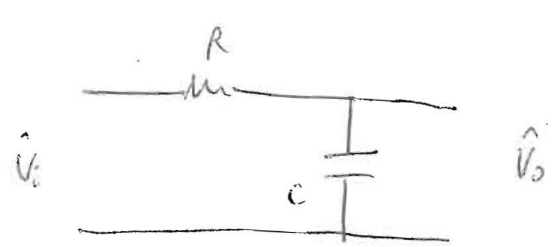
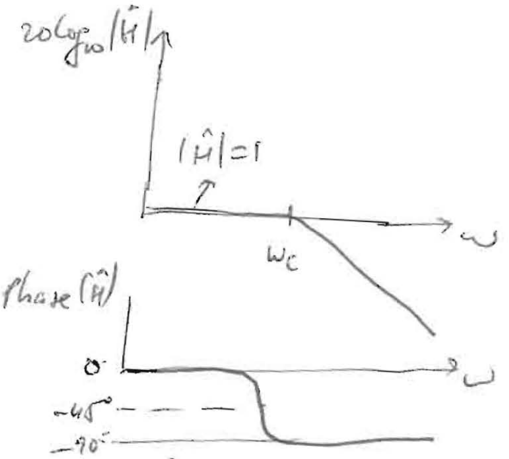
Passive Low-Pass



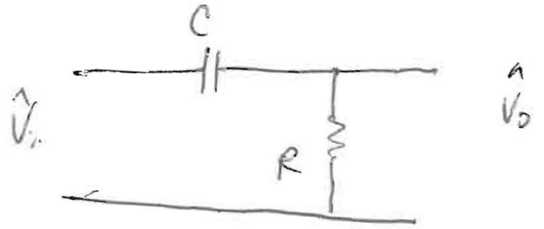
$$\hat{H}(j\omega) = \frac{\hat{V}_o}{\hat{V}_i}$$: passing $\hat{V}_i$ to $\hat{V}_o$ @ low frequencies.

$$= \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC} \rightarrow \boxed{\omega_c = \frac{1}{RC}}$$

(see PSPICE example for numerical confirmation)

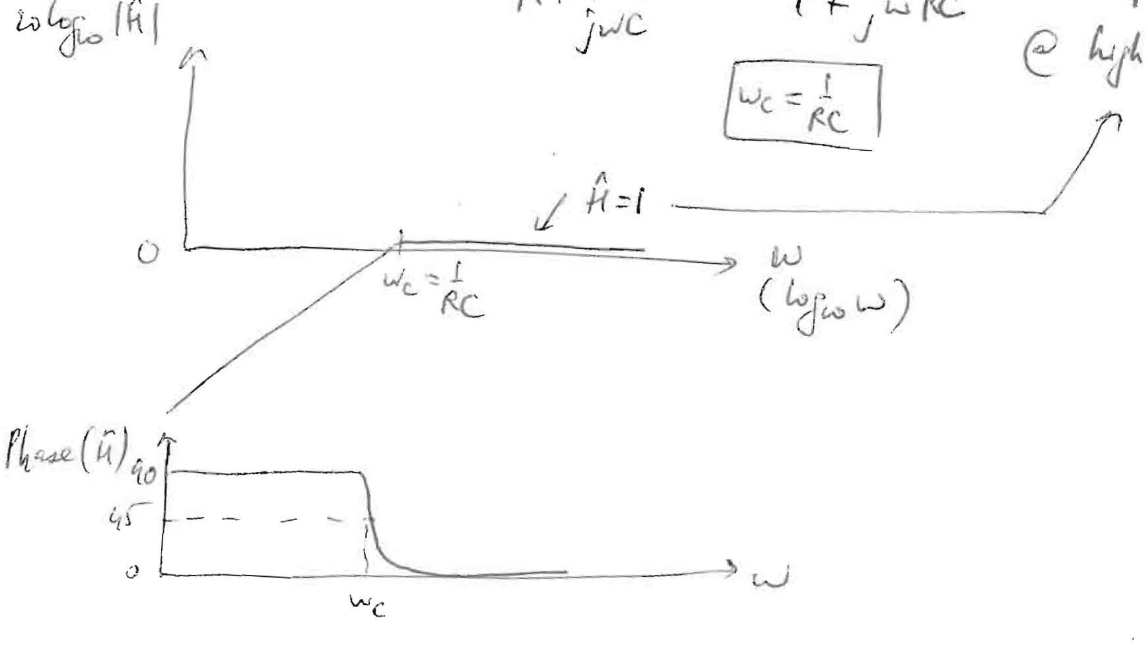


Passive High-Pass



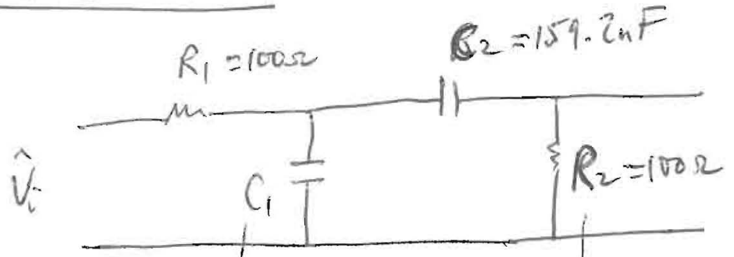
$$\hat{H}(j\omega) = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega RC}{1 + j\omega RC}$$ is passing $\hat{V}_i$ to $\hat{V}_o$ @ high frequencies.

$$\boxed{\omega_c = \frac{1}{RC}}$$



Passive Band-Pass :

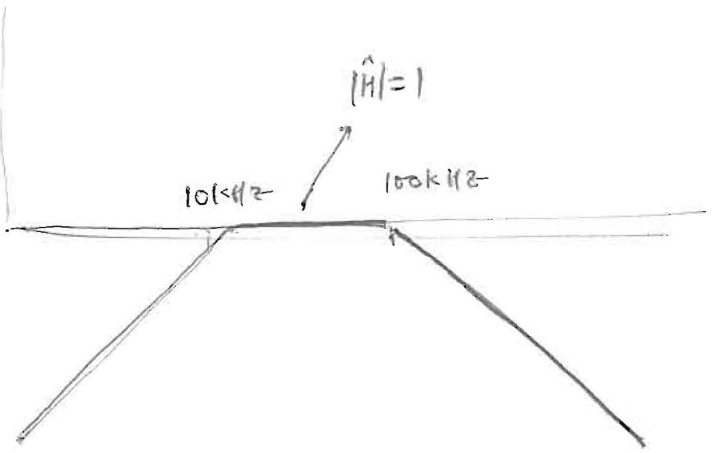
1)



$C_1 \rightarrow 15.92nF$

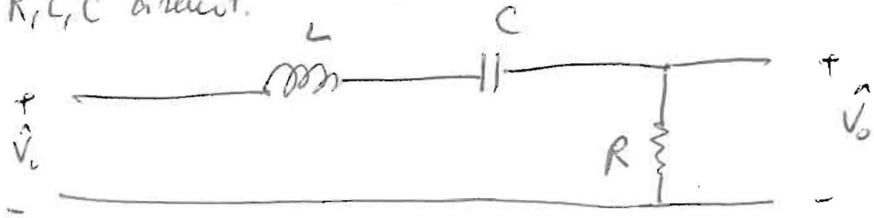
$$w_{c_{HP}} = \frac{w_{CLH}}{10} = 10K\ Hz.$$

$$w_{c_{LP}} = \frac{1}{RC} = \frac{1}{100 \times 15.92 \times 10^{-9}} = 100K\ Hz.$$



This circuit is passing $\hat{V}_i \rightarrow \hat{V}_o$ when $10K < \omega < 100K$

2) Using R, L, C circuit.



$$\hat{H} = \frac{R}{R + j\omega L + \frac{1}{j\omega C}} = \frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

$\boxed{\text{Resonant frequency:}}$ is a frequency @ which reactive impedance cancels to zero: $\omega L = \frac{1}{\omega C} \rightarrow \omega_0 \equiv \frac{1}{\sqrt{LC}}$. Also in this example @ resonant freq $\rightarrow$ $\hat{V}_o = \hat{V}_i$.

Quality factor: $Q \equiv \omega_0 \frac{L}{R} = \frac{1}{\sqrt{LC}} \frac{L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}$

$$\hat{H} = \frac{\cancel{R}}{\cancel{R}\left[1 + j\left(\omega\frac{L}{R} - \frac{1}{\omega RC}\right)\right]}$$

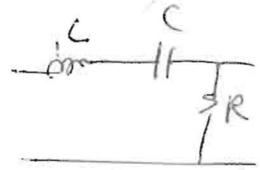
$$\omega\frac{L}{R} = \frac{1}{\omega RC} \rightarrow \omega = \omega_0$$

$$\omega_0\frac{L}{R} = \frac{1}{\omega_0 RC} = \boxed{\frac{\omega_c}{\omega_0} \equiv Q}$$

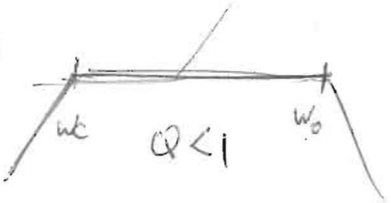
$Q > 1 \rightarrow \omega_c > \omega_0$

$Q < 1 \rightarrow \omega_c < \omega_0$

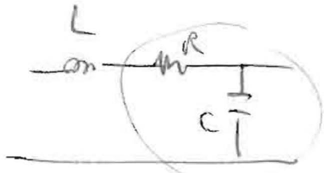
$\boxed{RC = \frac{1}{\omega_c}}$ → when there would be no L



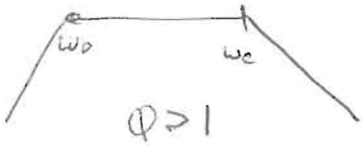
HP



→ $\omega_c = 0.01 \quad \omega_0 = 0.1$

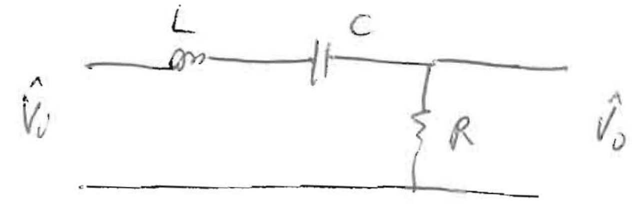


LP



$\boxed{\text{Meaning of } Q \text{ ?}}$

Bandpass:



$$\hat{H}(j\omega) = \frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)} = \frac{RC\omega}{RC\omega + j\left(\omega^2 LC - 1\right)}$$

Magnitude: $$|\hat{H}| = \frac{RC\omega}{\sqrt{(RC\omega)^2 + (\omega^2 LC - 1)^2}} \longrightarrow \omega^2 LC = \frac{\omega^2}{\frac{1}{LC}}$$

Phase: $$\angle\hat{H} = -\tan^{-1}\left(\frac{\omega^2 LC - 1}{RC\omega}\right)$$

Asymptotic analysis:

Small ω:

$\omega^2 \ll \frac{1}{LC}$ or $\omega^2 LC \ll 1 \longrightarrow$

$$|\hat{H}| = \frac{RC\omega}{\sqrt{(RC\omega)^2 + 1}} \xrightarrow{\omega \to 0} RC\omega \simeq 0$$

$$\angle\hat{H} = + \tan^{-1} \frac{+1}{RC\omega} \xrightarrow{\omega \to 0} \tan^{-1}\infty = 90^\circ$$

Large ω:

$\omega^2 \gg \frac{1}{LC}$ or $\omega^2 LC \gg 1 \longrightarrow$

$$|\hat{H}| = \frac{RC\omega}{\sqrt{(RC\omega)^2 + \omega^4 L^2 C^2}} \simeq \frac{RC\cancel{\omega}}{\omega^{\cancel{2}} L \cancel{C}} \xrightarrow{\omega \to \infty} 0$$

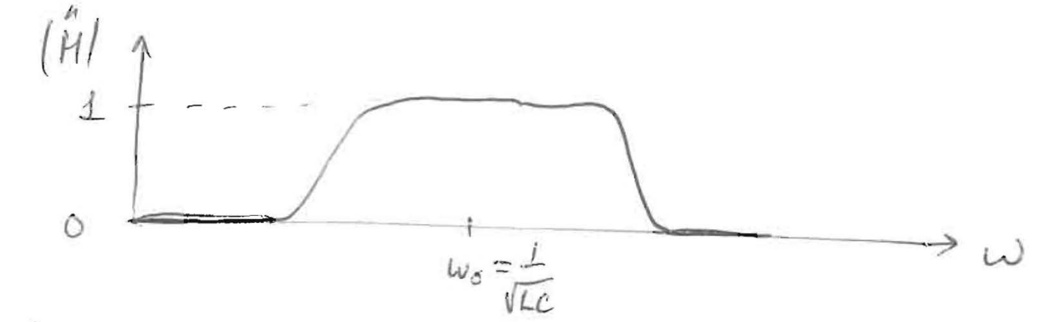
$$\angle\hat{H} = -\tan^{-1} \frac{\omega^{\cancel{2}} L\cancel{C}}{R\cancel{C}\cancel{\omega}} = -\tan^{-1}\frac{\omega L}{R} \xrightarrow{\omega \to \infty} -90^\circ$$

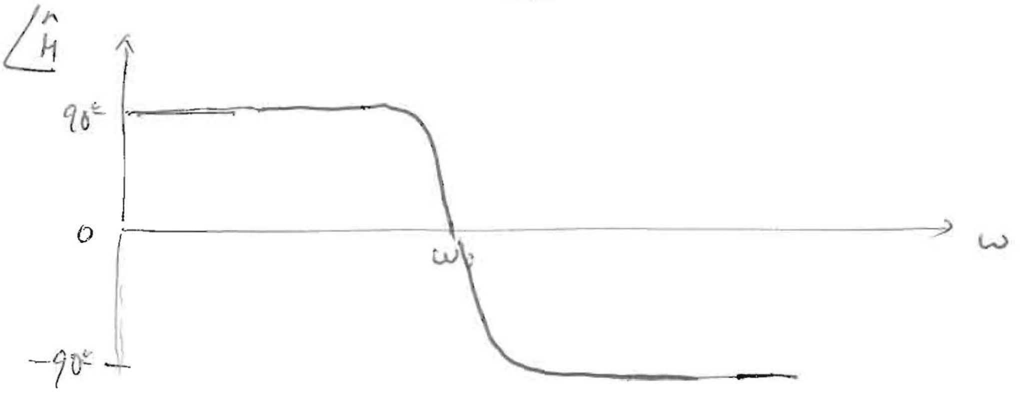
Resonant frequency:

$\omega^2 = \frac{1}{LC}$ $(\omega = \omega_0)$ $\longrightarrow$

$$|\hat{H}| = \frac{RC\omega}{RC\omega} = 1$$

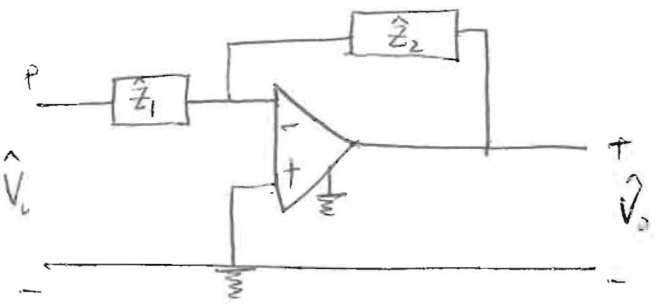
$$\angle\hat{H} = 0$$



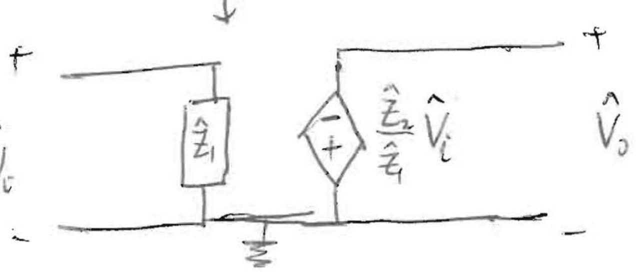


Op-Amp review:

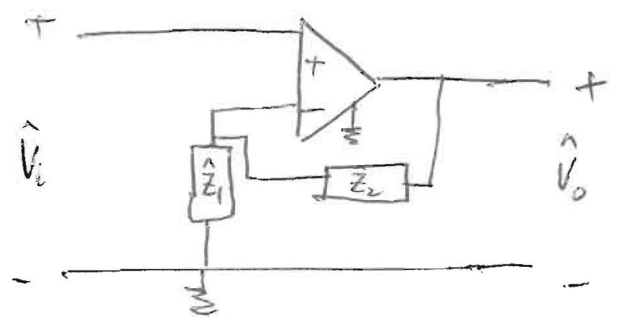
Inverting



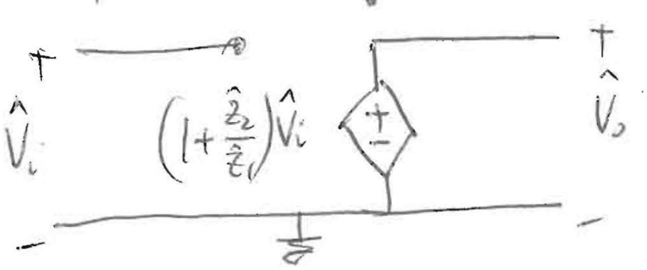
Equivalent circuit ↓ ‖



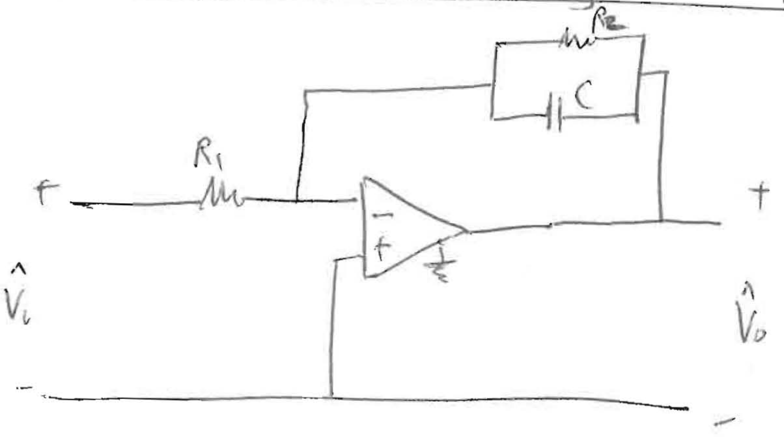
Non-inverting



Equivalent circuit ↓ ‖



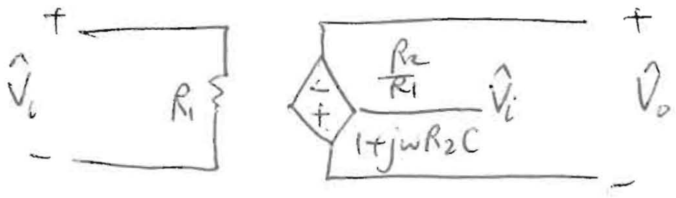
Active-Low-Pass Filter using Op-Amp :



$$\hat{Z}_1 \equiv R_1$$

$$\hat{Z}_2 = R_2 \parallel C = \frac{R_2 \frac{1}{j\omega C}}{R_2 + \frac{1}{j\omega C}} = \frac{R_2}{1 + j\omega R_2 C} \quad \Bigg\} \quad \frac{\hat{Z}_2}{\hat{Z}_1} = \frac{\frac{R_2}{R_1}}{1 + j\omega R_2 C}$$

Using the equivalent circuit for a ~~non~~inverting Op-Amp:



$$\hat{H} = \frac{\hat{V}_o}{\hat{V}_i} = \frac{-\frac{\frac{R_2}{R_1}}{1+j\omega R_2 C}\cancel{\hat{V}_i}}{\cancel{\hat{V}_i}}$$

$$\hat{H} = (-)\,\frac{\frac{R_2}{R_1}}{1+j\omega R_2 C}$$

inverting Op-Amp

Low-Pass Filter.

New compared to a simple RC Low-pass is $\frac{R_2}{R_1}$ which could give a gain in addition to passing $\hat{V}_i$ to $\hat{V}_o$ (when $R_2 > R_1$)

Connection b/w gain & bandwidth : since R_2 is in the

$w_c = \frac{1}{R_2 C}$: $\uparrow R_2 \rightarrow$ high gain but also lower w_c or shorter bandwidth!

Laplace Transform:

$f(t)$ $\longrightarrow$ integral in time: $\boxed{\hat{F}(s) \equiv \int_0^\infty dt\, f(t)\, e^{-st}}$

Time-domain (function of time)

Frequency-domain (function of frequency)

$s \equiv \alpha + j\omega \;;\; \alpha > 0$

$\omega \downarrow$ ang. frequency

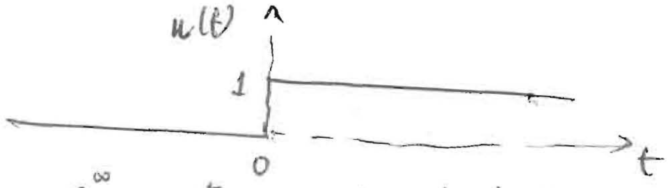
(when $\alpha \to 0 \to$ Laplace T. is related to Fourier T.)

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} ds\, \hat{F}(s)\, e^{+st} \longleftarrow \hat{F}(s)$$

integral in the complex plane | instead $\to$ Partial Fraction Expansion & Properties of Laplace Transform.

Simple functions:

1) Unit Step Function $u(t)$



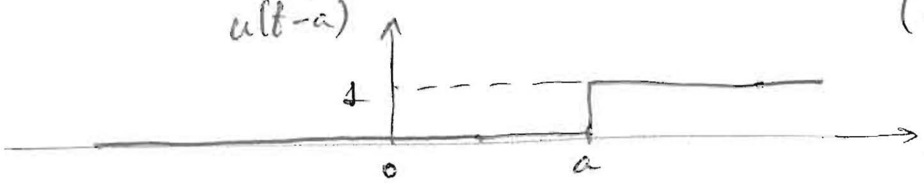
$$\hookrightarrow L[u(t)] = \int_0^\infty dt\, u(t)\, e^{-st} = \int_0^\infty dt\, e^{-st} = \left[\frac{e^{-st}}{-s}\right]_{t=0}^{t=\infty} =$$

$$= \left[\frac{e^{-\alpha\infty} e^{-j\omega\infty} - e^{-(\alpha+j\omega)\cdot 0}}{-(\alpha + j\omega)}\right]$$

$\boxed{s = \alpha + j\omega, \; \alpha > 0}$

$$\boxed{L[u(t)] = \frac{1}{s}}$$

2) Time-shifted unit step function: $u(t-a)$



(It's 0 b/w $-\infty$ & a, and 1 b/w a & $+\infty$)

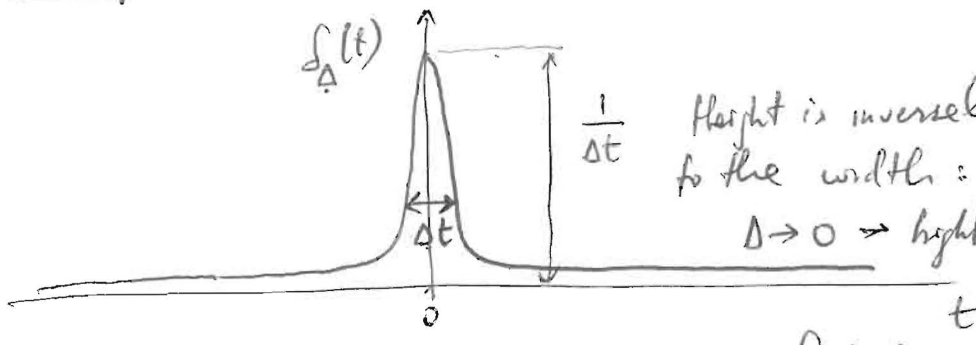
$$L[u(t-a)] = \int_0^\infty dt\, u(t-a)\, e^{-st} = \underbrace{\int_0^a dt\, 0 \cdot e^{-st}}_{0} + \int_a^\infty dt\, e^{-st}$$

$$= \left[\frac{e^{-st}}{-s} \right]_{t=a}^{t=\infty} = \frac{0 - e^{-sa}}{-s} = \frac{e^{-sa}}{s}$$

$$\boxed{L[u(t-a)] = \frac{e^{-sa}}{s} = \underbrace{L[u(t)]}_{\frac{1}{s}} \cdot e^{-sa}}$$

Generalization: when f is shifted by a in time, $L[f(t)]$ gets multiplied by e^{-sa}.

3) Delta function $\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$



Height is inversely proportional to the width: $\Delta \to 0 \Rightarrow$ height $\to \infty$

- → The real $\delta(t)$ is very thin & very tall!
- → Center of real $\delta(t)$ is same as center of $\delta_\Delta(t)$ → in this case center is 0
- → If the center is @ a: $\delta(t-a)$ (@ $-a$ → $\delta(t+a)$)

Important to remember about the delta function: when it is showing in an integral it will pick one value of the integrand to be the integral (since it is $\underbrace{\text{so thin}}_{\text{will pick only one value.}}$ & $\underbrace{\text{so tall!}}_{\text{dominate over anything}}$)

The value it picks is the value of the integrand @ $x=a$

so
$$\boxed{\int_0^\infty dt\ \underbrace{\delta(t-a)}_{\substack{\text{center}\\ @\,a}}\ \underbrace{f(t)}_{\text{integrand}} = f(a)}$$

$$L[\delta(t)] = \int_0^\infty dt\ \underbrace{\delta(t)}_{\substack{\text{center}\\ @\,0}}\ \underbrace{e^{-st}}_{\text{integrand}} = e^{-s\cdot 0} = 1$$

$$L[\delta(t-a)] = \int_0^\infty dt\ \underbrace{\delta(t-a)}_{\substack{\text{center}\\ @\,a}}\cdot \underbrace{e^{-st}}_{\text{integrand}} = e^{-sa}$$

(since we are shifting the signal $f(t)=\delta(t)$ by a its Laplace Transform which is 1 gets multiplied by e^{-sa}.)

4) $f(t) = t$

$$L[t] = \int_0^\infty dt\ t\ e^{-st} = -\frac{d}{ds}\Bigg[\underbrace{\int_0^\infty dt\ e^{-st}}_{\frac{1}{s}}\Bigg] = \frac{1}{s^2}$$

alternative trick to the integration by parts

5) $f(t) = \cos\omega t$

$$\cos(\omega t) = \frac{e^{-j\omega t} + e^{j\omega t}}{2}$$

$$\left(\text{Recall: } \begin{array}{l} e^{j\theta} = \cos\theta + j\sin\theta \\ \text{or} \quad e^{-j\theta} = \cos\theta - j\sin\theta \\ \hline e^{j\theta} + e^{-j\theta} = 2\cos\theta \end{array} \right)$$

$$L[\cos(\omega t)] = \frac{1}{2}\int_0^\infty dt \left(e^{-j\omega t} + e^{j\omega t}\right) e^{-st} = \frac{1}{2}\int_0^\infty dt \left[e^{-(s+j\omega)t} + e^{-(s-j\omega)t}\right]$$

$$= \frac{1}{2}\left[\frac{e^{-(s+j\omega)t}}{-(s+j\omega)}\right]_{t=0}^{t=\infty} + \frac{1}{2}\left[\frac{e^{-(s-j\omega)t}}{-(s-j\omega)}\right]_{t=0}^{t=\infty}$$

$s = \alpha + j\omega$
$\alpha > 0$

$$= \frac{1}{2}\left[\frac{1}{s+j\omega} + \frac{1}{s-j\omega}\right]$$

$$= \frac{1}{2}\,\frac{s - \cancel{j\omega} + s + \cancel{j\omega}}{(s+j\omega)(s-j\omega)} = \boxed{\frac{s}{s^2+\omega^2}}$$

6) $f(t) = \sin\omega t$

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$L[\sin(\omega t)] = \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] = \frac{1}{2j}\,\frac{\cancel{s} + j\omega - \cancel{s} + j\omega}{s^2+\omega^2}$$

$$= \boxed{\frac{\omega}{s^2+\omega^2}}$$