Relevant properties of L.T. for doing inverse L.T. using P.F.E (partial fraction expansion)

Property # 4 :

if $f(t) \xrightarrow{L.T.} \hat{F}(s)$

then $f(t-t_0)\, u(t-t_0)\,;\ t_0 \geq 0 \longrightarrow e^{-st_0}\hat{F}(s)$

Proof:

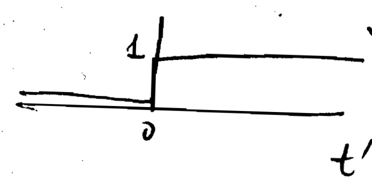
$$L.T.[f(t-t_0)\,u(t-t_0)] = \int_0^\infty dt\ f(t-t_0).\,u(t-t_0)\, e^{-st}\, e^{-st_0}\, e^{st_0}$$

$$= e^{-st_0} \underbrace{\int_0^\infty dt\ f(t-t_0)\, u(t-t_0)\, e^{-s(t-t_0)}}$$

$t - t_0 \to t'$ $\quad (t=\infty;\ t'=\infty)$

$dt = dt'$

$$\int_{-t_0}^\infty dt'\ f(t')\, \underbrace{u(t')}\, e^{-st'} = \underbrace{\int_0^\infty dt'\, f(t')\, e^{-st'}}$$



$$\downarrow$$

$$LT(f(t')) = \hat{F}(s)$$

$$= e^{-st_0}\hat{F}(s) \ ✓$$

Property #8:

$$\text{if } f(t) \longrightarrow \hat{F}(s)$$

$$LT[t f(t)] \longrightarrow -\frac{d}{ds}\hat{F}(s)$$

$$LT[t f(t)] = \int_0^\infty dt\ t f(t) e^{-st} = -\frac{d}{ds}\underbrace{\int_0^\infty dt\ f(t) e^{-st}}_{\hat{F}(s)} \checkmark$$

Partial Fraction Expansion (PFE)

1) Simple poles:
$$\frac{\hat{P}}{\hat{Q}} = \frac{\hat{K}_1}{s+p_1} + \frac{\hat{K}_2}{s+p_2} + \cdots + \frac{\hat{K}_n}{s+p_n}$$

($\hat{Q}$ is a polynomial of order n; $-p_1, -p_2, \cdots -p_n$ are the roots or solutions of $\hat{Q} = 0$

In this case: $\hat{K}_i = \left.\frac{(s+p_i)\hat{P}}{\hat{Q}}\right|_{s=-p_i}$ — obvious algorithm to find $\hat{K}_i$

$$(s+p_i)\frac{\hat{P}}{\hat{Q}} = (s+p_i)\left(\frac{\hat{K}_1}{s+p_1} + \frac{\hat{K}_2}{s+p_2} + \cdots \frac{\hat{K}_i}{s+p_i} + \cdots + \frac{\hat{K}_n}{s+p_n}\right)$$

All terms with $(s+p_i)$ will vanish after substitution $s=-p_i$. ~~except~~
One term carries no $(s+p_i)$ will remain, which give $\hat{K}_i$.

2) Complex conjugate poles:

$$\frac{\hat{P}}{\hat{Q}} = \frac{\hat{k}_1}{s+\alpha-j\beta} + \frac{\hat{k}_1^*}{s+\alpha+j\beta}$$

$-\alpha \pm j\beta$ are complex conjugate roots of $\hat{Q}=0$

In this case: $\hat{k}_1 = \left[(s+\alpha-j\beta)\frac{\hat{P}}{\hat{Q}}\right]_{s=-\alpha+j\beta}$

$\hat{k}_1^* =$ complex conjugate of $\hat{k}_1$

3) Multiple poles:

$$\frac{\hat{P}}{\hat{Q}} = \frac{\hat{k}_{11}}{s+p_1} + \frac{\hat{k}_{12}}{(s+p_1)^2} + \cdots + \frac{\hat{k}_{1r}}{(s+p_1)^r}$$

$\hat{Q}=0$ has one root (in this expression) p_1 with multiplicity r ($(x+1)^3$ has one root at -1 with multiplicity or order 3)

$\hat{k}_{1r} = \left[(s+p_1)^r \frac{\hat{P}}{\hat{Q}}\right]_{s=-p_1}$ (same as before)

$\hat{k}_{1r-1} = \left[\frac{d}{ds}\left((s+p_1)^r \frac{\hat{P}}{\hat{Q}}\right)\right]_{s=-p_1}$

$\vdots$

$$(s+p_1)^r\left(\frac{\hat{P}}{\hat{Q}}\right) = (s+p_1)^{r-1}\hat{k}_{11} + (s+p_1)^{r-2}\hat{k}_{12} + \cdots + (s+p_1)\hat{k}_{1r-1} + \hat{k}_{1r}$$

PFE for $\hat{F}(s) = \dfrac{10(s+3)}{(s+1)^3(s+2)}$:

$$= \frac{\hat{k}_{11}}{s+1} + \frac{\hat{k}_{12}}{(s+1)^2} + \frac{\hat{k}_{13}}{(s+1)^3} + \frac{\hat{k}_2}{s+2}$$

$$\hat{k}_{13} = \left[(s+1)^3 \hat{F}(s)\right]_{s=-1} = \left[\frac{10(s+3)}{s+2}\right]_{s=-1} = \frac{10.2}{1} = 20$$

$$\hat{k}_{12} = \left[\frac{d}{ds}\left(\frac{10(s+3)}{s+2}\right)\right]_{s=-1} = \left[\frac{10}{s+2} - \frac{10(s+3)}{(s+2)^2}\right]_{s=-1} = 10 - 20 = -10$$

$$\hat{k}_{11} = \left[\frac{d}{ds}\left(\frac{10}{s+2} - \frac{10(s+3)}{(s+2)^2}\right)\right]_{s=-1} = \left[\frac{-10}{(s+2)^2} - \frac{10}{(s+2)^2} + \frac{20(s+3)}{(s+2)^3}\right]_{s=-1}$$

$$= 20$$

$$\hat{k}_2 = \left[(s+2)\hat{F}(s)\right]_{s=-2} = \left[\frac{10(s+3)}{(s+1)^3}\right]_{s=-2} = -10$$

$$\frac{10(s+3)}{(s+1)^3(s+2)} = \frac{20}{s+1} + \frac{-10}{(s+1)^2} + \frac{20}{(s+1)^3} + \frac{-10}{s+2}$$

What is $LT^{-1}\left[\hat{F}(s)\right]$?

$$f(t) = 20\,LT^{-1}\left(\frac{1}{s+1}\right) - 10\,LT^{-1}\frac{1}{(s+1)^2} + 20\,LT^{-1}\frac{1}{(s+1)^3} - 10\,LT^{-1}\frac{1}{s+2}$$

$$= 20e^{-t}u(t) - 10te^{-t}u(t) + 10t^2e^{-t}u(t) - 10e^{-2t}u(t)$$

Property #4: $f(t-t_0)\,u(t-t_0) \longrightarrow e^{-st_0}\hat{F}(s)$

Dual property: (time & freq are equivalent)

$$e^{-at}f(t) \longleftarrow \hat{F}(s+a)$$

$$e^{-t}u(t) \longleftarrow \frac{1}{s+1} \quad \left(\text{since } u(t) \leftarrow \frac{1}{s}\right)$$

Property #8 :

$$t\, f(t) \longleftarrow -\frac{d}{ds}\hat{F}(s)$$

$$te^{-t}u(t) \longleftarrow \frac{1}{(s+1)^2}$$

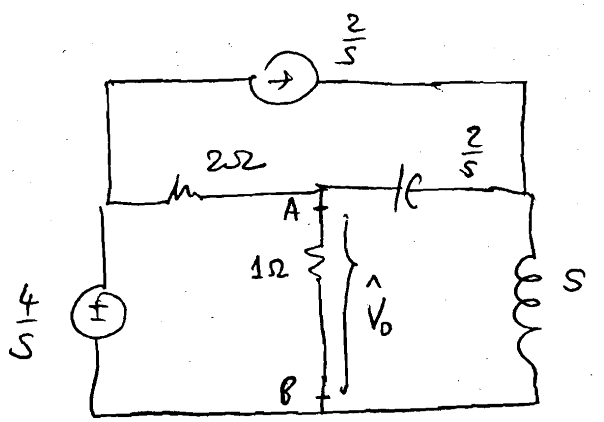
$$\frac{1}{(s+1)^3} = -\frac{1}{2}\frac{d}{ds}\frac{1}{(s+1)^2}$$

$$\frac{1}{2}t^2e^{-t}u(t) \longleftarrow \frac{1}{(s+1)^3}$$

$$f(t) = u(t)\left[20e^{-t} - 10te^{-t} + 10t^2e^{-t} - 10e^{-2t}\right]$$

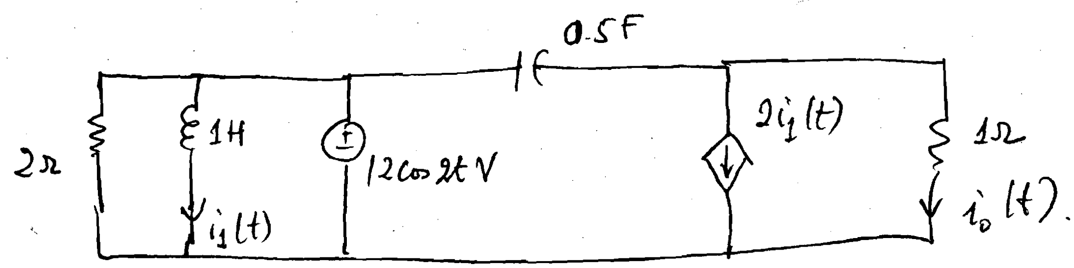
$$= \cancel{u(t)[10e^{t} - 10te^{t} + 10t^2e^{-t}}$$

13.26 :



Find $\hat{V}_0$ then find $v_0(t)$

13.62 :



Find $\hat{I}_0$, then $i_0(t)$

If $\hat{Q}(s)=0$ has p_1 & p_2 as roots $\Rightarrow \hat{Q}(s)=(s-p_1)(s-p_2)$ (91)

(12.18)

a) $F(s)=\dfrac{10}{s^2+2s+2}$; b) $F(s)=\dfrac{10(s+2)}{s^2+4s+5}$

a) $s^2+2s+2=0 \rightarrow s=\dfrac{-2\pm\sqrt{4-8}}{2}=-1\pm j$

$$\frac{10}{s^2+2s+2}=\frac{\hat{k}_1}{s+1-j}+\frac{\hat{k}_1^*}{s+1+j}$$

$$\hat{k}_1=\left[(s+1-j)\,\hat{F}(s)\right]_{s=-1+j}=\left[\frac{10}{s+1+j}\right]_{s=-1+j}=\frac{10}{2j}=-5j$$

$$\left[\cancel{(s+1-j)}\frac{10}{\cancel{(s+1-j)}(s+1+j)}\right]_{s=-1+j}$$

$$\rightarrow \frac{10}{s^2+2s+2}=\frac{-5j}{s+1-j}+\frac{5j}{s+1+j}$$

<u>Back to time domain</u>:

$$-5j\,e^{-(1-j)t}u(t)+5j\,e^{-(1+j)t}u(t)$$

$$=-5j\,e^{-t}u(t)\underbrace{\left[e^{jt}-e^{-jt}\right]}_{2j\sin t}=\boxed{10\,e^{-t}\sin(t)\,u(t)}$$

$$\left.\begin{aligned}e^{j\alpha}&=\cos\alpha+j\sin\alpha\\ e^{-j\alpha}&=\cos\alpha-j\sin\alpha\end{aligned}\right\}\quad \begin{aligned}e^{j\alpha}-e^{-j\alpha}&=2j\sin\alpha\\ e^{j\alpha}+e^{-j\alpha}&=2\cos\alpha\end{aligned}$$

13.26 13.62 11.54 11.57. 11.59.

b) $F(s) = \dfrac{10(s+2)}{s^2+4s+5}$: $s^2+4s+5=0 \rightarrow s = \dfrac{-4 \pm \sqrt{16-20}}{2}$

$$= -2 \pm j$$

$$\frac{10(s+2)}{s^2+4s+5} = \frac{\hat{k}_1}{s+2-j} + \frac{\hat{k}_1^*}{s+2+j}$$

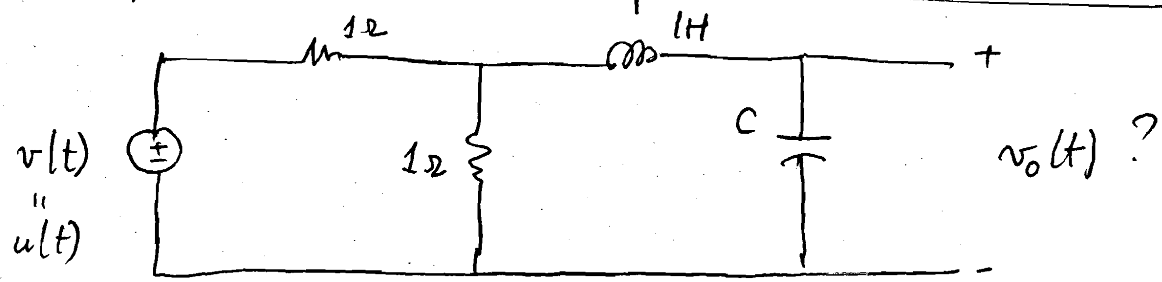
$$\hat{k}_1 = \left[\cancel{(s+2-j)}\frac{10(s+2)}{\cancel{(s+2-j)}(s+2+j)}\right]_{s=-2+j} = \frac{10j}{2j} = 5$$

$$\hat{k}_1^* = 5$$

$$\rightarrow f(t) = 5e^{-(2-j)t}u(t) + 5e^{-(2+j)t}u(t)$$

$$= 5e^{-2t}u(t)\underbrace{\left[e^{jt}+e^{-jt}\right]}_{2\cos t} = 10e^{-2t}\cos t\, u(t)$$

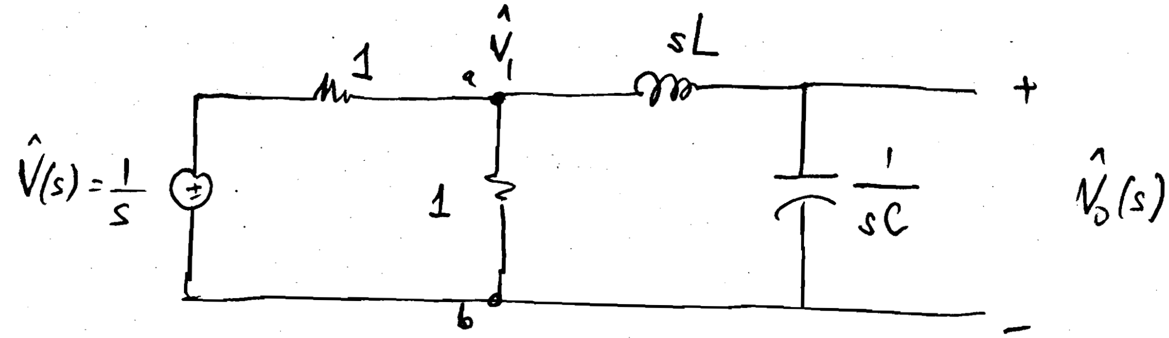
Application of Laplace Transform to AC circuit analysis.



Find $\hat{V}_0(s)$, then $LT^{-1}[\hat{V}_0(s)] = v_0(t)$

↓

Redraw circuit with phasors



$(j\omega = s)$

$$\hat{V}_1 = \frac{\dfrac{1\left(sL+\frac{1}{sC}\right)}{1+sL+\frac{1}{sC}}}{1+\dfrac{sL+\frac{1}{sC}}{1+sL+\frac{1}{sC}}}\;\frac{1}{s}$$

$$\hat{V}_o = \frac{\frac{1}{sC}}{sL+\frac{1}{sC}}\,\hat{V}_1 = \frac{\frac{1}{sC}}{\cancel{sL+\frac{1}{sC}}}\;\frac{1}{s}\;\frac{\dfrac{\cancel{sL+\frac{1}{sC}}}{1+sL+\frac{1}{sC}}}{1+\dfrac{sL+\frac{1}{sC}}{1+sL+\frac{1}{sC}}}$$

$$= \frac{1}{sC}\cdot\frac{1}{s}\;\frac{1}{\left(1+sL+\frac{1}{sC}+sL+\frac{1}{sC}\right)}$$

$$= \frac{1}{sC}\cdot\frac{1}{s+2s^2L+\frac{2}{C}} \overset{L=1}{=} \frac{1}{sC\left(s+2s^2+\frac{2}{C}\right)}$$

$$= \frac{\frac{1}{2}}{sC\underbrace{\left(s^2+\frac{s}{2}+\frac{1}{C}\right)}} \rightarrow \text{P.F.E.}$$

$$s^2+\frac{s}{2}+\frac{1}{C}=0 \rightarrow s = \frac{-\frac{1}{2}\pm\sqrt{\frac{1}{4}-\frac{4}{C}}}{2}$$

$C = 4F$ $s = -\frac{1}{4}(1 \pm j\sqrt{3})$ (complex conj.)

$C = 16F$ $s = -\frac{1}{4}$ (double)

$C = 32F$ $s = -\frac{1}{4}\left(1 \pm \frac{1}{\sqrt{2}}\right)$ (two simple poles)

E.g. if $C = 16F$ $\quad \left(s^2 + \frac{s}{2} + \frac{1}{16}\right) = \left(s + \frac{1}{4}\right)^2$

$$\hat{V}_o(s) = \frac{\frac{1}{2}}{s16\left(s+\frac{1}{4}\right)^2} \overset{P.F.E.}{=} \left(\frac{k_2}{s} + \frac{k_{11}}{\left(s+\frac{1}{4}\right)} + \frac{k_{12}}{\left(s+\frac{1}{4}\right)^2}\right)$$

$$k_2 = \left[s\hat{V}_o(s)\right]_{s=0} = \left[\frac{\frac{1}{32}}{\left(s+\frac{1}{4}\right)^2}\right]_{s=0} = \frac{1}{2}$$

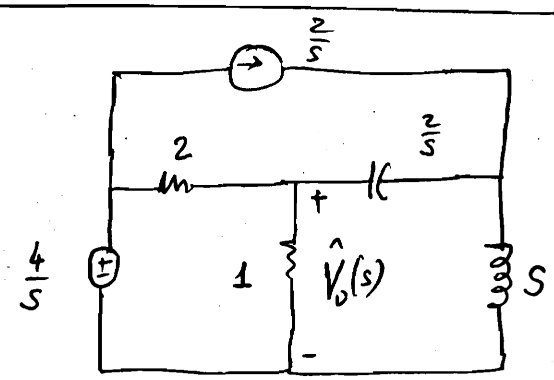
$$k_{12} = \left[\left(s+\frac{1}{4}\right)^2\hat{V}_o(s)\right]_{s=-\frac{1}{4}} = \left[\frac{\frac{1}{32}}{s}\right]_{s=-\frac{1}{4}} = -\frac{1}{8}$$

$$k_{11} = \left[\frac{1}{32}\frac{d}{ds}\left(\frac{1}{s}\right)\right]_{s=-\frac{1}{4}} = \left[-\frac{1}{32}\frac{1}{s^2}\right]_{s=-\frac{1}{4}} = -\frac{1}{2}$$

$$\rightarrow \hat{V}_o(s) = \frac{\frac{1}{2}}{s} - \frac{\frac{1}{2}}{s+\frac{1}{4}} - \frac{\frac{1}{8}}{\left(s+\frac{1}{4}\right)^2}$$

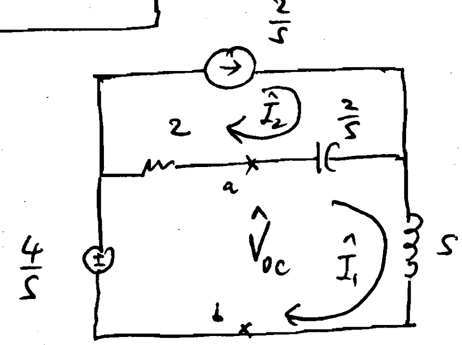
$$\boxed{v(t) = \frac{1}{2}u(t) - \frac{1}{2}e^{-\frac{1}{4}t}u(t) - \frac{1}{8}te^{-\frac{1}{4}t}u(t)}$$

13.26



$\hat{V}_o(s)$? $\rightarrow$ $v_o(t)$

Thevenin Equiv. $\quad \hat{V}_{oc}$:



1) $\hat{I}_2 = \frac{2}{s}$

2) $\frac{4}{s} = \left(2+\frac{2}{s}\right)\left(\hat{I}_1 - \hat{I}_2\right) + \hat{I}_1 s$

$\hat{V}_{oc} = \frac{4}{s} - 2\left(\hat{I}_1 - \hat{I}_2\right)$

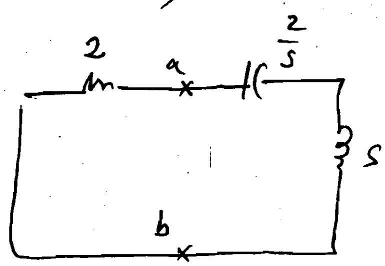
Solve for $\hat{I}_1$: $\frac{4}{s} = \left(2+s+\frac{2}{s}\right)\hat{I}_1 - \left(2+\frac{2}{s}\right)\frac{2}{s}$

$$\hat{I}_1 = \left(\frac{\frac{4}{s} + \frac{2}{s}\left(2+\frac{2}{s}\right)}{2+s+\frac{2}{s}}\right) = \frac{8s+4}{s(s^2+2s+2)}$$

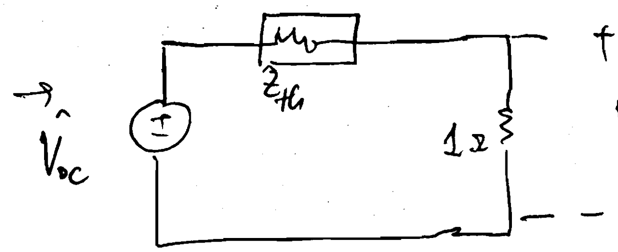
$$\hat{V}_{oc} = \frac{4}{s} - 2\left(\frac{8s+4}{s(s^2+2s+2)} - \frac{2}{s}\right) = \frac{8}{s} - \frac{8(2s+1)}{s(s^2+2s+2)}$$

$$= \frac{8s^2+8}{s(s^2+2s+2)}$$

$\hat{Z}_{th} =$



$$\hat{Z}_{th} = \frac{2\left(\frac{2}{s}+s\right)}{2+\frac{2}{s}+s}$$



$$\rightarrow \hat{V}_o = \frac{4(s^2+1)}{\underbrace{s(s^2+s+3)}}$$

$s^2+s+3=0$

$s = \frac{-1 \pm \sqrt{1-12}}{2}$

$s = -\frac{1}{2} \pm j\frac{\sqrt{11}}{2}$

$\rightarrow$ PFE: $\hat{V}_o(s) = \frac{k_2}{s} + \frac{\hat{k}_1}{s+\frac{1}{2}-j\frac{\sqrt{11}}{2}} + \frac{\hat{k}_1^*}{s+\frac{1}{2}+j\frac{\sqrt{11}}{2}}$

$$V_O = \frac{\mathbf{V}_{OC}(1)}{1+Z_{TH}} = \frac{8S+8/S}{S^2+2S+2+2S^2+4}$$

$$V_O = \frac{(8/3)(S^2+1)}{S(S^2+(2/3)S+2)}$$

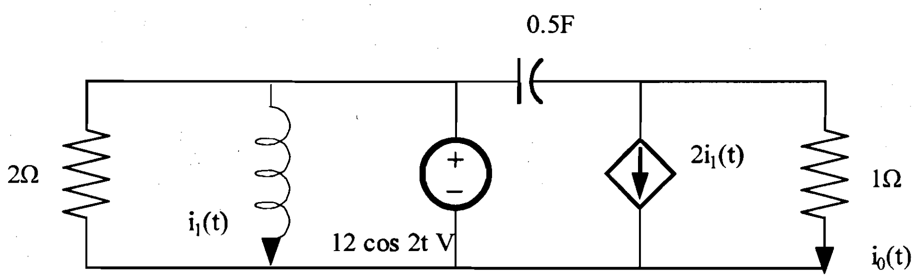
$$A = 4/3$$

$$K\underline{|\theta} = 1.273\underline{|10.05^0}$$

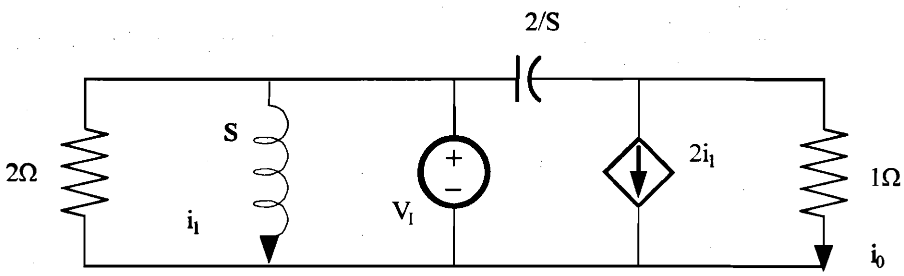
$$V_O(t) = (\frac{4}{3} + 2.546e^{-t/3}COS(\sqrt{17t} + 10.05^0))V$$

Problem 13.62

Find the steady state response Io(t) for the network shown the fig.



Suggested Solution



KCL

$$\frac{V_0}{1}+2I_1+(V_0-V_I)\frac{s}{2}=0$$

$$I_0=V_0/1, I_1=V_I/s$$

$$I_0+\frac{2V_I}{s}+I_0(\frac{s}{2})-V_I(\frac{s}{2})=0$$

or

$$I_0(1+\frac{s}{2})=V_I(\frac{s}{2}-\frac{2}{s})=V_I(\frac{s^2-4}{2s})$$

$$I_0(\frac{s+2}{2})=V_I(\frac{s^2-4}{2s})$$

Finally

$$\frac{I_0}{V_I}=\frac{s-2}{s}$$

at steady state s=j2

$V_i(t) = 12\cos(2t)$

$$\left.\frac{I_0}{V_I}\right|_2 = \frac{j2-2}{j2} = \sqrt{2}\angle 45^o$$

and

$$I_0 = |V_I|\sqrt{2}\angle 45^o = 12\sqrt{2}\angle 45^o$$

$$I_0 = 12\sqrt{2}\cos(2t + \angle 45^o)A$$