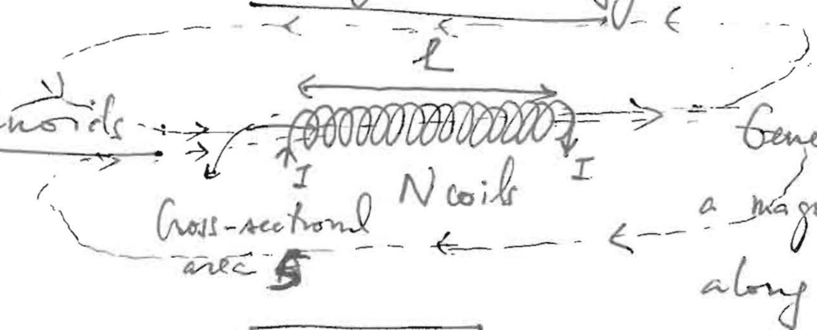
Self & Mutual Inductance (Magnetic Coupling) & Magnetic Energy

Solenoids



Generate & confine a magnetic field B along its axis

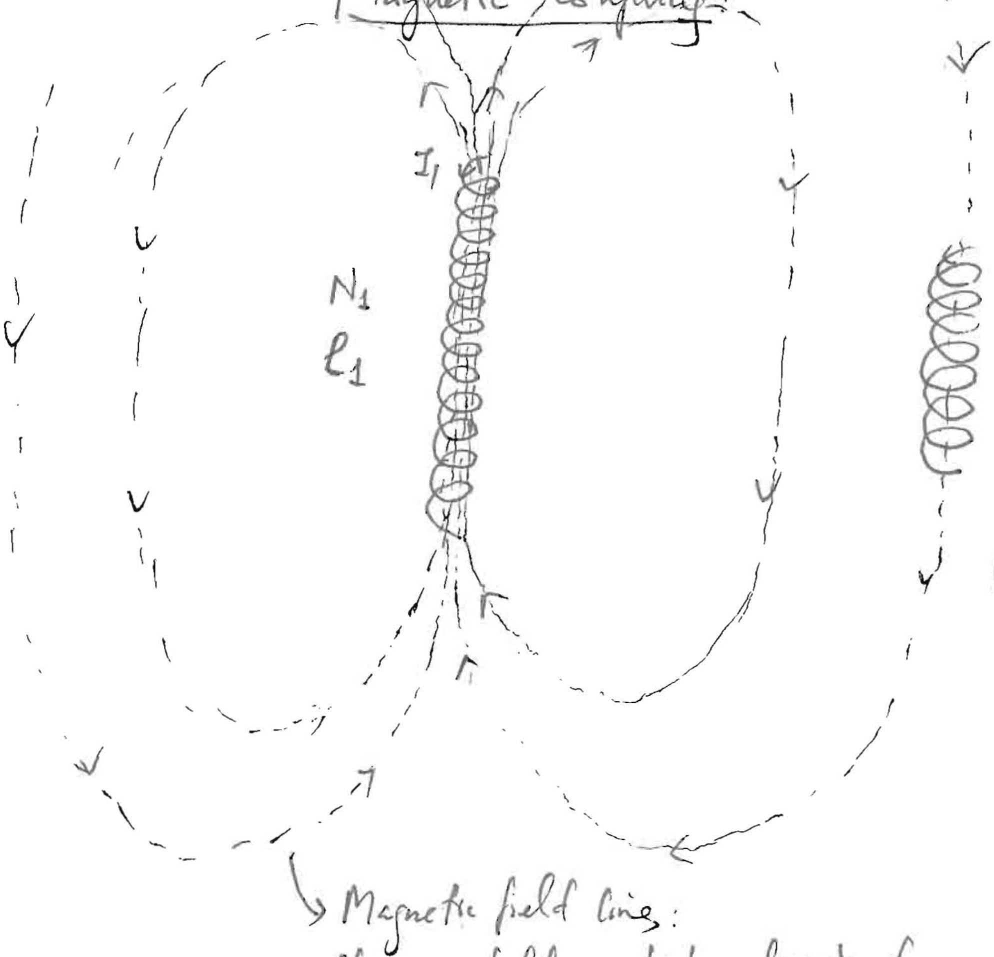
$$\boxed{B = \mu n I}$$

$n \equiv \frac{N}{\ell}$ or density of coils per unit length

magnetic permeability

Direction by RHR (RH fingers wrapping as current flows inside coil → thumb gives direction of $\vec{B}$)

Magnetic coupling:



$B_2 = \mu n_2 I_2$; $n_2 \equiv \frac{N_2}{\ell_2}$

(Field lines for B_2 are not shown)

- Magnetic field lines created by solenoid #1 affects solenoid #2. And vice versa! ↔ mutual inductance. M (unit in S.I. H for Henry)

↳ Magnetic field lines: Stronger field ↔ higher density of field lines.

$n_1 \equiv \frac{N_1}{\ell_1}$ $\quad B_1 = \mu n_1 I_1$

inductance.

→ Use correct sign for the mutual inductance term..

Mutual inductance in our equation :

If solenoids 1 & 2 are ∞ far apart:

$$V_1 = \frac{d\Phi_1}{dt} = \underbrace{\mu n_1 S_1}_{L_1 \to \text{self-inductance}} \frac{dI_1}{dt} \leftrightarrow \boxed{V_1 = L_1 \frac{dI_1}{dt}}$$

Faraday's law or law of induction: $\mathcal{E} = \frac{d\Phi}{dt}$

(a change of magnetic flux Φ in time induces an e.m.f. or voltage)

$\Phi = \oint \vec{B} \cdot d\vec{S}$ For our solenoid: $\vec{B}$ is uniform throughout cross-sectional area & perpendicular to these areas. $\rightarrow \Phi_1 = B_1 \cdot S_1$

$= \mu n_1 I_1 S_1$

$$\begin{cases} V_1 = L_1 \frac{dI_1}{dt} \\ V_2 = L_2 \frac{dI_2}{dt} \end{cases}$$

If the solenoids are not ∞ far apart: (change of current in #1 will affect voltage in #2 & viceversa.)

$$\begin{cases} V_1 = L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt} \\ V_2 = L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt} \end{cases}$$

Magnetic coupling (more than one solenoid in a circuit)

→ Use modified equation for voltages to include mutual inductance.

→ Use correct sign for the mutual inductance terms.

↳ $\boxed{\text{Use the DOT Convention.}}$

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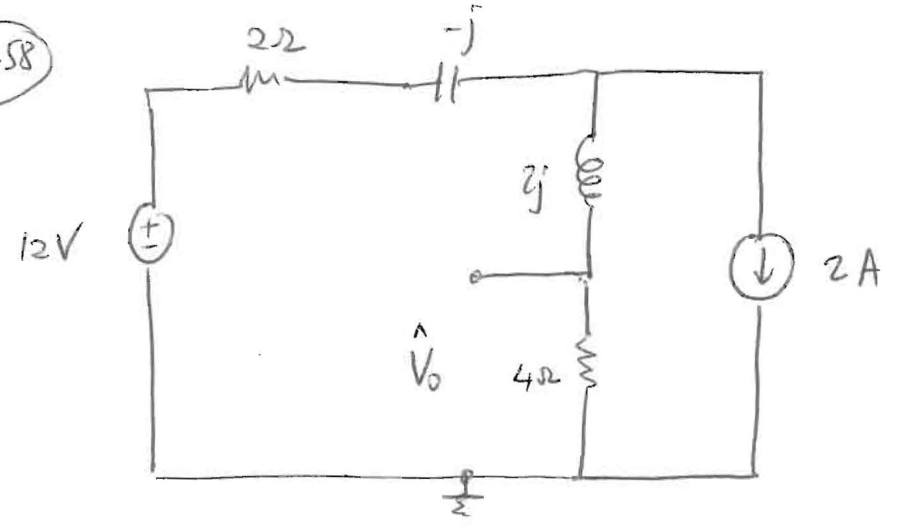
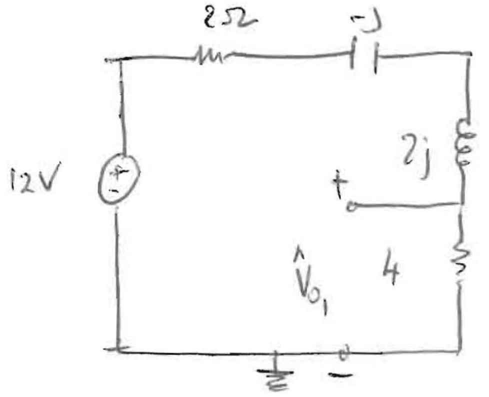


Find $\hat V_0$ using superposition

$$\hat V_0 = \hat V_{o_1} + \hat V_{o_2}$$

$\hat V_{o_1}$: Current source is open-circuited

$\hat V_{o_2}$: Voltage source is short-circuited.



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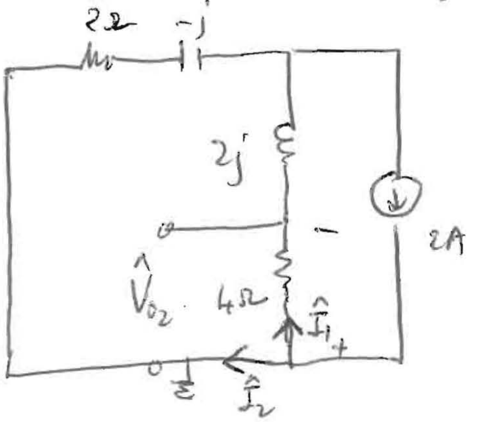


Voltage Division:

$$\hat V_{o_1} = 12 \frac{4}{4 + 2 + j} = \frac{48}{6 + j}$$

Current division:

$$\hat V_{o_2} = -4\hat I_1$$

$$\hat I_1 = 2A \frac{2 - j}{2 - j + 4 + 2j} = \frac{4 - 2j}{6 + j}$$

$$\hat V_0 = \frac{48}{6 + j} - \frac{16 - 8j}{6 + j} = \frac{32 + 8j}{6 + j} = 8\,\frac{4 + j}{6 + j}$$

$$= 8\frac{\sqrt{17}}{\sqrt{37}} \angle \tan^{-1}\tfrac{1}{4} - \tan^{-1}\tfrac{1}{6} = 5.42 \angle 4.53^\circ \checkmark$$

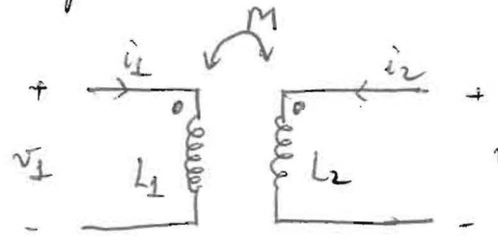
$$V_0(t) = 5.42 \cos(377t + 4.53^\circ) \quad V$$

Self & Mutual Inductance: Dot Convention:

$\downarrow$

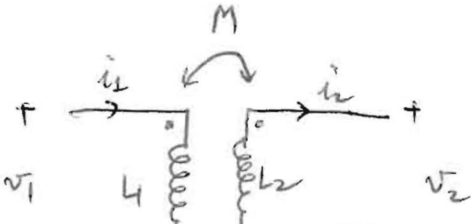
Provides information about internal properties of the inductors: how the wire is wrapped around solenoid determines direction of the magnetic field. $\leftrightarrow$ a reference for correct signs for the mutual inductance terms.

3 possible scenarios involving 2 inductors:

1) 

$\leftrightarrow$ $$\begin{cases} v_1 = L_1 \dfrac{di_1}{dt} + M \dfrac{di_2}{dt} \\ v_2 = L_2 \dfrac{di_2}{dt} + M \dfrac{di_1}{dt} \end{cases}$$ (Time-domain)

$$\begin{cases} \hat{V}_1 = j\omega L_1 \hat{I}_1 + j\omega M \hat{I}_2 \\ \hat{V}_2 = j\omega L_2 \hat{I}_2 + j\omega M \hat{I}_1 \end{cases}$$ (Frequency-domain)

2) 

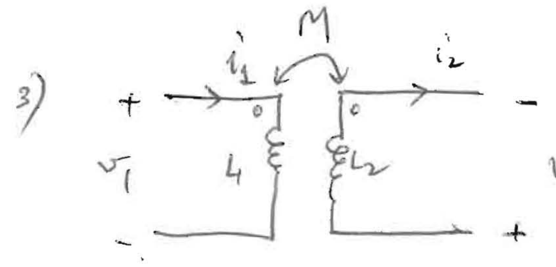
$\leftrightarrow$ $$\begin{cases} v_1 = L_1 \dfrac{di_1}{dt} - M \dfrac{di_2}{dt} \\ v_2 = -L_2 \dfrac{di_2}{dt} + M \dfrac{di_1}{dt} \end{cases}$$

$\downarrow$

terms involving i_2 will carry a minus sign

$$\begin{cases} \hat{V}_1 = j\omega L_1 \hat{I}_1 - j\omega M \hat{I}_2 \\ \hat{V}_2 = -j\omega L_2 \hat{I}_2 + j\omega M \hat{I}_1 \end{cases}$$

3)



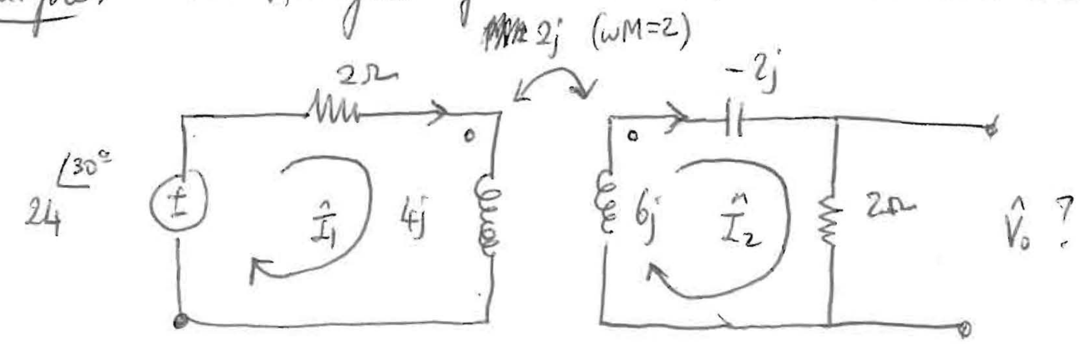
$\longleftrightarrow$

$$\begin{cases} v_1 = L_1 \dfrac{di_1}{dt} - M \dfrac{di_2}{dt} \\ v_2 = L_2 \dfrac{di_2}{dt} - M \dfrac{di_1}{dt} \end{cases}$$

compared with 2) terms involving v_2 will carry a minus sign

$$\begin{cases} \hat{V}_1 = j\omega L_1 \hat{I}_1 - j\omega M \hat{I}_2 \\ \hat{V}_2 = j\omega L_2 \hat{I}_2 - j\omega M \hat{I}_1 \end{cases}$$

Example: AC Analysis of circuits with more than one inductor:



Loop Analysis: choose direction for current in each loop (2 loops)
→ CW $\hat{I}_1$ & $\hat{I}_2$ → 2nd scenario! → Mutual inductance term has the opposite sign as the self inductance term.

Left loop: 1) $24\angle 30^\circ - \hat{I}_1(2+\underbrace{4j}_{L_1}) + \hat{I}_2 \underbrace{2j}_{M} = 0$

Right loop: 2) $-\hat{I}_2(\underbrace{6j}_{L_2} - 2j + 2) + 2j\,\hat{I}_1 = 0$

$$\hookrightarrow \hat{I}_1 = \hat{I}_2 \frac{(4j+2)}{2j} = \hat{I}_2 \frac{2j+1}{j} = \hat{I}_2 (+2-j)$$

1) $24\angle 30^\circ - \hat{I}_2(2-j)(2+4j) + \hat{I}_2\, 2j = 0$

$24\angle 30^\circ - \hat{I}_2\left[(2-j)(2+4j) - 2j\right] = 0$

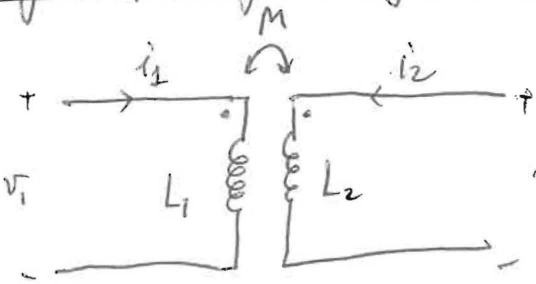
$$24\angle 30^\circ - \hat{I}_2\left[8 + 6j - 2j\right] = 0$$

$$\hat{I}_2 = \frac{24\angle 30^\circ}{8+4j} = \frac{6\angle 30^\circ}{2+j}$$

$$= \frac{6}{\sqrt{5}}\angle\left(30^\circ - \tan^{-1}\tfrac{1}{2}\right) = 2.68\angle 3.43^\circ \text{ A}$$

$$\boxed{\hat{V}_o = 2\hat{I}_2 = 2\times 2.68\angle 3.43^\circ \text{ V} = 5.36\angle 3.43^\circ \text{ V}}$$

Energy Analysis of Magnetically Coupled Circuits



Scenario #1:

$$\begin{cases} v_1(t) = L_1\dfrac{di_1}{dt} + M\dfrac{di_2}{dt} \\ v_2(t) = L_2\dfrac{di_2}{dt} + M\dfrac{di_1}{dt} \end{cases}$$

Power consumption : $P(t) = i_1(t)\,v_1(t) + i_2(t)\,v_2(t)$

Energy spent : $\mathcal{E} = \int_0^{t_2} dt\, P(t)$

$0 \longrightarrow t_1 \longrightarrow t_2$

- 0: $\boxed{i_1 \text{ increasing},\ i_2 = 0}$
- t_1: when current in L_1 starts becoming steady or constant
- $\boxed{i_2 \text{ increasing},\ i_1 \text{ constant}} \to i_1 = I_1$
- t_2: when the current in L_2 starts becoming a constant

$$\mathcal{E} = \int_0^{t_1} dt\,\underbrace{\left(i_1 L\frac{di_1}{dt} + i_1 M \overbrace{\cancel{\frac{di_2}{dt}}}^{0}\right)}_{\substack{i_1\cdot v_1 \\ (i_2=0)}} + \int_{t_1}^{t_2} dt\left[\underbrace{\left(i_1 L_1\cancel{\frac{di_1}{dt}} + i_1 M\frac{di_2}{dt}\right)}_{i_1 v_1} + \underbrace{\left(i_2 L_2\frac{di_2}{dt} + i_2 M\cancel{\frac{di_1}{dt}}\right)}_{i_2 v_2}\right]$$

$$\mathcal{E} = L_1 \underbrace{\int_0^{t_1} i_1\, di_1}_{\frac{1}{2} i_1^2} + M I_1 \int_{t_1}^{t_2} di_2 + L_2 \int_{t_1}^{t_2} i_2\, di_2$$

$$\mathcal{E} = \boxed{\frac{1}{2} L_1 I_1^2} + M I_1 I_2 + \boxed{\frac{1}{2} L_2 I_2^2}$$

$(i_1(t=t_1) = I_1)$

$(i_2(t=t_1) = 0$
$i_2(t=t_2) = I_2)$

Magnetic energy in L_1

Coupling energy b/w L_1 & L_2

Magnetic energy in L_2

$KE = \frac{1}{2} m v^2$

inertia of motion: mass opposes any instantaneous change in velocity.

$ME = \frac{1}{2} L I^2$

inertia of current: inductance opposes any instantaneous change in current. using magnetic energy it stored when circuit was on

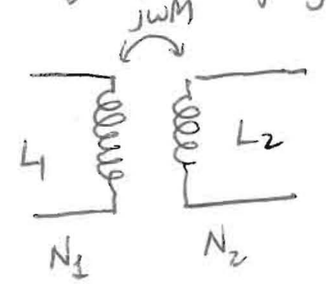
$EE = \frac{1}{2} C V^2$

inertia of voltage: capacitance opposes any instantaneous change in voltage using electric energy it stored when circuit was on.

$\boxed{\text{inertia} \leftrightarrow \text{energy storage}}$

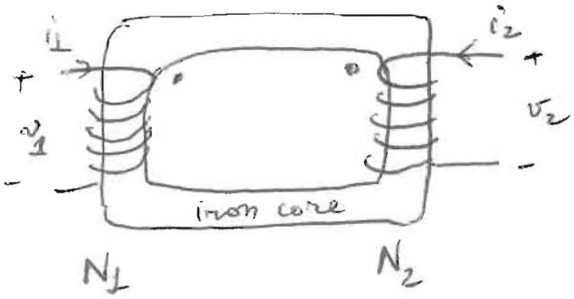
Ideal Transformer:

Magnetic coupling: ⟶ **Ideal Transformer:**



→ Two solenoids with different inductances L_1 & L_2 (different cross sections, numbers of turns, etc.)

Two solenoids of same cross sections: using an iron core:



Two inductors or solenoids next to each other → Faraday's Law: voltage v_1 comes from a change of magnetic flux through solenoid #1: $N_1 \phi_1$

$$v_1 = \frac{d(N_1 \phi_1)}{dt} = N_1 \frac{d\phi_1}{dt}$$

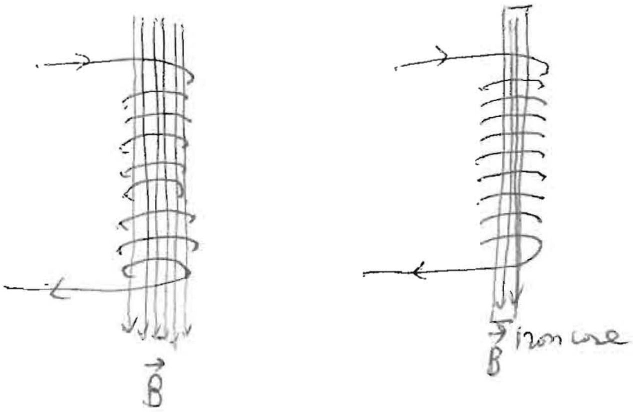
ϕ_1 = m. flux through one loop of inductor or solenoid #1

$= B \cdot S_1$

$$v_2 = \frac{d(N_2 \phi_2)}{dt} = N_2 \frac{d\phi_2}{dt}$$

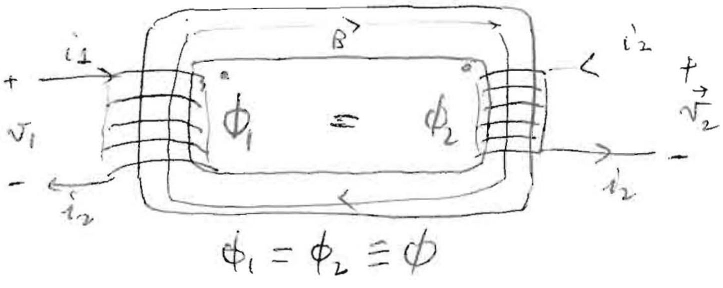
$$\phi_2 = B \cdot S_2$$

ideal Transformers (same cross section)



can set a fixed cross-section for the magnetic flux by using an iron core

↓



$$\phi_1 = \phi_2 \equiv \phi$$

$$v_1 = N_1 \frac{d\phi_1}{dt} = N_1 \frac{d\phi}{dt}$$

$$v_2 = N_2 \frac{d\phi_2}{dt} = N_2 \frac{d\phi}{dt}$$

$$\boxed{\frac{v_1}{v_2} = \frac{N_1}{N_2}}$$ voltage transformer.

Any connection b/w i_1 & i_2 in an ideal transformer?

Yes, using Ampere's Law.

$$\oint_{\text{Amp. loop}} \vec{H} \cdot \vec{dl} = \text{current enclosed by loop}$$

$$\vec{B} = \mu \vec{H}$$

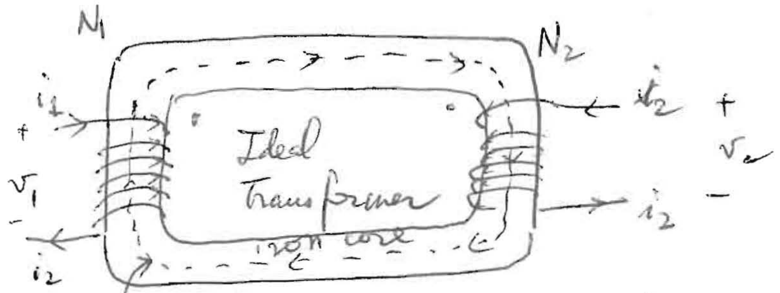
$\vec{B}$ → magnetic induction

μ → magnetic permeability

$\vec{H}$ → magnetic field

(most of the time we call $\vec{B}$ the magnetic field)

$$\oint_{\text{Amp loop}} \vec{B} \cdot \vec{dl} = \mu I_{\text{enclosed by loop}}$$



Amperian loop:

$$\oint_{A.\text{ loop}} \vec{H} \cdot \vec{dl} = i_1 N_1 + i_2 N_2$$

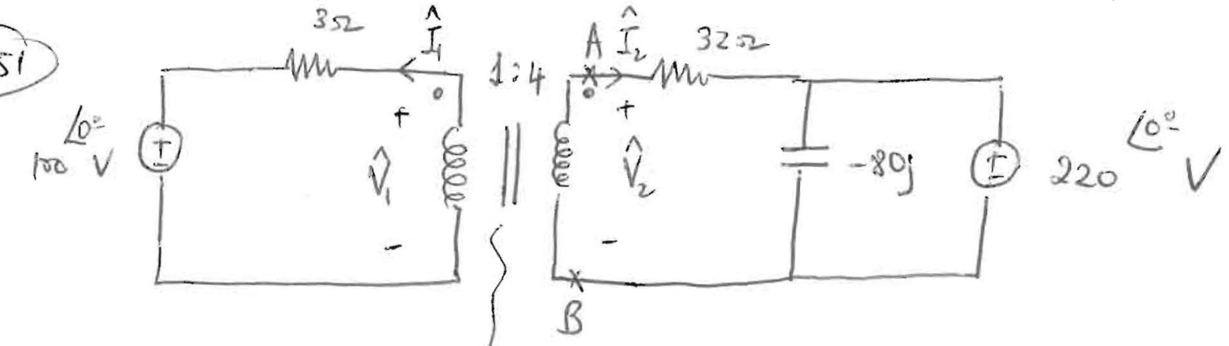
$$\frac{1}{\mu} \oint_{A.\text{ loop}} \vec{B} \cdot \vec{dl}$$

Ideal transformer → iron core : $\mu = \infty \rightarrow \frac{1}{\mu_{\text{iron core}}} = 0$

↳ $0 = i_1 N_1 + i_2 N_2 \longrightarrow \boxed{\frac{i_1}{i_2} = -\frac{N_2}{N_1}}$

Ideal transformer.

8.51

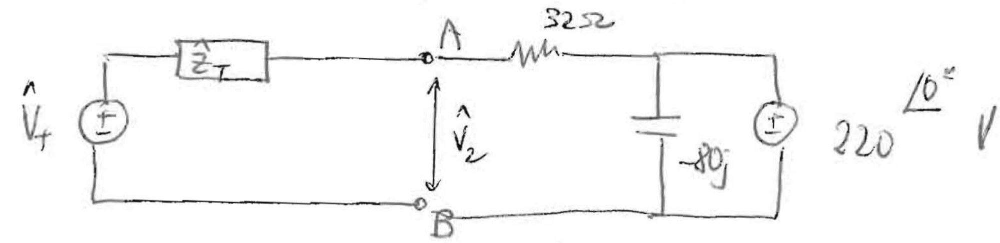


iron core
ideal transformer coil ratio: $\frac{N_1}{N_2} = \frac{1}{4}$

$\boxed{\hat{I}_1 ?\ \hat{I}_2 ?;\ \hat{V}_1 ?;\ \hat{V}_2\ ?}$

Ideal Transformer: $\frac{\hat V_2}{\hat V_1} = 4$; $\frac{\hat I_2}{\hat I_1} = -\frac{1}{4}$ Only need to solve for two variables, e.g. $\hat I_2$ & $\hat V_2$

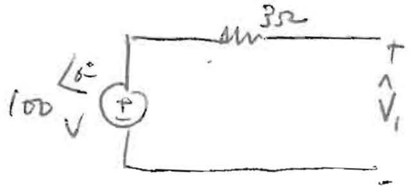
Solve using Thevenin equivalent → b/w two points → A & B
(we need to find $\hat V_2$)
will replace circuit to the left of AB by its Thevenin's equivalent



Find this
$\hat V_T = \hat V_{o.c}$ across AB
(open circuit)
$\hat Z_T$ = impedance b/w A & B when sources are eliminated { Voltage source → short-circuit; current source → open-circuit

$\hat V_{o.c.}$ across AB :

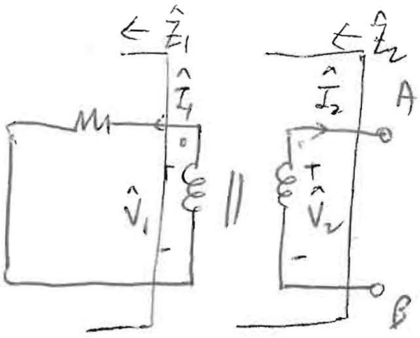
$\hat V_{o.c} = \hat V_2$
$\hat I_2 = 0 \to \hat I_1 = -4\hat I_2 = 0$
→ Left loop is open!



→ $\hat V_1 = 100\angle 0^\circ$ V → $\hat V_2 = 4\hat V_1 = 400\angle 0^\circ$ V

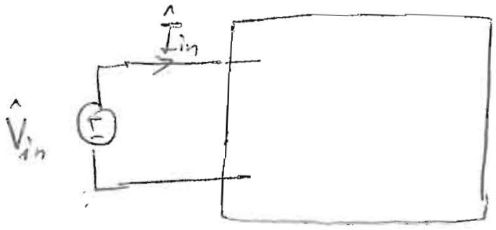
→ $\boxed{\hat V_{o.c} = 400\angle 0^\circ\ V = \hat V_T}$

$\hat{Z}_T = \hat{Z}_{AB}\Big|_{\text{no source}}$



$\hat{Z}_{AB} = ? = \dfrac{\hat{V}_2}{-\hat{I}_2}$

In general the input impedance is found:

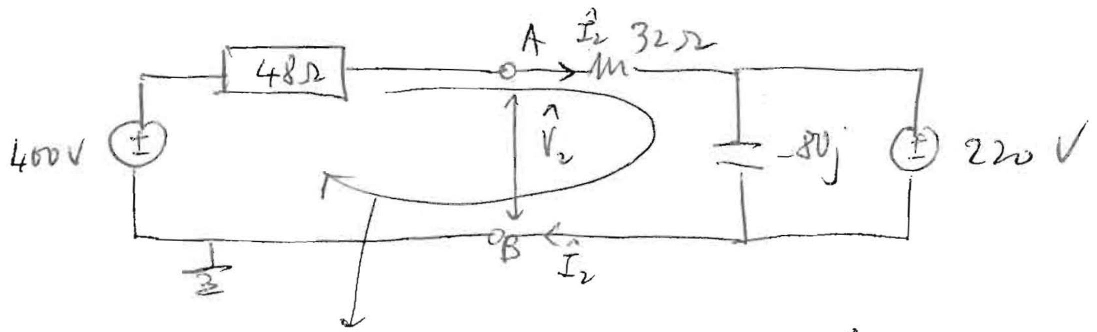


$\hat{Z}_{in} = \dfrac{\hat{V}_{in}}{\hat{I}_{in}}$

$$\hat{Z}_T = \frac{\hat{V}_2}{-\hat{I}_2} = \frac{4\hat{V}_1}{-\left(-\frac{1}{4}\hat{I}_1\right)} = 16\underbrace{\frac{\hat{V}_1}{\hat{I}_1}}_{} = 16.3 = 48\,\Omega$$

input impedance from the ~~voltage source of 100∠0°~~ inductor #1 to the left.

16 fold increase due to the ideal transformer.



$$\text{KVL:}\quad 400 - \underbrace{(48+32)}_{80}\hat{I}_2 - \underbrace{220}_{} = 0$$

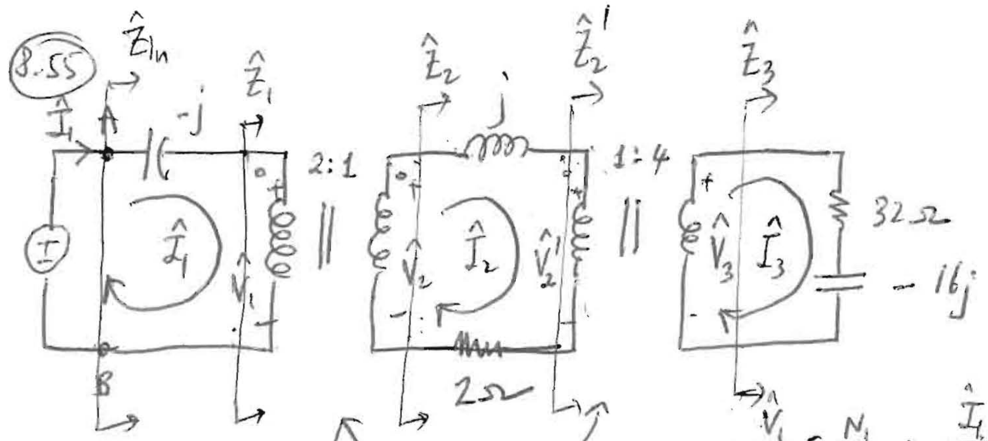
Voltage drop across $(-80j)$

$$\hat{I}_2 = \frac{400-220}{80} = \frac{180}{80} = \boxed{2.25\,A}$$

$$\hat{V}_2 = \hat{V}_{AB} = 400 - 48 I_2 = \boxed{292\,V} \rightarrow \hat{V}_1 = \frac{\hat{V}_2}{4} = \frac{292}{4} = \boxed{73\,V}$$

$$\hat{I}_1 = -4\hat{I}_2 = -4 \times 2.25 = \boxed{-9\,A}$$

HW4: 8.55



$\frac{\hat{V}_1}{\hat{V}_2} = \frac{N_1}{N_2}$; $\frac{\hat{I}_1}{\hat{I}_2} = -\frac{N_2}{N_1}$ w/ dots at tops (both currents go down @ inductors)

ideal transformers

Find input impedance b/w A & B from the source $\hat{V}_s$:

$\hat{Z}_{in} = \frac{\hat{V}_s}{\hat{I}_1}$ (low $\hat{I}_1 \Rightarrow$ high $\hat{Z}_{in}$ as in meters)

Strategy: find $\hat{I}_1$ in term of $\hat{V}_s$

1) Name voltages across inductors as part of the ideal transformers.
Name currents in each of the three loops: $\hat{V}_1$; $\hat{V}_2$; $\hat{V}_2'$; $\hat{V}_3$;
$\hat{I}_1$; $\hat{I}_2$; $\hat{I}_3$

2) Starting w/ loop #3: $\boxed{\frac{\hat{V}_3}{\hat{I}_3} = 32-16j = \hat{Z}_3} \rightarrow \frac{4\hat{V}_2'}{\frac{\hat{I}_2}{4}} = 32-16j$

Since $\frac{\hat{V}_3}{\hat{V}_2'} = 4 \rightarrow \hat{V}_3 = 4\hat{V}_2'$

$\frac{-\hat{I}_3}{\hat{I}_2} = -\frac{1}{4} \rightarrow \hat{I}_3 = \frac{\hat{I}_2}{4}$

$\Rightarrow \boxed{\frac{\hat{V}_2'}{\hat{I}_2} = 2-j = \hat{Z}_2'}$

3) $\hat{Z}_2 = 4$ ✓ (series combination of j & 2-j & 2)

$\hat{Z}_2 = \frac{\hat{V}_2}{\hat{I}_2}$ How does $\hat{V}_2$ relate to $\hat{V}_2'$?

KVL for middle loop : $-\hat{V}_2 + I_2 j + \hat{V}_2' + 2I_2 = 0$

$\hat{V}_2 = (2+j)\hat{I}_2 + \hat{V}_2' = \hat{I}_2 \left[2+j + \underbrace{\frac{\hat{V}_2'}{\hat{I}_2}}_{2-j} \right] = 4\hat{I}_2$

$\rightarrow \hat{Z}_2 = \frac{\hat{V}_2}{\hat{I}_2} = 4$ ✓

4) Find $\hat{Z}_1 = \frac{\hat{V}_1}{\hat{I}_1} = \longrightarrow \frac{2\hat{V}_2}{\frac{\hat{I}_2}{2}} = 4\frac{\hat{V}_2}{\hat{I}_2} = 4\hat{Z}_2 = 4\times4 = 16\,\Omega$

Ideal Transformer $\frac{\hat{V}_1}{\hat{V}_2} = 2 \rightarrow \hat{V}_1 = 2\hat{V}_2$

$\frac{-\hat{I}_2}{\hat{I}_1} = -2 \rightarrow \hat{I}_1 = \frac{\hat{I}_2}{2}$

$$\boxed{\hat{Z}_1 = \frac{\hat{V}_1}{\hat{I}_1} = 16}$$

5) Find $\hat{Z}_{in} =$

Method #1: series combination of $-j$ & $\hat{Z}_1 = -j + \hat{Z}_1 = -j + 16\ \Omega$

Method #2: $\boxed{\hat{Z}_{in} = \frac{\hat{V}_s}{\hat{I}_1} = \frac{\hat{V}_1 - \hat{I}_1 j}{\hat{I}_1} = \frac{\hat{V}_1}{\hat{I}_1} - j = 16 - j\ \Omega}$

Relate $\hat{V}_s$ to $\hat{V}_1$: $\hat{V}_s = \hat{V}_1 + (-j\hat{I}_1)$

(KVL on Loop # 1)

Power Analysis:

- What load would extract the most power?
- What is the average power consumed in an AC circuit?
- What are the root-mean-square (RMS) or effective values for current & voltages?
- What is the complex power?

Instantaneous Power: p (lower case letter p)

$$p(t) = i(t) \cdot v(t) = I_M \cos(\omega t + \theta_i) \cdot V_M \cos(\omega t + \theta_v)$$

AC circuits: $\underbrace{\left.\begin{array}{l} i(t) = I_M \cos(\omega t + \theta_i) \\ v(t) = V_M \cos(\omega t + \theta_v) \end{array}\right\}}_{i \,\&\, v \text{ are time varying}}$ I_M & V_M magnitude of alternating current & voltage. θ_i & θ_v are phase of AC current & voltage. $\underbrace{}_{\text{time independent}}$

Average power: P (upper case letter p)

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} dt \cdot p(t)$$

(integrating the instantaneous power over one period, then divide by the period T)

Any connection b/w T & ω: Yes: $T = \frac{2\pi}{\omega}$

Need to integrate a product of 2 cosines: →

$$\left\{\begin{array}{l} \cos\alpha \cos\beta = \frac{1}{2}\left[\cos(\alpha - \beta) + \cos(\alpha + \beta)\right] \quad \underline{\text{Trig. Id}} \\ \alpha \equiv \omega t + \theta_i \\ \beta \equiv \omega t + \theta_v \end{array}\right.$$

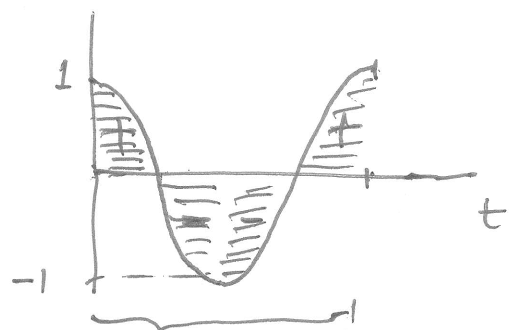
$\rightarrow \alpha - \beta = \theta_i - \theta_v$; $\alpha + \beta = 2\omega t + \theta_i + \theta_v$

$$\rightarrow \underline{P} = \frac{1}{T} I_m V_m \frac{1}{2}\left[\int_T dt \underbrace{\cos(\theta_i - \theta_v)}_{\text{time independent}} + \int_T dt \underbrace{\cos(2\omega t + \theta_i + \theta_v)}_{\text{period is } \frac{2\pi}{2\omega} = \frac{T}{2}}\right]$$

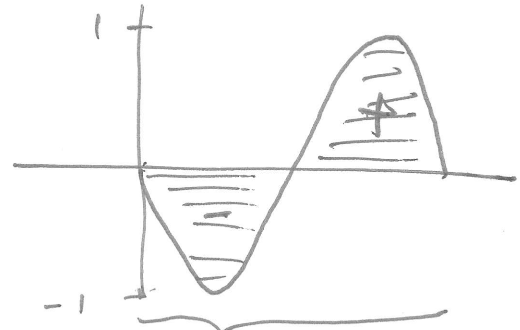
$$= \frac{1}{2} I_m V_m \frac{1}{T}\left[\cos(\theta_i - \theta_v) \underbrace{\int_T dt}_{T} + \text{integral of a sinusoid over two of its periods of } \frac{T}{2}\right]$$

$$\Downarrow 0$$

* Integral of a sinusoid over one of its period is: 0



one period.



one period.

$$\Rightarrow \boxed{\underline{P} = \frac{1}{2} I_m V_m \cos(\theta_i - \theta_v)}$$

Average Power is not time-varying!

Consequences:

1) At a pure resistive element (a resistor) : $\theta_i = \theta_v$
→ max. ave. power , $\underline{P}_{max} = \frac{1}{2} I_m V_m$

2) At a pure reactive element (inductor or a capacitor): $\theta_i - \theta_v = \pm\frac{\pi}{2}$

$$\underline{P} = 0$$

(they are energy storage device → makes sense)

$\boxed{\text{At a resistor}}$ $\hat{Z} = \dfrac{\hat{V}}{\hat{I}} = \dfrac{V_M \angle \theta_v}{I_M \angle \theta_i} = \dfrac{V_M}{I_M} \angle \theta_v - \theta_i$

Ohm's Law works with phasors

Because we know $\hat{Z} = R \rightarrow$ a real number $\rightarrow$ phase is zero $\rightarrow$ $\theta_v - \theta_i = 0$ or $\boxed{\theta_v = \theta_i}$

$\boxed{\text{At an inductor}}$

$$\hat{Z}_L \begin{cases} = \dfrac{\hat{V}}{\hat{I}} = j\omega L = \omega L \angle 90^\circ \\ = \dfrac{\hat{V}}{\hat{I}} = \dfrac{V_M}{I_M} \angle \theta_v - \theta_i \end{cases} \quad \boxed{\theta_v - \theta_i = 90^\circ}$$

$\boxed{\text{At a capacitor:}}$

$$\hat{Z}_C \begin{cases} = \dfrac{1}{j\omega C} = \dfrac{1}{\omega C} \angle -90^\circ \\ = \dfrac{\hat{V}}{\hat{I}} = \dfrac{V_M}{I_M} \angle \theta_v - \theta_i \end{cases} \quad \boxed{\theta_v - \theta_i = -90^\circ}$$

What are the R-M-S values or effective values for current & voltage? (Root of the Mean of the square)

$$I_{RMS} = \sqrt{\underbrace{\frac{1}{T}\int_T dt \; i^2(t)}_{\text{mean of square of current.}}}$$

AC current: $i(t) = I_M \cos(\omega t + \theta_i)$

$$I_{RMS} = I_M \sqrt{\frac{1}{T}\int_T dt \cos^2(\omega t + \theta_i)}$$

Trig. Id.: $\cos^2\alpha = \frac{1}{2}(1 + \cos 2\alpha)$

$$I_{RMS} = \frac{I_M}{\sqrt{2}}\sqrt{\frac{1}{T}\int_T dt \left(1 + \cos(2\omega t + 2\theta_i)\right)} = \frac{I_M}{\sqrt{2}}\sqrt{\underbrace{\frac{1}{T}\int_T dt}_{1} + \underbrace{\frac{1}{T}\int_T dt \cos(2\omega t + 2\theta_i)}_{\text{period of } \frac{T}{2} \;\; 0}}$$

$$\boxed{I_{RMS} = \frac{I_M}{\sqrt{2}}}$$

likewise the RMS value for voltage is its magnitude divided by $\sqrt{2}$: $\boxed{V_{RMS} = \frac{V_M}{\sqrt{2}}}$

Consequences:

1) Average Power in term of the RMS values:

$$\left.\begin{aligned} P &= \frac{1}{2} I_m V_M \cos(\theta_i - \theta_v) \\ &= \frac{I_m}{\sqrt{2}} \cdot \frac{V_M}{\sqrt{2}} \cdot pf \end{aligned}\right\} \boxed{P = I_{RMS} \cdot V_{RMS}\, pf}$$

$\left\{ \begin{array}{l} \text{Power factor: } pf \text{ (always} < 1) \\ pf \triangleq \cos(\theta_i - \theta_v) \end{array} \right.$

2) Pure-resistive element: $\theta_i = \theta_v$ $\quad pf = \cos(0) = 1 \to P = I_{RMS} \cdot V_{RMS}$ (reminds us about DC analysis)

Pure-reactive element: $\theta_i - \theta_v = \pm \frac{\pi}{2}$ $\quad pf = \cos(\pm \frac{\pi}{2}) = 0 \to P = 0$

Complex Power: upper case letter $\hat{S}$

Not a measurable quantity, but a definition that facilitates the analysis of power in AC circuits (similar to the complex phasors for the analysis of AC circuits)

$$\hat{S} \equiv \hat{V}_{RMS} \cdot \hat{I}_{RMS}^{*} = V_{RMS}\angle\theta_v \cdot I_{RMS}\angle -\theta_i = V_{RMS} \cdot I_{RMS}\angle\theta_v - \theta_i$$

Complex conjugate

$$= \underbrace{V_{RMS} \cdot I_{RMS} \cos(\theta_v - \theta_i)}_{} + j\, V_{RMS} \cdot I_{RMS} \sin(\theta_v - \theta_i)$$

de Moivre formula

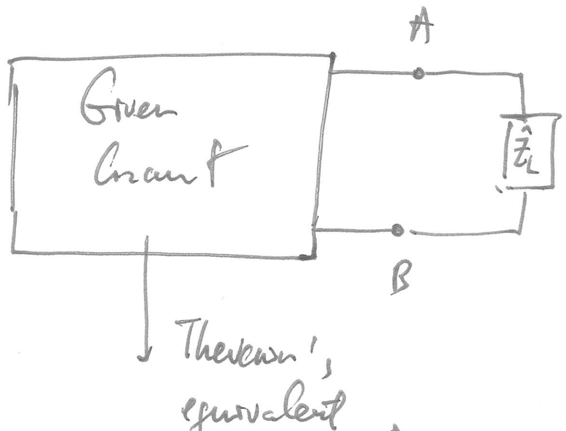
The real part of the complex power is just

$I_{RMS} \cdot V_{RMS}\, pf$ ($\cos(\theta_v - \theta_i) = \cos(\theta_i - \theta_v) = pf$ since cos is symmetric w/r/t the origin or cos is a even function)

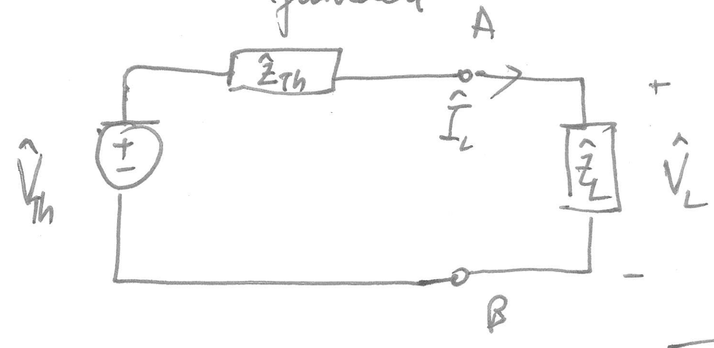
which is the average power P.

$$\Rightarrow \hat{S} \equiv \underbrace{P}_{\text{Real part}} + j \underbrace{V_{RMS} I_{RMS} \sin(\theta_v - \theta_i)}_{\text{Imaginary part}}$$

Which load would draw the max power given a circuit?



what $\hat{Z}_L$ will draw the most power out of this given circuit?



Average power consumed at load is P_L

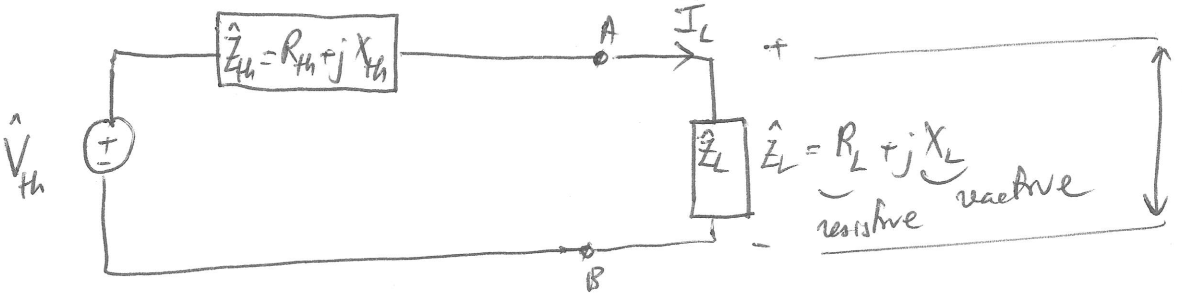
Max P_L can be achieved when $\boxed{\hat{Z}_L = \hat{Z}_{th}^{*}}$

Proof of the Max Power Theorem:

In general, for the circuit shown in the previous figure, the average power consumed at the load is $P_L = \frac{1}{2} I_L V_L \cos(\theta_{v_L} - \theta_{i_L})$ (time-independent).

Using the Thevenin's equivalent for the given circuit, we can write I_L & V_L in terms of V_{th}, R_{th}, X_{th} where

$$V_{th} = |\hat{V}_{th}|; \quad \hat{Z}_{th} = \underbrace{R_{th}}_{\text{resistive}} + j \underbrace{X_{th}}_{\text{reactive}}$$



a) $$\hat{I}_L = \frac{\hat{V}_{th}}{\hat{Z}_{th} + \hat{Z}_L} \longrightarrow \underset{(\text{to use in } P_L)}{I_L} = \frac{V_{th}}{\sqrt{(R_{th}+R_L)^2 + (X_{th}+X_L)^2}}$$

(real part & imaginary part are perpendicular in the complex plane) → can use Pythagoras Theorem to find the magnitude of the total impedance.

$$\hat{V}_L = \hat{V}_{th} \frac{\hat{Z}_L}{\hat{Z}_L + \hat{Z}_{th}}$$

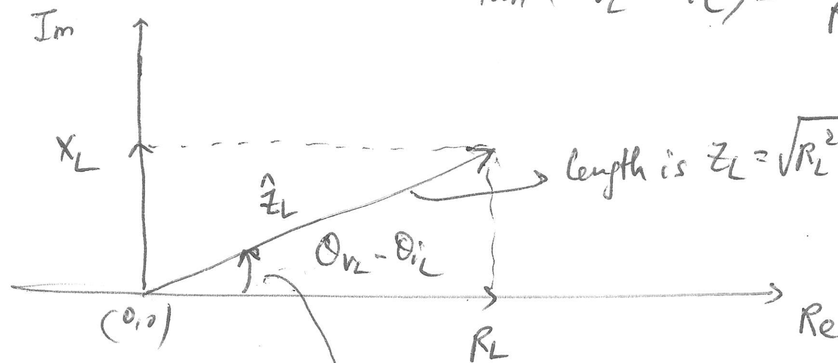
$$\longrightarrow \underset{(\text{to be used in } P_L)}{V_L} = \frac{V_{th}\sqrt{R_L^2 + X_L^2}}{\sqrt{(R_{th}+R_L)^2 + (X_{th}+X_L)^2}}$$

b) $\cos(\theta_{v_L} - \theta_{i_L})$:

$$\left[\hat{Z}_L = \frac{\hat{V}_L}{\hat{I}_L} = \boxed{\frac{V_L}{I_L} \angle \theta_{V_L} - \theta_{i_L}}\right] \rightarrow \hat{Z}_L = \underbrace{R_L + jX_L}_{\text{Cartesian format}} = \underbrace{\sqrt{R_L^2 + X_L^2}\,\angle \tan^{-1}\frac{X_L}{R_L}}_{\text{Polar format}}$$

$\hat{Z}_L \downarrow$ Ôhm's Law

$$\tan(\theta_{V_L} - \theta_{i_L}) = \frac{X_L}{R_L}$$



length is $Z_L = \sqrt{R_L^2 + X_L^2}$

Phase is $\tan^{-1}\left(\frac{X_L}{R_L}\right)$

which is also $\boxed{\theta_{V_L} - \theta_{i_L}}$

Then: $\cos(\theta_{V_L} - \theta_{i_L}) = \frac{R_L}{\sqrt{R_L^2 + X_L^2}}$

Then plug a) & b) into $P_L = \frac{1}{2} I_L V_L \cos(\theta_{V_L} - \theta_{i_L})$

$$P_L = \frac{1}{2} \underbrace{\frac{V_{th}}{\sqrt{(R_{th}+R_L)^2 + (X_{th}+X_L)^2}}}_{I_L} \cdot \underbrace{\frac{V_{th}\cancel{\sqrt{R_L^2 + X_L^2}}}{\sqrt{(R_{th}+R_L)^2 + (X_{th}+X_L)^2}}}_{V_L} \cdot \underbrace{\frac{R_L}{\cancel{\sqrt{R_L^2 + X_L^2}}}}_{\cos(\theta_{V_L} - \theta_{i_L})}$$

$$\boxed{P_L = \frac{1}{2} V_{th}^2 \frac{R_L}{(R_{th}+R_L)^2 + (X_{th}+X_L)^2}}$$

Ave. power @ load in terms of the Thevenin equivalent for circuit & load information.

What X_L & R_L will maximize P_L? Use Calculus to find the maximum of a function:

i) Larger P_L needs smaller denominator: if we drop a parenthesis (since they are both squared or +)

$$\rightarrow \quad (X_{th} + X_L)^2 = 0 \quad \text{or} \quad X_{th} + X_L = 0 \quad \text{or} \quad \boxed{X_L = -X_{th}}$$

This can be realized in practice: we should be able to select a load whose reactive part of the impedance is the same & opposite of that of the Thevenin impedance of the given circuit.

ii) Then what R_L will maximize P_L? $\rightarrow$ Use Calculus:

$$\frac{\partial P_L}{\partial R_L} = 0$$

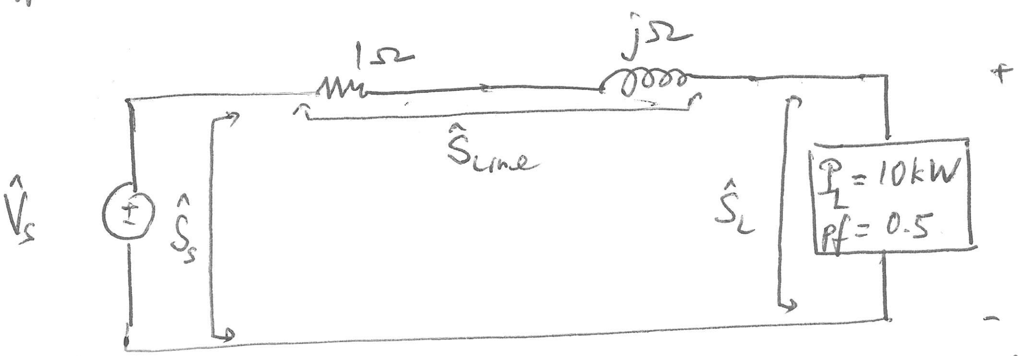
$$\frac{\partial\left[\frac{1}{2} V_{th}^2 \frac{R_L}{(R_{th}+R_L)^2}\right]}{\partial R_L} = \frac{1}{2} V_{th}^2 \frac{\partial\left[\frac{R_L}{(R_{th}+R_L)^2}\right]}{\partial R_L} = \frac{1}{2} V_{th}^2 \underbrace{\left\{\frac{1}{(R_{th}+R_L)^2} - \frac{2R_L}{(R_{th}+R_L)^3}\right\}}_{\text{has to be } 0.}$$

$$\frac{1}{(R_L+R_{th})^2} \underbrace{\left\{1 - \frac{2R_L}{R_{th}+R_L}\right\}}_{\text{has to be } 0} \longrightarrow \quad 2R_L = R_{th} + R_L$$

$$\boxed{R_L = R_{th}}$$

Summary: $\left.\begin{matrix} R_L = R_{th} \\ X_L = -X_{th} \end{matrix}\right\rangle \rightarrow \boxed{\hat{Z}_L = \hat{Z}_{th}^*}$

Application of Complex Power $\hat{S}$: scenarios in which $\hat{S}$ is helpful



$\hat{V}_{RMS} = 110\angle 0^\circ$ ✓

Calculate $\hat{V}_s$ so the circuit can deliver the specified load.

Strategy using complex power: 1) & 2)

1) Find $\hat{S}_s$ (complex power delivered at source):

$\hat{S}_s \equiv \hat{V}_s \hat{I}_s^*$ since can then find 2) $\hat{V}_s = \dfrac{\hat{S}_s}{\hat{I}_s^*}$ (and $\hat{I}_s = \hat{I}_L$)

1) i) $\hat{S}_s = \hat{S}_{Line} + \hat{S}_L$ (power delivered is to be consumed along the line & at the load)

Find $\hat{S}_L$: Recall: $Re[\hat{S}_L] = P_L = 10\,kW$

Also: $\hat{S}_L = \hat{V}_{L_{RMS}} \cdot \hat{I}_{L_{RMS}}^* = V_{L_{RMS}} I_{L_{RMS}} \angle \theta_{V_L} - \theta_{i_L}$

or the angle of $\hat{S}_L$ (phase) is simply $\theta_{V_L} - \theta_{i_L}$

$= \cos^{-1}(pf)$

$(pf = \cos(\theta_v - \theta_i))$

Now what is S_L? (magnitude of the complex power $\hat{S}_L$)

It is $V_{RMS_L} I_{RMS_L} = \dfrac{P_L}{pf}$ (since $P = V_{RMS} I_{RMS}\, pf$)

$\hookrightarrow \hat{S}_L = S_L \angle \theta_{V_L} - \theta_{i_L} = \dfrac{P_L}{pf} \angle \cos^{-1} pf = \dfrac{10000}{0.5} \angle \cos^{-1} 0.5$

$$\boxed{\hat{S}_L = 20000 \angle 60^\circ \; V \cdot A}$$

→ save W for real power (average power)

Complex Power @ load can be obtain from its average power & pf :

$$\boxed{\hat{S}_L = \frac{P_L}{pf} \angle \cos^{-1} pf}$$

1 ii) Find $\hat{S}_{Line}$

$$\hat{S}_{Line} = \hat{V}_{\substack{Line\\RMS}} \hat{I}^*_{\substack{line\\RMS}} \underset{\substack{\downarrow\\\text{Ohm's law}\\\text{for phasors}}}{=} \hat{Z}_{line} \cdot \underbrace{\hat{I}_{\substack{Line\\RMS}} \cdot \hat{I}^*_{\substack{Line\\RMS}}}_{I^2_{\substack{Line\\RMS}}} = I^2_{\substack{Line\\RMS}} \underset{\substack{\downarrow\\(1+j)}}{\hat{Z}_{Line}}$$

$$\hat{S}_{Line} = I^2_{\substack{Line\\RMS}} (1+j)$$

Now $I_{\substack{Line\\RMS}} = I_{\substack{L\\RMS}}$ for this circuit.

Recall: $P_L = I_{L_{RMS}} \underbrace{V_{L_{RMS}}}_{\substack{\Downarrow\\\text{given as } 110}} pf \rightarrow I_{\substack{L\\RMS}} = \frac{P_L}{V_{L_{RMS}}\, pf} = \frac{10000}{110 \times 0.5}$

$$I_{\substack{L\\RMS}} = \frac{20000}{110} = 181.82 \text{ A}$$

$$\hat{S}_{Line} = 181.82^2 (1+j)$$

1 iii) $\hat{S}_S = \hat{S}_{Line} + \hat{S}_L = 181.82^2(1+j) + 20000(\cos 60° + j \sin 60°)$

$= (181.82^2 + 20000 \cos 60°) + j(181.82^2 + 20000 \sin 60°)$

$= 43058.5 \quad + j\, 50379$

$$\boxed{\hat{S}_S = 66272.8 \angle 49.5° \text{ V.A}}$$

(59)

2) $$\boxed{\hat{V}_s = \frac{\hat{S}_s}{\hat{I}_s^*} = \frac{66272.8\angle 49.5^\circ}{181.82\angle 60^\circ} = 364.49\angle -10.5^\circ \; V}$$

$\hat{I}_s = \hat{I}_{line} = \hat{I}_L$ for our circuit

$$\hat{I}_L^* = \frac{\hat{S}_L}{\hat{V}_L} = \frac{20000\angle 60^\circ}{110\angle 0^\circ} = 181.82\angle 60^\circ$$

$$pf_{source} = \cos(\text{phase of } \hat{S}_s) = \cos(49.5^\circ) = 0.65$$