→ For other code: add a parity bit:

8421	Parity bit (so total number of ones is always odd)
0000	1
0001	0
0010	0
0011	1
0100	0
0101	1
0111	0
...	

→ Can detect a simple error: 00001 → 01001

There's an error (# of ones is even!)

→ Can't detect a double error 00001 → 11001

→ Can detect a triple error ...

Ch3 Boolean Algebra & Combinational Networks

Boolean Algebra:

a) Commutative

$A + B = B + A$

$A \cdot B = B \cdot A$

b) Associative

$(A + B) + C = A + (B + C)$

$A \cdot (B \cdot C) = (A \cdot B) \cdot C$

c) Neutral Element

$A + 0 = A$

$A \cdot 1 = A$

d) Distributive

$A \cdot (B + C) = A \cdot B + A \cdot C$

$\boxed{A + B \cdot C = (A+B) \cdot (A+C)}$ *

$\boxed{+ \quad = \quad OR}$

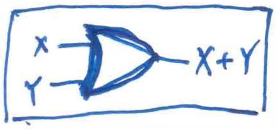
$\boxed{\cdot \quad = \quad AND}$

$* \quad \underbrace{A + 1 \cdot 1}_{(A+1)} = (A+1) \cdot (A+1) \Rightarrow A + 1 = 1$

Truth Tables:

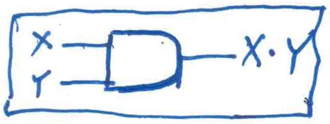
X OR Y

X	Y	X+Y
0	0	0
1	0	1
0	1	1
1	1	1



X AND Y

X	Y	X·Y
0	0	0
1	0	0
0	1	0
1	1	1



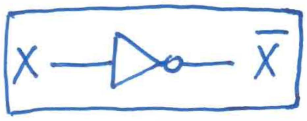
NOT or

Complement of A is $\bar{A}$

$A + \bar{A} = 1$

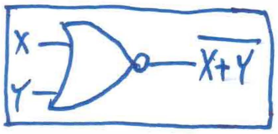
$A \cdot \bar{A} = 0$

X	$\bar{X}$
0	1
1	0



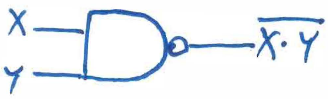
(NOR)

X	Y	$\overline{X+Y}$
0	0	1
1	0	0
0	1	0
1	1	0



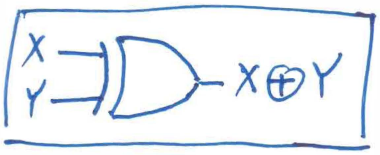
(NAND)

X	Y	$\overline{X \cdot Y}$
0	0	1
1	0	1
0	1	1
1	1	0



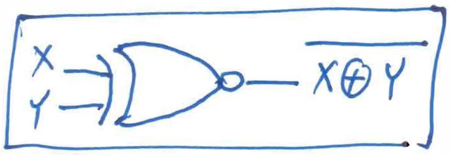
(XOR)

X	Y	$X \oplus Y$
0	0	0
1	0	1
0	1	1
1	1	0



(XNOR)

X	Y	$\overline{X \oplus Y}$
0	0	1
1	0	0
0	1	0
1	1	1



Corollaries from Boolean Algebra:

a) $\bar{\bar{A}} = A$

Proof: $A + \bar{A} = 1 \rightarrow \bar{A} = 1 - A$

$$\rightarrow \bar{\bar{A}} = 1 - \bar{A} = 1 - (1 - A) = A$$

b) $\overline{A+B} = \bar{A} \cdot \bar{B}$ DeMorgan's Law #1

Proof: use truth tables:

A	B	$\overline{A+B}$
0	0	1
1	0	0
0	1	0
1	1	0

A	B	$\bar{A}$	$\bar{B}$	$\bar{A} \cdot \bar{B}$
0	0	1	1	1
1	0	0	1	0
0	1	1	0	0
1	1	0	0	0

A more general proof:

$$\overline{A+B} = \bar{A} \cdot \bar{B} \longleftrightarrow A+B + \bar{A} \cdot \bar{B} = 1$$ ← Need To prove This

Since: $A+B + \overline{A+B} = 1$

Distributive Property of Boolean Algebra

$$(\underbrace{A+B+\bar{A}}_{1}) \cdot (A+\underbrace{B+\bar{B}}_{1})$$

$$= (\underbrace{B+1}_{1}) \cdot (\underbrace{A+1}_{1})$$

$$= 1 \checkmark$$

De Morgan's Law #2:

$$\overline{A \cdot B} = \bar{A} + \bar{B}$$

Since $A \cdot B + \overline{A \cdot B} = 1$

Different strategy: using a different property for complements:

$$\begin{cases} A + \bar{A} = 1 \\ A \cdot \bar{A} = 0 \end{cases}$$

$$\overline{A \cdot B} = \bar{A} + \bar{B} \longrightarrow A \cdot B \cdot \overline{A \cdot B} = 0$$

$$A \cdot B \cdot (\bar{A} + \bar{B}) = 0$$ Need to prove this.

$$A \cdot B \cdot \bar{A} + A \cdot B \cdot \bar{B}$$

$$B \cdot \underbrace{(A \cdot \bar{A})}_{0} + A \cdot \underbrace{(B \cdot \bar{B})}_{0} = 0 \checkmark$$

Absorption Law:

a) $A + A \cdot B = A$

Proof: $A = A \cdot 1$

$A \cdot 1 + A \cdot B = A \cdot \underbrace{(1+B)}_{1} = A$ ✓

b) $A \cdot (A+B) = A$

Proof: $A \cdot (A+B) = A \cdot A + A \cdot B = A + A \cdot B = A$ ✓

Since: $A \cdot A = A \cdot (1 - \bar{A}) = A - \underbrace{A \cdot \bar{A}}_{0} \rightarrow \boxed{A \cdot A = A}$

$\boxed{A + A = A}$ ✓

Since $A + A \cdot 1 = A$ ($B = 1$ in absorption Law).

Boolean Formulas & Functions:

Formulas: boolean combinations of literals (a literal is a variable or its complement: x or $\bar{x}$)

$xyz + x\bar{y}z + \bar{x}yz$, etc.

Functions: map a set of variables (eg. x, y, z) to a unique formula.

$f(x, y, z) = (\bar{x} + y)z$

$f(w, x, y, z) = z(x + \bar{y}) \cdot (w + \bar{x} + \bar{y})$

Canonical Formulas from a truth table: $\begin{cases} \text{Minterm} \\ \text{Maxterm} \end{cases}$

Truth Table :⟺ Function

2^3 entries

m	x	y	z	f(x,y,z)
m=0	0	0	0	0
m=1	0	0	1	1
m=2	0	1	0	0
m=3	0	1	1	1
m=4	1	0	0	1
m=5	1	0	1	0
m=6	1	1	0	0
m=7	1	1	1	0

⟷ $\bar{x}\cdot\bar{y}\cdot z$: this term only gives one for entry #2 (m=1)

⟷ $\bar{x}\cdot y\cdot z$: this term only gives one for entry #4 (m=3)

⟷ $x\cdot\bar{y}\cdot\bar{z}$: this term only gives one for entry #5 (m=4)

Minterm Canonical Formula for this function or truth table:

Look at the one: there are three ones → 3 terms:

$$f(x,y,z) = \underbrace{\bar{x}\cdot\bar{y}\cdot z}_{\text{entry \#2 only.}} + \underbrace{\bar{x}\cdot y\cdot z}_{\text{entry \#4 only}} + \underbrace{x\cdot\bar{y}\cdot\bar{z}}_{\text{entry \#5 only}} \qquad \text{(Standard)}$$

$$f(x,y,z) = \sum m(1,3,4) \qquad \text{(m-notation)}$$

Maxterm Canonical Formula:

Look at the zeroes: five zeroes in the truth table
→ five factors.

M=	x	y	z	f(x,y,z)	
0	0	0	0	0	→ $x+y+z$: only gives zero for entry #1 (gives 1 for rest of entries)
1	0	0	1	1	
2	0	1	0	0	→ $x+\bar{y}+z$: gives zero for entry #3, (gives 1 for the rest)
3	0	1	1	1	
4	1	0	0	1	
5	1	0	1	0	→ $\bar{x}+y+\bar{z}$ (only for entry #6, gives 0)
6	1	1	0	0	→ $\bar{x}+\bar{y}+z$
7	1	1	1	0	→ $\bar{x}+\bar{y}+\bar{z}$

$$f(x,y,z) = (x+y+z)(x+\bar{y}+z)(\bar{x}+y+\bar{z})(\bar{x}+\bar{y}+z)(\bar{x}+\bar{y}+\bar{z})$$
(Standard)

$$f(x,y,z) = \prod M(0,2,5,6,7)$$
(M-notation)

Formula manipulation: Converting to a Standard Minterm Canonical

$$f(x,y,z) = \overline{(x+y)} + (y+xz).(x+\bar{y})$$

Use Boolean Algebra properties in Table 3.1 (p. 70):

De Morgan: $\overline{(x+y)} = \bar{x}\cdot\bar{y}$

Distributive: $(y+xz).(x+\bar{y}) = y\cdot x - \underbrace{y\cdot\bar{y}}_{0} + x.z.x + x.z.\bar{y}$ $\boxed{x.x=x}$

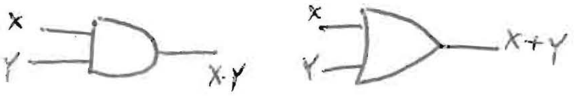
$$f(x,y,z) = \bar{x}.\bar{y}\underset{\substack{1\\(z+\bar{z})}}{\uparrow} + x.y\underset{\substack{1\\(z+\bar{z})}}{\uparrow} + x.\underset{\substack{1\\(y+\bar{y})}}{\uparrow}z + x\bar{y}.z$$

$$= \bar{x}.\bar{y}.z + \bar{x}.\bar{y}.\bar{z} + x.y.z + x.y.\bar{z} + \underset{(dup)}{x.y.z} + x.\bar{y}.z + \underset{(dup.)}{x.\bar{y}.z}$$

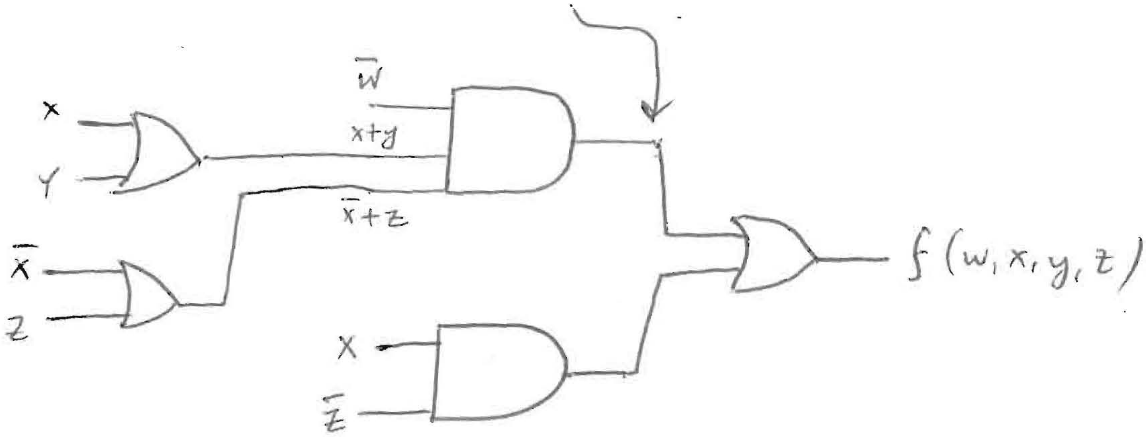
Dropping duplicates:

$$\boxed{f(x,y,z) = \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z} + x\cdot y\cdot z + xy\bar{z} + x\bar{y}\cdot z.}$$

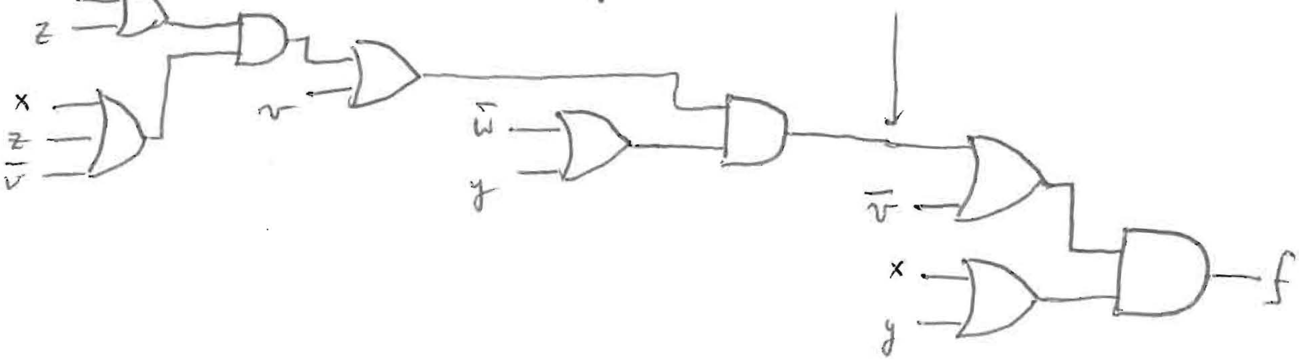
3.19 Draw logic diagram using gates: AND (·) & OR (+)



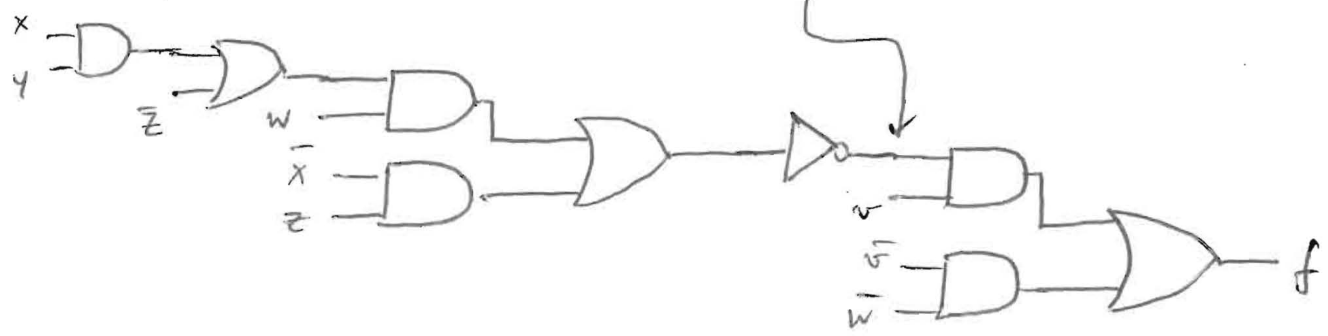
a) $f(w,x,y,z) = \underbrace{\bar{w}\cdot(x+y)\cdot(\bar{x}+z)} + x\cdot\bar{z}$



b) $f(v,w,x,y,z) = (x+y)\cdot\left\{\bar{v} + \underbrace{(\bar{w}+y)\cdot\left[v + (\bar{w}+z)\cdot(\bar{v}+x+z)\right]}\right\}$



c) $f(v,w,x,y,z) = v\cdot\overline{\underbrace{\left[w\cdot(x\cdot y+\bar{z})+\bar{x}\cdot z\right]}} + \bar{v}\cdot\bar{w}$



$\boxed{\text{Formula Manipulation (Cont.)}}$ Converting to a Standard Maxterm Canonical.

$f(x,y,z) = \overline{(x+y)} + (y + x\cdot z)\cdot(x+\bar{y})$

Use Boolean Algebra properties in Table 3.1 (p. 70):

- Distributive on $(y + x\cdot z) = (y+x)\cdot(y+z)$
- De Morgan on $\overline{(x+y)} = \bar{x}\cdot\bar{y}$

$f(x,y,z) = \bar{x}\cdot\bar{y} + (y+x)\cdot(y+z)\cdot(x+\bar{y})$

$= (\bar{x}\cdot\bar{y} + y + x)\cdot(\bar{x}\cdot\bar{y} + y + z)\cdot(\bar{x}\cdot\bar{y} + x + \bar{y})$

Distributive

$= \underbrace{(\bar{x} + y + x)}_{1}\cdot\underbrace{(\bar{y} + y + x)}_{1}\cdot(\bar{x} + y + z)\underbrace{(\bar{y} + y + z)}_{1}\cdot\underbrace{(\bar{x} + x + \bar{y})}_{1}(\bar{y} + x + \bar{y})$

Distributive

$1 + y = 1$ ✓; $1 + \bar{y} = 1$? Absorption law: $A + A\cdot B = A$

$1 + \bar{y} = 1 + 1\cdot\bar{y} = 1$

$\bar{y} + \bar{y} = \bar{y}$ Absorption Law: $A + A\cdot 1 = A$

$f(x,y,z) = (\bar{x} + y + z)(x + \bar{y})$ $+ z\cdot\bar{z}$ $= (\bar{x} + y + z)(x + \bar{y} + z\cdot\bar{z})$

$= (\bar{x} + y + z)(x + \bar{y} + z)(x + \bar{y} + \bar{z})$ ✓

Formula Manipulation:

Expansion about a variable:

a) $f(x_1, x_2, \cdots x_n) = x_i \cdot f(x_1, x_2, \cdots x_n)\Big/_{x_i=1} + \overline{x_i} \cdot f(x_1, x_2, \cdots x_n)\Big/_{x_i=0}$

b) $f(x_1, x_2, \cdots x_n) = \left[x_i + f(x_1, x_2, \cdots x_n)\Big/_{x_i=0}\right] \cdot \left[\overline{x_i} + f(x_1, x_2, \cdots x_n)\Big/_{x_i=1}\right]$

Example. $f(w, x, y, z) = \overline{w} \cdot \overline{z} + (wx + y) \cdot z$

Expand f about x: use a):

$$f(w, x, y, z) = x \cdot \left[\overline{w} \cdot \overline{z} + (w \cdot 1 + y) \cdot z\right] + \overline{x} \cdot \left[\overline{w} \cdot \overline{z} + (w \cdot 0 + y) \cdot z\right]$$

$$= x \cdot \left[\overline{w} \cdot \overline{z} + (w + y) \cdot z\right] + \overline{x} \cdot \left[\overline{w} \cdot \overline{z} + y \cdot z\right]$$

Reduction Theorems; or Shannon's Reduction Theorems

a) $x_i \cdot f(x_1, x_2, \cdots x_i, \cdots x_n) = x_i \cdot f(x_1, x_2, \cdots \underset{\uparrow}{1}, \cdots x_n)$

replaced x_i by 1

b) $x_i + f(x_1, x_2, \cdots x_i, \cdots x_n) = x_i + f(x_1, x_2, \cdots \underset{\uparrow}{0}, \cdots x_n)$

replaced x_i by 0

c) $\overline{x_i} \cdot f(x_1, x_2, \cdots x_i, \cdots x_n) = \overline{x_i} \cdot f(x_1, x_2, \cdots \underset{\uparrow}{0}, \cdots x_n)$

replaced x_i by 0

d) $\overline{x_i} + f(x_1, x_2, \cdots x_i, \cdots x_n) = \overline{x_i} + f(x_1, x_2, \cdots \underset{\uparrow}{1}, \cdots x_n)$

replaced x_i by 1

Boolean Algebra
Formula manipulation: standard minterm/maxterm canonical reduction theorems.

And/Or gates.

→ Logic Design: logic circuit to accomplish certain goal.

↳ 1st Example. Odd-Parity Bit Generator.

(Total number of 1's including parity bit should be odd → error detection)

Step 1:

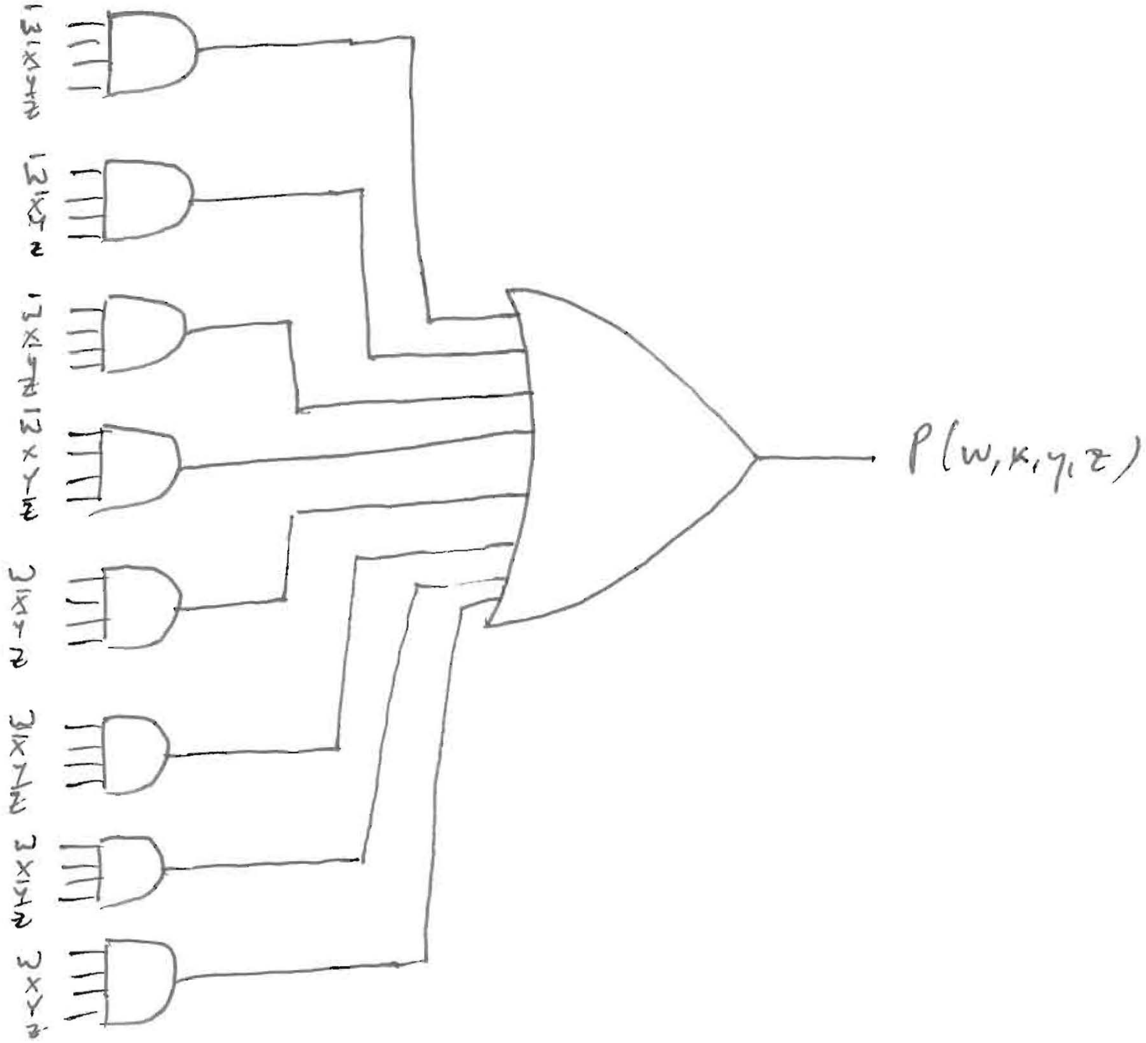
Truth Table.

8	4	2	1	
w	x	y	z	P	
0	0	0	0	1	$\bar{w}\cdot\bar{x}\cdot\bar{y}\cdot\bar{z}$
0	0	0	1	0	
0	0	1	0	0	
0	0	1	1	1	$\bar{w}\cdot\bar{x}\cdot y\cdot z$
0	1	0	0	0	
0	1	0	1	1	$\bar{w}\cdot x\cdot\bar{y}\cdot z$
0	1	1	0	1	$\bar{w}\cdot x\cdot y\cdot\bar{z}$
0	1	1	1	0	
1	0	0	0	0	
1	0	0	1	1	$w\cdot\bar{x}\cdot\bar{y}\cdot z$
1	0	1	0	1	$w\cdot\bar{x}\cdot y\cdot\bar{z}$
1	0	1	1	0	
1	1	0	0	1	$w\cdot x\cdot\bar{y}\cdot\bar{z}$
1	1	0	1	0	
1	1	1	0	0	
1	1	1	1	1	$w\cdot x\cdot y\cdot z$

Step 2: minterm canonical for this truth table: (focusing on the ones') → 8 terms.

$$P(w,x,y,z) = \bar{w}\cdot\bar{x}\cdot\bar{y}\cdot\bar{z} + \bar{w}\bar{x}yz + \bar{w}x\bar{y}z + \bar{w}xy\bar{z} + w\bar{x}\bar{y}z + w\bar{x}y\bar{z} + wx\bar{y}\bar{z} + wxyz$$

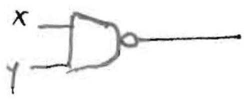
Step 3: Logic Gate Implementation



Additional Boolean & Gates Operations :

X	Y	AND $X \cdot Y$	NAND $\overline{X \cdot Y}$	OR $X+Y$	NOR $\overline{X+Y}$
0	0	0	1	0	1
0	1	0	1	1	0
1	0	0	1	1	0
1	1	1	0	1	0

NAND : $nand(x,y) = \overline{x \cdot y}$ $\longrightarrow$



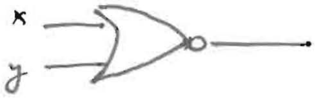
$\overline{x \cdot y}$

De Morgan $= \bar{x} + \bar{y}$ $\longrightarrow$

$\bar{x} + \bar{y}$

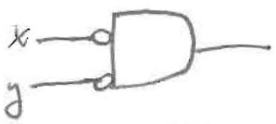
2 Alternate symbols for a nand gate.

NOR : $nor(x,y) = \overline{x + y}$ $\longrightarrow$



$\overline{x+y}$

De Morgan $= \bar{x} \cdot \bar{y}$ $\longrightarrow$



$\bar{x} \cdot \bar{y}$

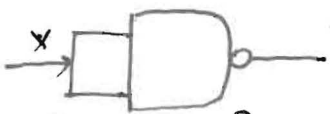
2 Alternate symbols for a nor gate

NAND & NOR gates are universal: it is possible to write ANY combinational network in terms of only NAND's or NOR's

NAND gates :

a) How to get a complement with only NAND gates ?

Boolean Algebra : $X \cdot X = X$ $\rightarrow$



$\overline{X.X} = \bar{X}$

b) How to make an OR out of NAND's ?

?

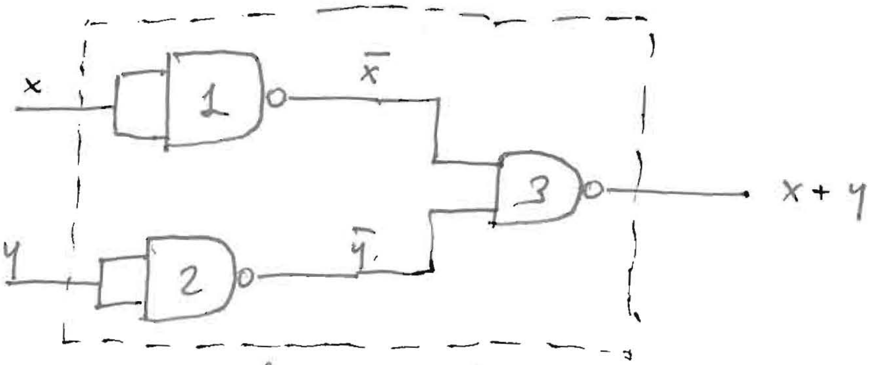
Boolean Algebra: De Morgan's: $\overline{x \cdot y} = \bar{x} + \bar{y}$

$$\overline{\bar{x} \cdot \bar{y}} = \bar{\bar{x}} + \bar{\bar{y}}$$

$$= x + y$$

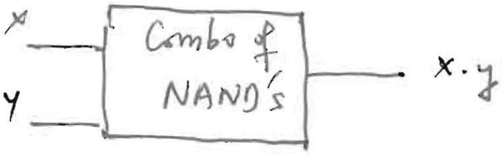
Find $\bar{x}$ & $\bar{y}$ then use a NAND

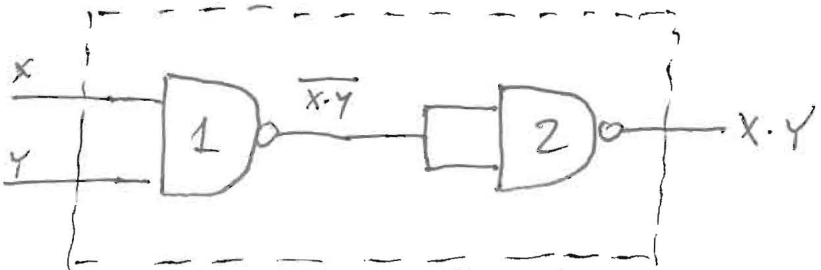
1 1 1



Equivalent OR using 3 NAND's

6) How to make an AND out of NAND's ?



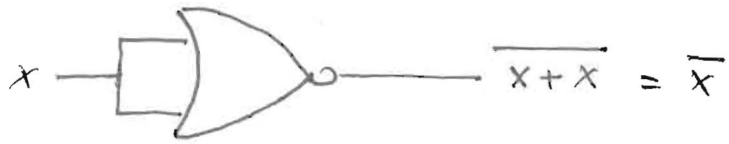


Equivalent AND using 2 NAND's

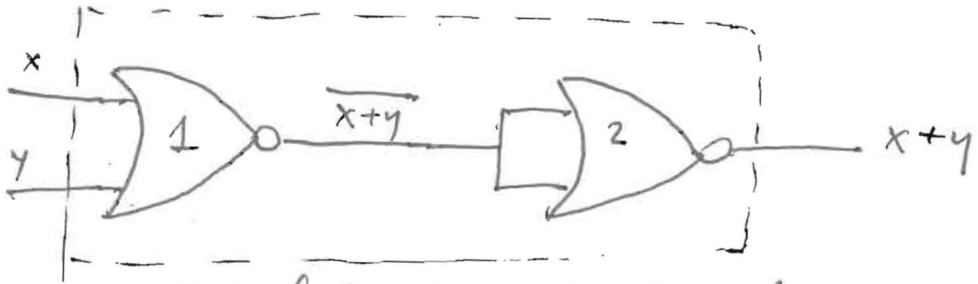
NOR Gates:

a) How to get a complement using NOR gates?

<u>Boolean Algebra</u>: $x + x = x$ (Absorption law)



b) OR using NOR's:



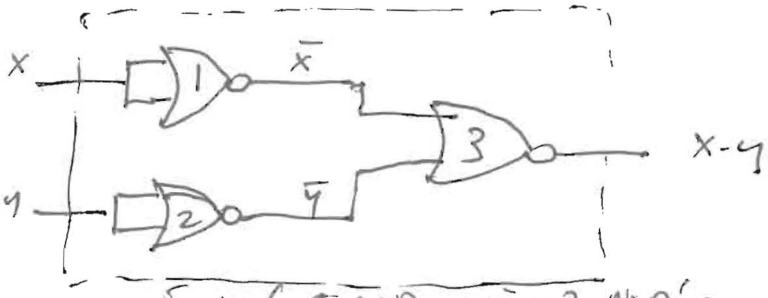
Equivalent OR using 2 NOR's

c) AND using NOR's:

$\hookrightarrow \quad \overline{x+y} = \bar{x} \cdot \bar{y}$

$\overline{\bar{x}+\bar{y}} = x \cdot y$:

Find $\bar{x}$ & $\bar{y}$ then use a NOR:



Equivalent AND using 3 NOR's

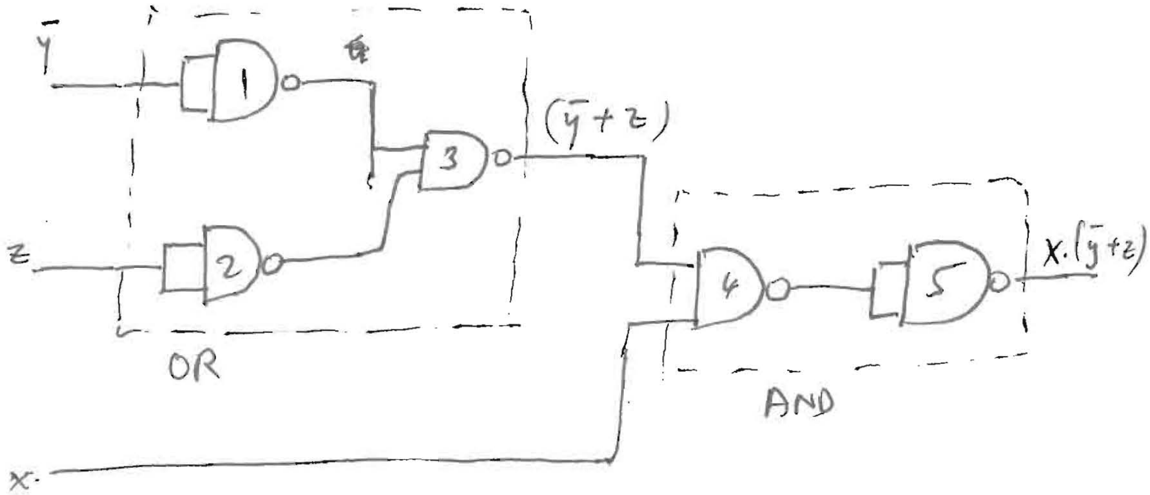
Create a Combinational Network using only NAND's:

Example: $f(x,y,z) = x \cdot (\bar{y} + z)$

$x \cdot$ → AND → 2 NAND's

$(\bar{y} + z)$ → OR → 3 NAND's

1st:



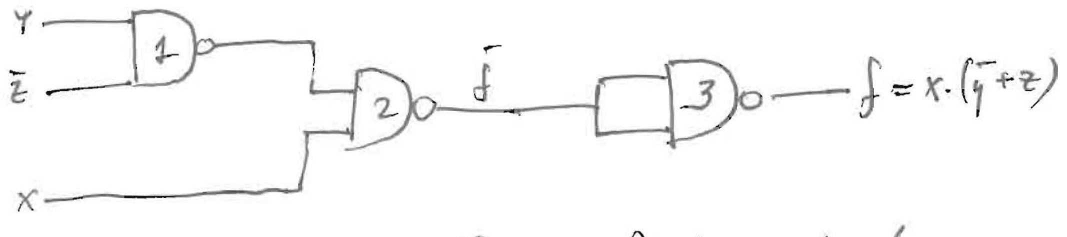
2nd: Since NAND includes a complement, let's start by complementing our expression:

$$\bar{f} = \overline{x \cdot (\bar{y}+z)} = \text{NAND}(x,\ \bar{y}+z)$$

Note: $\bar{y} + z = \overline{y \cdot \bar{z}}$ (De Morgan's law)

$$= \text{NAND}(y, \bar{z})$$

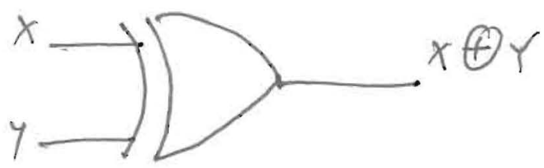
One more NAND will be used to complement $\bar{f}$



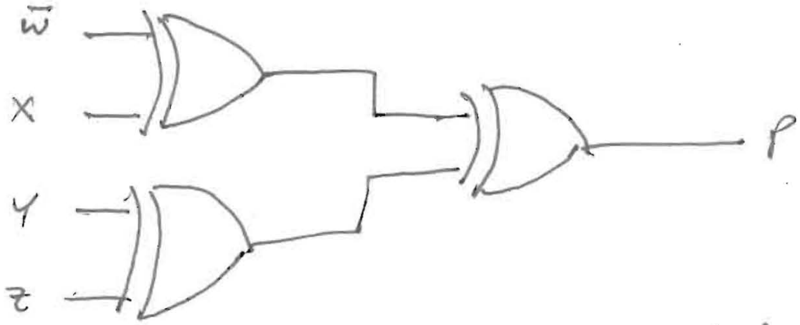
* Complement your expression & use De Morgan's law when possible.

Exclusive - OR

X	Y	XOR $X \oplus Y$
0	0	0
0	1	1
1	0	1
1	1	0 ←



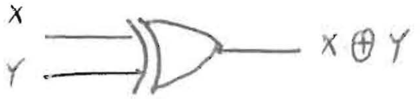
↳ Odd-Parity can be generated with only 3 XOR's !



(instead of 8 AND's + 1 OR before)

Properties of X-OR see Table 3.16 (p. 112)

Exclusive - OR (Cont.)
(XOR)



X	Y	X ⊕ Y
0	0	0
0	1	1
1	0	1
1	1	0

$$\boxed{X \oplus Y = \bar{X}Y + X\bar{Y}}$$

X	Y	$\bar{X}Y + X\bar{Y}$
0	0	0
0	1	1
1	0	1
1	1	0

Properties (Table 3.16, pg. 112)

(i) a) $\boxed{X \oplus Y = \bar{X}Y + X\bar{Y} \overset{?}{=} (X+Y).(\bar{X}+\bar{Y})}$

Boolean algebra properties

Distributive property:

$$(X+Y).(\bar{X}+\bar{Y}) = (X+Y).\bar{X} + (X+Y).\bar{Y} = Y.\bar{X} + X.\bar{Y}$$
$$= X.\bar{Y} + Y.\bar{X} \checkmark$$

($X.\bar{X} = Y.\bar{Y} = 0 \rightarrow$ Table 3.1 pg 70

b) $\overline{X \oplus Y} = \overline{\bar{X}.Y + X.\bar{Y}} \overset{?}{=} \overline{\bar{X}.Y} \cdot \overline{X.\bar{Y}} = (X+\bar{Y}).(\bar{X}+Y)$

De Morgan's Laws:

$$\begin{cases} \overline{A+B} = \bar{A}.\bar{B} \\ \overline{A.B} = \bar{A}+\bar{B} \end{cases}$$

Distributive Property:

$$(X+\bar{Y}).(\bar{X}+Y) = (X+\bar{Y}).\bar{X} + (X+\bar{Y}).Y = \bar{Y}.\bar{X} + X.Y$$
$$= \bar{X}.\bar{Y} + X.Y$$

$$\boxed{\overline{X \oplus Y} = \bar{X}.\bar{Y} + X.Y = (X+\bar{Y}).(\bar{X}+Y)}$$

$X \oplus Y$: commutative, associative; distributive; neutral element is 0 ($X \oplus 0 = X$)

iv) a) $\bar{X} \oplus \bar{Y} \overset{?}{=} X \oplus Y$

use i)

$$\bar{X} \oplus \bar{Y} = X \cdot \bar{Y} + \bar{X} \cdot Y = \bar{X} \cdot Y + X \cdot \bar{Y} = X \oplus Y \checkmark$$

b) $\bar{X} \oplus Y \overset{?}{=} X \oplus \bar{Y} \overset{\checkmark}{=} \overline{X \oplus Y}$

use i)

$$\bar{X} \oplus Y = X \cdot Y + \bar{X} \cdot \bar{Y} = \overline{X \oplus Y}$$

$$X \oplus \bar{Y} = \bar{X} \cdot \bar{Y} + X \cdot Y = \overline{X \oplus Y}$$

} equal.

<u>Use Exclusive-OR (XOR) to simplify the odd-parity bit formula:</u>

$$p(w,x,y,z) = \bar{w}\bar{x}\bar{y}\bar{z} + \bar{w}\bar{x}yz + \bar{w}x\bar{y}z + \bar{w}xy\bar{z} + w\bar{x}\bar{y}z + w\bar{x}y\bar{z} + wx\bar{y}\bar{z} + wxyz$$

$$= \bar{w}\bar{x}\,\underbrace{(\bar{y}\cdot\bar{z} + yz)}_{\overline{y \oplus z}} + \bar{w}x \cdot \underbrace{(\bar{y}z + y\bar{z})}_{y \oplus z} + w\bar{x}\,\underbrace{(\bar{y}z + y\bar{z})}_{y \oplus z} + wx\,\underbrace{(\bar{y}\bar{z} + yz)}_{\overline{y \oplus z}}$$

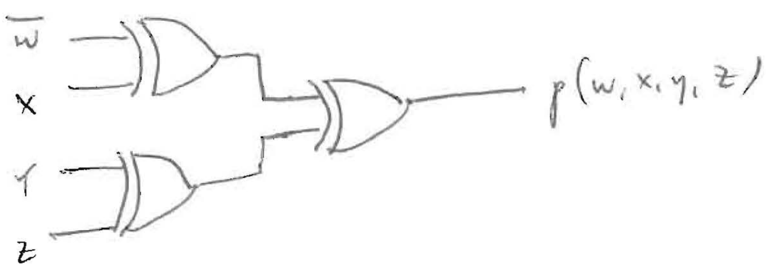
$$= \underbrace{(\bar{w}\bar{x} + wx)}_{\overline{w \oplus x}} \cdot \overline{(y \oplus z)} + \underbrace{(\bar{w}x + w\bar{x})}_{(w \oplus x)} \cdot (y \oplus z)$$

$$= \overline{(w \oplus x)} \cdot \overline{(y \oplus z)} + (w \oplus x) \cdot (y \oplus z)$$

$$= \overline{(w \oplus x) \oplus (y \oplus z)} \overset{iv\,b)}{=} \overline{(w \oplus x)} \oplus (y \oplus z)$$

$$\overset{iv\,b)}{=} (\bar{w} \oplus x) \oplus (y \oplus z)$$

(3 XOR's → odd-parity bit)



$\left\{ \begin{array}{l} 3 \text{ XOR's} \rightarrow 3 \text{ 7486 IC's} \quad \text{versus} \\ 8 \text{ four-input AND's} \ \& \ 1 \text{ eight-input OR} \end{array} \right\}$ 3 gates instead of 9 for a same result.

$= \underbrace{\bar{X} \cdot Y \cdot Z}$ + $X \cdot Y$

Ch 4 Simplification of Boolean Expressions

Implies:

Truth tables for two functions f_1 & f_2:

X	Y	Z	f_1	f_2
0	0	0	0	0
0	0	1	0	1
0	1	0	0	0
0	1	1	1	1
1	0	0	0	0
1	0	1	0	0
1	1	0	1	1
1	1	1	1	1

$$\boxed{f_i \text{ implies } f_j \text{ if } \left\{ \begin{array}{l} \text{No entries for which} \\ f_i = 1 \ \& \ f_j = 0 \end{array} \right.}$$

In the example above $(XYZ) = (001)$: $f_2 = 1$ & $f_1 = 0$
so f_2 does not imply f_1.
However when $f_1 = 1$, f_2 is never 0 $\Rightarrow$ $\underline{f_1 \text{ implies } f_2}$

Minterm canonical for $f_1(X,Y,Z) = \bar{X}YZ + \underbrace{XY\bar{Z} + XYZ}_{XY\cdot\underbrace{(\bar{Z}+Z)}_{1}}$

$$= \underbrace{\bar{X}\cdot Y\cdot Z}_{\text{implementing}} + X\cdot Y$$

implementing XYZ 0 1 1, f_1 1 $\rightarrow$ Also in a combination of 2 literals: YZ gives same result.

$$f_1(x,y,z) = Y\cdot Z + X\cdot Y$$

Min-term Canonical for $f_2(x,y,z) = \underbrace{\bar{x}\bar{y}z}_{\text{drop } \bar{y}} + \bar{x}yz + \underbrace{xy\bar{z}}_{\text{drop } \bar{z}} + xyz$

$$= \bar{x}\cdot z + xy + \underbrace{(\bar{x}+x)}_{1}\cdot yz$$

$$= \bar{x}\cdot z + x\cdot y + y\cdot z.$$

Since f_1 implies f_2 $\Rightarrow$ $\underbrace{x\cdot y + y\cdot z}_{f_1}$ implies $\underbrace{x\cdot y + y\cdot z + \bar{x}\cdot z}_{f_2}$

Ch 4 Simplification of Boolean Expressions (Cont.)

Implicants & Implicates:

Product term (in a minterm canonical expression of a function) is an implicant of the function (since it implies the function)

x	y	z	f(x,y,z)
0	0	0	0
0	0	1	1 ←
0	1	0	0
0	1	1	1 ←
1	0	0	1 ←
1	0	1	0
1	1	0	0
1	1	1	0

$f(x,y,z) = \underbrace{\bar{x}\bar{y}z}_{\text{product term}} + \bar{x}yz + x\bar{y}\bar{z}$

product term → gives a 1 as value for function since when a product term is 1 the function can't be 0 → product term implies function or is an implicant of the function.

Sum term (5 in a maxterm canonical for this example)

x	y	z	f(x,y,z)
0	0	0	0 ←
0	0	1	1
0	1	0	0 ←
0	1	1	1
1	0	0	1
1	0	1	0 ←
1	1	0	0 ←
1	1	1	0 ←

$f(x,y,z) = (x+y+z)(x+\bar{y}+z)(\bar{x}+y+\bar{z})(\bar{x}+\bar{y}+z)(\bar{x}+\bar{y}+\bar{z})$

Here the function implies the sum term → sum term is an implicate of the function.

Irredundant Disjunctive & Conjunctive Normal Formulas

Can be achieved by using prime implicantes or implicates.

A prime implicant is a product term that if any further literal is removed, it no longer implies the function.

Example:

x	y	z	f(x,y,z)
0	0	0	1 ←
0	0	1	1 ←
0	1	0	1 ←
0	1	1	1 ←
1	0	0	0
1	0	1	1 ←
1	1	0	0
1	1	1	0

5 product terms:

$f(xyz) = \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z + \bar{x}y\bar{z} + \bar{x}yz + x\bar{y}z$

all five are implicants of f.

Look at $\bar{y}z$: is it an implicant of this function? or whenever $\bar{y}z = 1$, can the function be 0? No → so $\bar{y}z$ is an implicant of the function

$y=0 \quad z=1$

→ $x\bar{y}z$ is not a prime implicant of f. Since if we drop literal x, $\bar{y}z$ is still an implicant of f.

An irredundant disjunctive normal formula (IDNF) is achieved when consists of product terms that are prime implicants and if a term is dropped the function is not the same.

A prime implicate: is a sum term that if any further literal is removed it is no longer implied by the function (or no longer an implicate of the function).

Karnaugh Maps:

Different representation of the Truth Table or the Boolean function.

x	y	f(x,y)
0	0	0
0	1	1
1	0	1
1	1	0

$\leftrightarrow$

x \ y	0	1
0	f(0,0)	f(0,1)
1	f(1,0)	f(1,1)

$\leftrightarrow$

x \ y	0	1
0	0	1
1	1	0

Example of a Karnaugh map for a 2-variable Boolean function.

x	y	z	f(x,y,z)
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

$\longleftrightarrow$

x \ yz	00	01	11	10
0	1	0	0	1
1	1	1	0	0

Karnaugh Map for a 3-variable Boolean function

Note: 11 appears before 10

so there is always a flip of one value between adjacent cells, including the last & the first cells horizontally (also vertically)

w	x	y	z	f(w, x, y, z)
0	0	0	0	1
0	0	0	1	0
0	0	1	0	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	0
1	0	0	0	0
1	0	0	1	1
1	0	1	0	1
1	0	1	1	0
1	1	0	0	1
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1

↓

Odd-parity bit

same z (01, 11)

same y (11, 10)

wx \ yz	00	01	11	10
00	1	0	1	0
01 (same x)	0	1	0	1
11 (same x, same w)	1	0	1	0
10 (same w)	0	1	0	1

Karnaugh map for a 4-variable Boolean function.

Karnaugh Maps & Canonical Formulas:

$$f(w,x,y,z) = \bar w\bar x\bar y\bar z + \bar w\bar x\bar y z + \bar w\bar x y\bar z + \bar w x\bar y\bar z + \bar w x\bar y z + w\bar x\bar y\bar z + w\bar x y\bar z$$

(Minterm canonical or SOP formula) → each product gives a one or an implicant of the function.

wx \ yz	00	01	11	10
00	1 (0)	1 (1)	0 (3)	1 (2)
01	1 (4)	1 (5)	0 (7)	0 (6)
11	0 (12)	0 (13)	0 (15)	0 (14)
10	1 (8)	0 (9)	0 (11)	1 (10)

Canonical Formula → Karnaugh Map.

↓

$\Sigma m(0,1,2,4,5,8,10)$

Karnaugh Map → Canonical Formula → Maxterm Canonical or POS: focus on the zeroes: product of nine sums:

$$f(w,x,y,z) = (w+x+\bar y+\bar z)(w+\bar x+\bar y+\bar z)(w+\bar x+\bar y+z)(\bar w+\bar x+y+z)(\bar w+\bar x+y+\bar z)(\bar w+\bar x+\bar y+\bar z)$$
$$(\bar w+\bar x+\bar y+z)(\bar w+x+y+\bar z)(\bar w+x+\bar y+\bar z)$$

↓

$\Pi M(3,7,6,9,11,12,13,14,15)$

3.33 : Use Truth tables or use properties of Boolean operations.

$$f_d(x_1 x_2, \dots x_n) = \bar{f}(\bar{x}_1, \bar{x}_2, \dots \bar{x}_n)$$ { complement of function is assigned to the complement of the variables)

X-OR

x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

dual function →

x	y	dual of $(x \oplus y)$
1	1	1
1	0	0
0	1	0
0	0	1

x	y	OR $x + y$	NOR $\overline{x + y}$	X-NOR
0	0	0	1	1
0	1	1	0	0
1	0	1	0	0
1	1	1	0	1

Same.