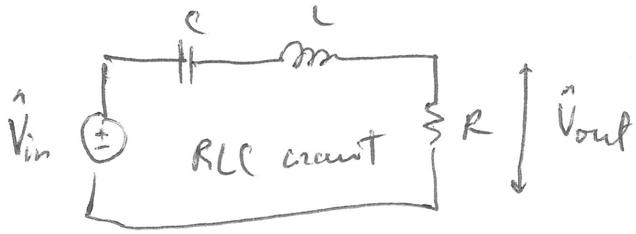
Variable Frequency Network (Cont.)

Classifying variable-frequency circuits based on their transfer function (denominator).

Transfer function is a rational function in general : denominator dictates the behavior of a rational function. (If a numerator is 0 it is still O.K, but if a denominator is 0 → the function blows up !)



$$\frac{\hat{V}_{out}}{\hat{V}_{in}} = \frac{R}{R + j\omega L + \frac{1}{j\omega C}}$$

It's not a linear response in frequency

Standard notations (Laplace Transforms are frequently used in AC circuit analysis) : $\boxed{j\omega \rightarrow s}$

$$\frac{\hat{V}_{out}}{\hat{V}_{in}} = \frac{R}{\left(R + sL + \frac{1}{sC}\right)} \frac{\times \frac{s}{L}}{\times \frac{s}{L}} = \frac{\frac{R}{L}s}{\frac{R}{L}s + s^2 + \frac{1}{LC}}$$

polynomial formats

Coefficient of highest order term in the polynomial has to be 1 if possible.

Denominator: a quadratic polynomial in s :

$$s^2 + \frac{R}{L}s + \frac{1}{LC}$$

Classifying circuit or the transfer function → look at different solutions for this quadratic polynomial :

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$x^2 + bx + c = 0 \rightarrow \begin{cases} x_1 = \dfrac{-b + \sqrt{b^2 - 4c}}{2} \\ x_2 = \dfrac{-b - \sqrt{b^2 - 4c}}{2} \end{cases}$$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \rightarrow \begin{cases} s_1 = \dfrac{-\frac{R}{L} + \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2} = -\dfrac{R}{2L} + \dfrac{1}{2L}\sqrt{\underbrace{R^2 - \frac{4L}{C}}} \\ s_2 = -\dfrac{R}{2L} - \dfrac{1}{2L}\sqrt{\underbrace{R^2 - \frac{4L}{C}}_{\text{Discriminant}}} \end{cases}$$

Argument of the square root is the "discriminant" → it distinguishes

3 cases:

Case 1) $R^2 - \frac{4L}{C} = 0 \rightarrow$ one double root $s_1 = s_2 = -\frac{R}{2L}$

$$\frac{\hat{V}_{out}}{\hat{V}_{in}} = \frac{\frac{R}{L}s}{(s-s_1)(s-s_2)} = \frac{\frac{R}{L}s}{\left(s + \frac{R}{2L}\right)^{2}}$$

This transfer function has a pole of order 2 @ $s_1 = -\frac{R}{2L}$

Case 2) $R^2 - \frac{4L}{C} > 0 \rightarrow$ two different roots:

$$s_{1,2} = -\frac{R}{2L} \pm \frac{1}{2L}\sqrt{\left|R^2 - \frac{4L}{C}\right|}$$

$$\frac{\hat{V}_{out}}{\hat{V}_{in}} = \frac{\frac{R}{L}s}{\left(s + \frac{R}{2L} - \frac{1}{2L}\sqrt{\left|R^2 - \frac{4L}{C}\right|}\right)\left(s + \frac{R}{2L} + \frac{1}{2L}\sqrt{\left|R^2 + \frac{4L}{C}\right|}\right)}$$

↳ This transfer function has two poles of order 1 or two simple poles.

Case 3) $R^2 - \frac{4L}{C} < 0 \rightarrow$ two complex conjugate roots:

$$S_{1,2} = -\frac{R}{2L} \pm \frac{j}{2L}\sqrt{\left|\frac{4L}{C} - R^2\right|} \qquad (\text{Recall: } j \equiv \sqrt{-1})$$

$$\frac{\hat{V}_{out}}{\hat{V}_{in}} = \frac{\frac{R}{L}s}{\left(s + \frac{R}{2L} - \frac{j}{2L}\sqrt{\left|\frac{4L}{C} - R^2\right|}\right)\left(s - \frac{R}{2L} + \frac{j}{2L}\sqrt{\left|\frac{4L}{C} - R^2\right|}\right)}$$

$\hookrightarrow$ This transfer function has two complex conjugate poles.

In general: $\hat{H}(s) = \frac{\hat{V}_{out}(s)}{\hat{V}_{in}(s)}$ or $\hat{H}(j\omega) = \frac{\hat{V}_{out}(j\omega)}{\hat{V}_{in}(j\omega)}$

$$(1) \quad \hat{H}(j\omega) = \frac{K_0 (j\omega)^{\pm N} (1 + j\omega\tau_1)\left[1 + 2\xi_3 j\omega\tau_3 + (j\omega\tau_3)^2\right]\ldots}{(1 + j\omega\tau_a)\left[1 + 2\xi_b j\omega\tau_b + (j\omega\tau_b)^2\right]\ldots}$$

τ_1 or "tau sub one"; ξ_3 or "xi sub three"; etc...

In our example:

$$(2) \quad \hat{H}(j\omega) = \frac{\frac{R}{L}j\omega}{(j\omega)^2 + \frac{R}{L}j\omega + \frac{1}{LC}} \cdot \frac{\times LC}{\times LC} = \frac{RCj\omega}{LC(j\omega)^2 + RCj\omega + 1}$$

Bode plots: standard format: independent term should be 1.

Comparing (2) to (1): $K_0 = RC$; $N = +1$; $\tau_1 = 0$; $\tau_3 = 0$

$\tau_a = 0$; $\boxed{\tau_b = \sqrt{LC}}$; $\boxed{\xi_b = \frac{RC}{2\sqrt{LC}} = \frac{R}{2}\sqrt{\frac{C}{L}}}$

$2\xi_b\tau_b = RC \rightarrow \xi_b = \frac{RC}{2\tau_b}$

τ_b will define critical points in the Bode plot; ξ_b will define the

$\boxed{\text{Bode Plots:}}$ → visualizing the frequency response of the transfer function or the circuit itself.

2 types $\begin{cases} \text{Magnitude} = \text{plot of } 20\log_{10}|\hat{H}(j\omega)| \text{ vs. } \omega \\ \text{phase: plot of angle of } \hat{H} \text{ vs. } \omega \end{cases}$

Magnitude of a complex rational function:

Review: 1) $\left|\dfrac{\hat{z}_1 \cdot \hat{z}_2}{\hat{z}_3}\right| = \dfrac{|\hat{z}_1|\cdot|\hat{z}_2|}{|\hat{z}_3|}$

2) $\log_{10}\dfrac{|\hat{z}_1|\cdot|\hat{z}_2|}{|\hat{z}_3|} = \log_{10}|\hat{z}_1| + \log_{10}|\hat{z}_2| - \log_{10}|\hat{z}_3|$

3) $|j\omega| = \omega\cdot\underline{|j|} = \omega$

4) $\log_{10}\hat{z}_1^{a} = a\log_{10}\hat{z}_1^{1}$

$$20\log_{10}|\hat{H}(j\omega)| = 20\log_{10}\frac{|K_0|\cdot|j\omega|^{\pm N}\cdot|1+j\omega\tau_1|\cdot|1+2\zeta_3 j\omega\tau_3+(j\omega\tau_3)^2|\cdots}{|1+j\omega\tau_a|\cdot|1+2\zeta_b j\omega\tau_b+(j\omega\tau_b)^2|\cdots}$$

$$= 20\log_{10}|K_0| + 20\log_{10}\omega^{\pm N} + 20\log_{10}|1+j\omega\tau_1|$$

$$+ 20\log_{10}|1+2\zeta_3 j\omega\tau_3+(j\omega\tau_3)^2| + \cdots$$

$$- 20\log_{10}|1+j\omega\tau_a| - 20\log_{10}|1+2\zeta_b j\omega\tau_b+(j\omega\tau_b)^2|$$

$$- \cdots$$

To sketch Bodge Plot of any transfer function: learn how to sketch each of these typical terms:

1) $20\log_{10}|K_0|$

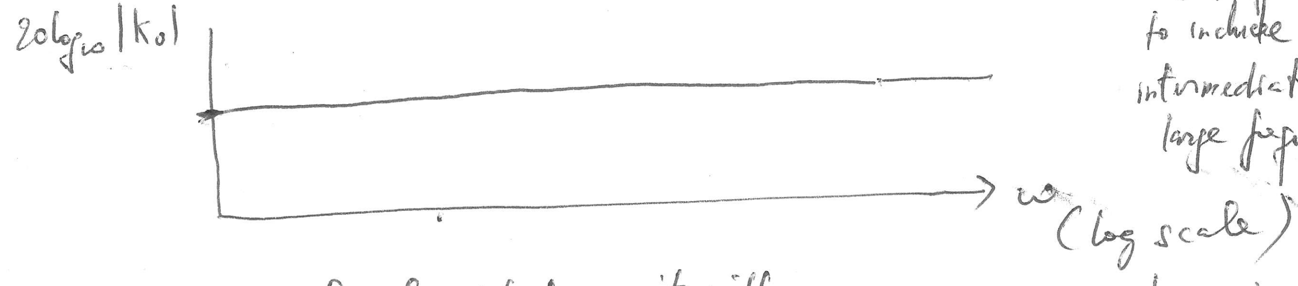
2) $\pm 20N\log_{10}\omega$

3) $\pm 20\log_{10}|1+j\omega\tau|$

4) $\pm 20\log_{10}|1+2\zeta_b j\omega\tau_b+(j\omega\tau_b)^2|$

1) $20\log_{10}|K_0|$ vs. ω → normally in log scale : to include a large range of frequency

to include very small, intermediate & very large frequencies.



Whatever initial value it has, it will keep the same value througout the entire ω-axis.

ω	$\log_{10}\omega$
0.001	−3
0.1	−1
1	0
10	1
1000	3.

2) $\pm 20N\log_{10}\omega$ vs. ω → log scale.