Magnetically Coupled Circuits

(7th ed.: Ch. 8 or 8th ed.: Ch. 10)

So far, inductance $L$ is actually self-inductance: we haven’t considered magnetic interaction, leading to mutual inductance.

Coil #1: $\mathbf{B}$
- Current: $I_1$
- # turns: $N_1$
- Cross section: $S_1$
- Length: $l_1$

Magnetic flux: $\Phi = \mathbf{B} \cdot \mathbf{S}$ (larger field and larger cross section $\Rightarrow$ larger flux)

Faraday’s law: \[ \text{induced e.m.f.} = \frac{d\Phi}{dt} \] potential

\[ V = \frac{d\Phi}{dt} = \frac{1}{\mu} \oint (\mathbf{B} \cdot \mathbf{S}) = \frac{d}{dt} (\Phi) = S \frac{d\mathbf{B}}{dt} = S \dot{B} (\mu I) \]
\[ V = \sum \frac{di}{dt} \quad \text{This is what we have been using so far.} \]

\[ V_i = L_1 \frac{dI_1}{dt} \quad \text{one coil, one current} \]

→ Two coils, two currents \( I_1 \) & \( I_2 \):

\[ V_i = L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt} \]

\[ \text{self-inductance} \]

\[ \text{mutual-inductance} \]

due to magnetic interaction between the two coils.

Similarly: \( V_2 = L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt} \quad (M_{12} = M_{21}) \)

Conclusion: for our purposes of doing circuit analysis, the implication of including magnetic interaction is the consideration of one extra term with the correct sign: using the "dot convention".

Correct sign for the mutual inductance term (dot convention):

\[ \begin{align*}
V_i &= L_1 \frac{di}{dt} + M \frac{dI_2}{dt} \\
V_2 &= L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt}
\end{align*} \]

\[ \begin{align*}
\text{Feed \( I_1 \) \& \( I_2 \) \quad \text{currents go into dots, then coils} \\
\text{potential polarities as shown}
\end{align*} \]

\[ \begin{align*}
\hat{V}_i &= j\omega L_1 \hat{I}_1 + j\omega M \hat{I}_2 \\
\hat{V}_2 &= j\omega L_2 \hat{I}_2 + j\omega M \hat{I}_1
\end{align*} \]
\[ v_1 = L_1 \frac{di_1}{dt} - M \frac{di_2}{dt}, \quad v_2 = -L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} \]

\[ \hat{V}_1 = j \omega L_1 \hat{I}_1 - j \omega M \hat{I}_2, \quad \hat{V}_2 = -j \omega L_2 \hat{I}_2 + j \omega M \hat{I}_1 \]

\[ v_1 = L_1 \frac{di_1}{dt} - M \frac{di_2}{dt}, \quad v_2 = L_2 \frac{di_2}{dt} - M \frac{di_1}{dt} \]

\[ \hat{V}_1 = j \omega L_1 \hat{I}_1 - M j \omega \hat{I}_2, \quad \hat{V}_2 = j \omega L_2 \hat{I}_2 - M j \omega \hat{I}_1 \]

Loop analysis:
1) \[ 24 = 2 \hat{I}_1 + 4 j \hat{I}_2 - 2 j \hat{I}_1 \]
   New!!

2) \[ 0 = 6 j \hat{I}_2 - 2 j \hat{I}_1 + (2 - 2 j) \hat{I}_2 \]
   New!!

25/11 HW3 \[ \hat{V}_0 = 2 \hat{I}_2 = \frac{3.43}{\sqrt{3}} \text{V} \]
\[ \dot{v} = 2 \Re - j \dot{z} = \frac{-2j}{z-j} = \frac{2}{2j+1} \]

\[ \frac{(2+6j)}{2j+1} \]

\[ -12 + \frac{2(2+6j)}{1+2j} \]

Voltage drop:
\[ \dot{V}_0 = \left[-12 + \frac{2(2+6j)}{1+2j} \right] \frac{2}{2+\frac{2}{2j+1}} \]

\[ = \left[-12(1+2j) + 2(2+6j) \right] \frac{2}{2+4j+2} \]
\[ V_0 = \left[ -8 - 12j \right] \frac{2}{4 \left( 1 + j \right)} \]

\[ = \frac{\left( -2 - 3j \right)^2}{1 + j} \]

\[ = 2 \frac{\left( 2 + 3j \right)}{\left( 1 + j \right)} \]

\[ = -2 \frac{180^\circ}{\sqrt{13}} \left( \tan^{-1} \frac{3}{2} \right)^{58^\circ} \frac{1 + j}{19.13^\circ} \]

7.63

Use Thévenin's equivalent.

\[ V_{\text{oc}} = V_{\text{o.c.}} = 4V \]

\[ Z_{\text{th}} = Z_{\text{y.w}} \text{ at } Z_{\text{oc}}, \text{ no sources.} \]

\[ V_{\text{oc}} = 6A \]

\[ V_{\text{o.c.}} = 12V \]

\[ V_{\text{oc}} = 22V \]

\[ \hat{V}_{\text{o.c.}} \]
\[ V_{oc'} = 0 \]

\[ V_{oc''} = 12 \left( \frac{2j}{1j} \right) = 24V \]

\[ V_{th} = 24 V \]

\[ Z_{th} = \]

\[ Z_{th} = 2 + \frac{2j(-j)}{2j} = 2 + \frac{2}{j} \]

\[ = 2 - 2j \]

\[ V_o = 24 \left( \frac{1}{3 - 2j} \right) = \frac{24}{\sqrt{13} - \text{tan}^{-1}\frac{2}{3}} = \frac{24}{\sqrt{13}} \]

\[ = 6.66 \sqrt{33.69^\circ} \]

(solution is good)
Mutual Induction (Cont.)

Energy analysis:

\[ + \quad i_2 \quad + \]
\[ \quad v_1 \quad L_1 \quad v_2 \quad \]

Power \( P = \text{energy per unit time} \implies \mathcal{E} = \int P(t) \, dt \)

We look at two time intervals:

\[ 0 \quad \rightarrow \quad t_1 \quad \rightarrow \quad t_2 \]
\[ I_1 \quad \rightarrow \quad \text{const} \rightarrow \quad I_1^{(\text{max})} \]
\[ i_2 \quad \rightarrow \quad 0 \quad \rightarrow \quad \text{const} \rightarrow \quad I_2^{(\text{max})} \]

At any time: \( P(t) = i_1 v_1 + i_2 v_2 \)

\[ = i_1(t) \left[ L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \right] + i_2(t) \left[ L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} \right] \]

\[ \mathcal{E} = \int_0^{t_1} P(t) \, dt + \int_{t_1}^{t_2} P(t) \, dt = \int_0^{t_1} L_1 \frac{di_1}{dt} \, dt + \int_{t_1}^{t_2} \left( I_1 M \frac{di_1}{dt} - L_2 \frac{di_2}{dt} \right) \, dt + L_1 i_2(t) \, dt \]

\[ \mathcal{E} = L_1 \int_0^{t_1} i_1 \, di_1 + M I_1 \int_0^{t_1} di_2 + L_2 \int_0^{t_2} i_2 \, di_2 \]

\[ = L_1 \frac{I_1^2}{2} + M I_1 I_2 + L_2 \frac{I_2^2}{2} \]

\( \rightarrow \mathcal{E} = \frac{1}{2} L_1 i_1^2 + \frac{1}{2} L_2 i_2^2 + \frac{1}{2} M (I_1 i_2 + I_2 i_1) \)
Ideal Transformer:

Two coils + iron core

To change the voltage using different number of turns

\[ v_1 = N_1 \frac{d\phi_1}{dt} \]
\[ v_2 = N_2 \frac{d\phi_2}{dt} \]

\[ \phi_1 = \phi_2 \text{ (iron core)} \]

Faraday's Law: change of magnetic flux w.r.t. time \( \rightarrow \) induced voltage.

Iron core: similar to a short-circuit for electrons (electric fields), the iron core channels all magnetic field lines.

\[ v_1 \rightarrow \frac{v_1}{v_2} = \frac{N_1}{N_2} \]

Ampere's Law:

\[ \oint \vec{H} \cdot d\vec{l} = \mu_0 \left( \text{current by loop} \right) = i_1 N_1 + i_2 N_2 \]

\[ \vec{B} = \mu \vec{H} \text{ m. field} \]

m. induction \( m. \) permeability

\[ \mu = \frac{1}{\mu_0} \text{ for vacuum} \]
\[ \mu = \infty \text{ for iron core} \]

If \( \mu = \infty \Rightarrow \) for a finite value for \( B \) \( H \) has to be \( 0 \).

\[ \Phi = \Phi_0 \text{ for a finite value of } B \]
Ideal transformer in a circuit:

\[
\frac{N_1}{N_2} = \left\{ \begin{array}{l}
\frac{v_1}{v_2} = \frac{N_1}{N_2} \\
\frac{i_1}{i_2} = \frac{N_2}{N_1}
\end{array} \right.
\]

HW 4: due 3/13

\(8.29\)

1) \(3j \hat{I}_1 + 2 \hat{I}_1 - \hat{I}_2 = 0\)
   \(\hat{I}_1\) goes into \(L_1\) then \(L_1\)
   \(\hat{I}_2\) goes into \(L_2\)

2) \(-j \hat{I}_2 + 10 \frac{L_{30}^2}{-j} \hat{I}_1 = 0\)

3) \(-10 \frac{L_{30}^2}{-j} + \hat{I}_3 (1-j) + 10 = 0\) \(\hat{I}_3 = \frac{-10 + 10 \frac{L_{30}^2}{1-j}}{-1-j} = 3.66 \frac{L_{30}^2}{152^2}\)

\[\hat{V}_0 = 3.66 \frac{L_{30}^2}{152} + 10 = \frac{152}{876} \frac{L_{30}^2}{158^2}\]

\(8 \cos(100t)\) \(V\) \(L_1\) \(Z\) \(320 \Omega\)

Then solve for \(\hat{I}_2 = \)

\[K = \frac{M^2}{L_1 L_2} : \text{coupling coeff.}\]

\[k = 0.8 ; L_1 = L_2 = 6H\]