}}$

WX \ YZ	00	01	11	10
00				
01		1	1	
11				
10				

These two adjacent one's:

$$AB + \bar{A}B = \underbrace{(A+\bar{A})}_{1}\cdot B = B$$

or $YZ + \bar{Y}Z = Z$

(Two adjacent ones on Karnaugh map allows elimination of one variable, Y in our example here).

Writing final version with all 4 variables: these two ones give:

$\bar{w}x(yz + \bar{y}z) = \bar{w}xz$. (among 4 variable, y was eliminated since we have these two adjacent ones.)

WX \ YZ	00	01	11	10
00				
01		1		
11		1		
10				

$X\bar{Y}Z$

WX \ YZ	00	01	11	10
00				
01	1			1
11				
10				

$\bar{W}X\bar{Z}$

$(\bar{w}x(\bar{y}\bar{z} + y\bar{z}) = \bar{w}x\bar{z})$

WX \ YZ	00	01	11	10
00			1	
01				
11				
10			1	

$\bar{X}YZ$

Karnaugh map: 4-variable Boolean function with $\boxed{4 \text{ ones}}$

wx \ yz	00	01	11	10
00				
01		1	1	
11		1	1	
10				

$$\text{SOP}: \quad \bar{w}x\bar{y}z + \bar{w}xyz + wx\bar{y}z + wxyz$$
$$= \bar{w}xz(\underbrace{\bar{y}+y}_{1}) + wxz(\underbrace{\bar{y}+y}_{1})$$
$$= (\underbrace{\bar{w}+w}_{1})xz$$

⇒ Conclusion: 4 ones together allows us to eliminate 2 variables. The variable that changes value is eliminated, in our case vertically w & horizontally y. The variable that does not change value stays, in our case x (vertically) & z (horizontally).

wx \ yz	00	01	11	10
00	1			1
01	1			1
11				
10				

$\bar{w}\,\bar{z}$

wx \ yz	00	01	11	10
00	1			1
01				
11				
10	1			1

$\bar{x}\,\bar{z}$

wx \ yz	00	01	11	10
00		1		
01		1		
11		1		
10		1		

$\bar{y}z$

wx \ yz	00	01	11	10
00				
01				
11	1	1	1	1
10				

wx

<u>Karnaugh map:</u> 4-variable Boolean function w/ $\boxed{\text{8 ones}}$

wx \ yz	00	01	11	10
00	1	1	1	1
01	1	1	1	1
11				
10				

$\bar{w}$

wx \ yz	00	01	11	10
00	1			1
01	1			1
11	1			1
10	1			1

$\bar{z}$

<u>What about simplifying POS or Maxterm Canonical Expressions?</u>

Karnaugh map: 4 variables with 2, 4, 8 zeroes.

2 → eliminate 1 variable

4 → eliminate 2 variables.

8 → eliminate 3 variables

wx \ yz	00	01	11	10
00				
01	0	0		
11				
10				

$w+\bar{x}+y$

POS or Maxterm Canonical:

$$(\underbrace{w+\bar{x}+y}_{A}+\underbrace{z}_{B})(\underbrace{w+\bar{x}+y}_{A}+\underbrace{\bar{z}}_{C})$$

<u>Distributive property:</u> $A + B\cdot C = (A+B)\cdot(A+C)$

$$\rightarrow w+\bar{x}+y + \underbrace{z\cdot\bar{z}}_{0} = \boxed{w+\bar{x}+y}$$

<u>Conclusion:</u> two adjacent zeroes allow us to eliminate one variable, the one that is changing value

4 zeros

wx \ yz	00	01	11	10
00	0	0		
01				
11				
10	0	0		

$(x + y)$

8 zeros.

wx \ yz	00	01	11	10
00			0	0
01			0	0
11			0	0
10			0	0

$(\bar{y})$

Simplyfying a Boolean Function Using Karnaugh Mapping:

wx \ yz	00	01	11	10
00	0	A 1	1	1 B
01	0	1	1	1
11	0	1 C	0	0
10	D 1	0	0	0

Minterm Canonical or SOP: 8 terms.

Simplifying using Karnaugh Map:

- Max number of ones together: 4 → 2 subcubes A & B → $\bar{w}z$, $\bar{w}y$
- look for next smaller groups of ones not entirely included in the previous step: 2 ones in C not included in A & B → $x\bar{y}z$
- look for next smaller group of ones not entirely included in previous groups: D → $w\bar{x}\bar{y}\bar{z}$

→ $f(w,x,y,z) = \bar{w}z + \bar{w}y + x\bar{y}z + w\bar{x}\bar{y}\bar{z}$ → can now be implemented in a logic circuit w/ min # of gates.

(irredundant disjunctive normal formula)

What if a Boolean function is not originally in canonical format? Can we still simplify it using Karnaugh map?

Yes, using "reverse Karnaugh map":

Assume we are given : $\boxed{f(w,x,y,z) = \bar{w}\bar{x} + \bar{w}xz + yz}$

(Not canonical since terms do not show all 4 variables)

Reverse Karnaugh Map:

wx \ yz	00	01	11	10
00	1	1	1	1
01	0	1	1	0
11	0	0	1	0
10	0	0	1	0

→ $\bar{w}\bar{x}$ (product of 2 variable is a group of 4 ones)

→ $\bar{w}xz$

→ yz

Now use Karnaugh map to simplify:

wx \ yz	00	01	11	10
00	1	1	1	1
01		1	1	
11			1	
10			1	

A

B

C

- Max number of one together : 4 : A & B & C

  $\bar{w}\bar{x}$ yz $\bar{w}z$

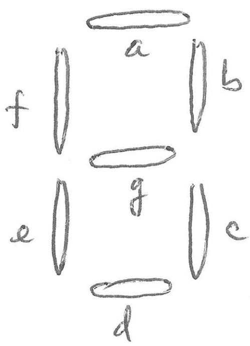
- Smaller groups of ones not entirely included in previous groups 0

→ $\boxed{f(w,x,y,z) = \bar{w}\bar{x} + yz + \bar{w}z}$ simplified with elimination of x out of $\bar{w}xz$

Karnaugh Map & Incomplete Functions:

7-Segment functions

Truth Table:



wxyz	a	b	c
0000	1	1	1
0001	0	1	0
0010	1	1	1
0011	1	1	1
0100	0	1	1
0101	1	0	1
0110	1	0	1
0111	1	1	1
1000	1	1	1
1001	1	1	1

Don't care about function values for the other 6 inputs. → a is an incomplete function.

Segment a Karnaugh Map: (incomplete function)

wx \ yz	00	01	11	10
00	1	0	1	1
01	0	1	1	1
11	-	-	-	-
10	1	1	-	-

A

B

To find simplest SOP:

1) Max number of ones (including unknowns) together: 8 → (eliminating 3 variables)
→ 2 subcubes A & B
A → y, B → w

2) Next smaller group of ones (including unknowns) together: 4 (not entirely included in the previous larger group) → eliminating 2 variables.

WX \ YZ	00	01	11	10
00	1	0	1	1
01	0	1	1	1
11	–	–	–	–
10	1	1	–	–

(C) XZ

(D) $\bar{X}\bar{Z}$

3) No smaller group of ones including unknowns

$$a(w,x,y,z) = y + w + xz + \bar{x}\bar{z}$$

Segment c Karnaugh Map : (another incomplete function)

WX \ YZ	00	01	11	10
00	1	1	1	0
01	1	1	1	1
11	–	–	–	–
10	1	1	–	–

(A) (B) (C) (D)

Simplest POS :

1) Max number of zeroes including unknowns together: 2

eliminate one variable: w

$c(w,x,y,z) = x + \bar{y} + z$ (No smaller group, min is 2)

(POS)

$x = 0$
$\bar{y} = 0$ $(y=1)$
$z = 0$
wo 10

Simplest SOP :

1) Max number of ones including unknowns : 8 → 4 subcubes :

(A) → x, (B) → w, (C) → $\bar{y}$, (D) → z

$c(w,x,y,z) = x + w + \bar{y} + z$

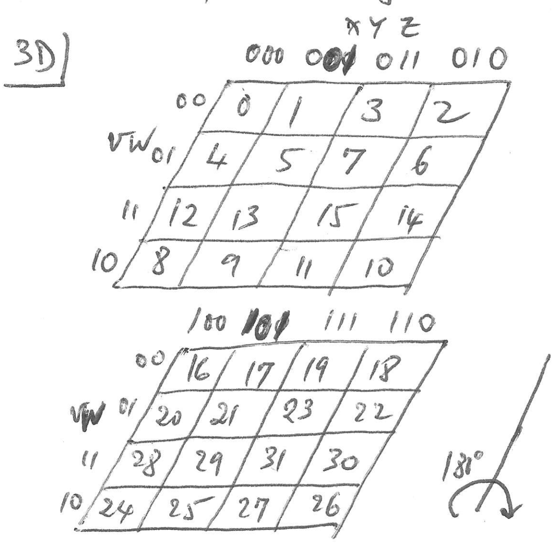
(SOP)

when $x=0$, $y=1$, $z=0$ this agrees with the simplest POS when $w=0$, but when $w=1$ the function is unknown (see Karnaugh map)

Five-variable Karnaugh Map: $f(\underbrace{v, w}_{4}, \underbrace{x, y, z}_{8})$

4x8 table in a special arrangement so between consecutive cells there is only a change of one variable:

3D)



2D)

vw \ xyz	000	001	011	010	110	111	101	100
00	0	1	3	2	18	19	17	16
01	4	5	7	6	22	23	21	20
11	12	13	15	14	30	31	29	28
10	8	9	11	10	26	27	25	24

- For Karnaugh map method to work: only a change of one variable when going b/w consecutive cells in any direction including vertically
- There are 32 cells $(= 2^5)$
- Karnaugh map: consecutive cells are not so obvious:

Example:

vw \ xyz	000	001	011	010	110	111	101	100
00	1			1	1		1	1
01		1	1			1	1	
11		1	1			1	1	
10	1	1	1	1			1	

(A) These 4 ones are together $\rightarrow \bar{v}\bar{w}\bar{z}$

(B) These 8 ones are together (top & bottom)

ZW