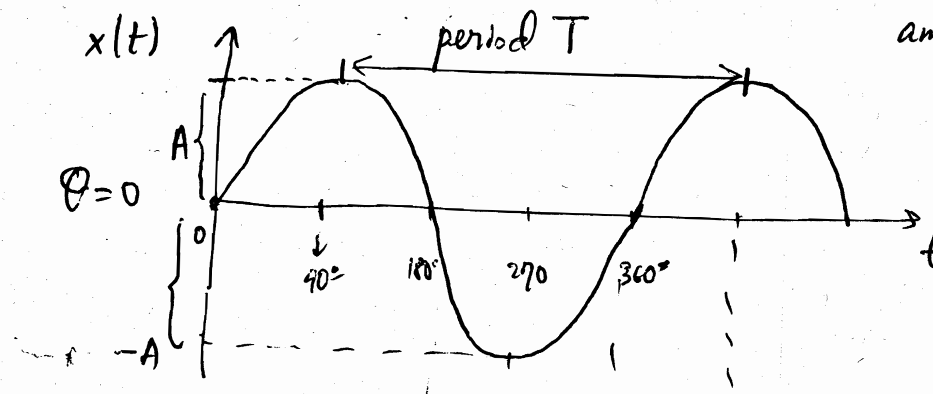
Circuit Analysis II   Sp' 07

AC Steady-state Analysis

AC → Alternate current: Sinusoid: $x(t) = \underbrace{A}_{\text{amplitude}} \sin(\underbrace{\omega}_{\substack{\text{angular}\\\text{frequency}}} t + \underbrace{\theta}_{\text{phase}})$

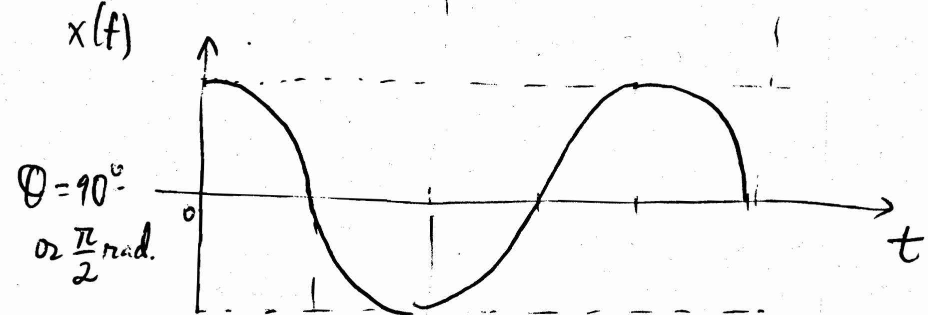


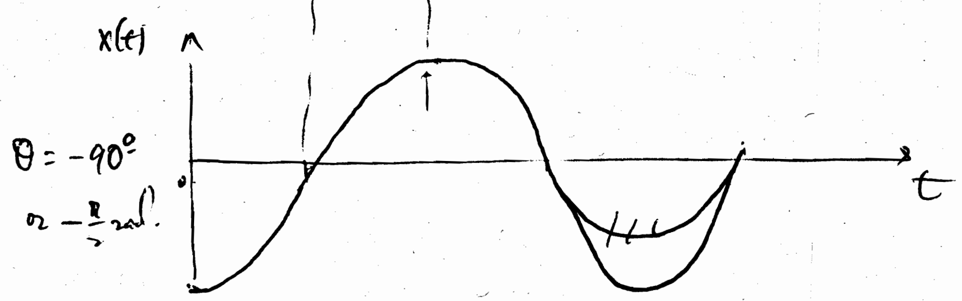
T: how many seconds per cycle ; f: frequency : how many cycles per second → $f = \frac{1}{T}$

ω: angular frequency = $2\pi f$ (omega)

Standard AC : $f = 60$ Hz ↔ $\omega = 377\ s^{-1}$

θ: phase (in radian) (theta)



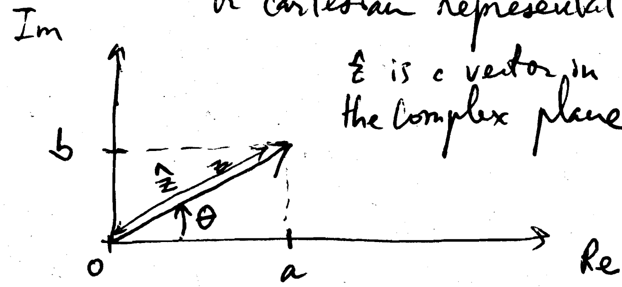


a $\theta = 180°$ or π rad is a minus sign.

→ Sinusoids : A, ω, θ

→ Complex numbers : $\hat{z} = a + jb$ $\begin{cases} a = Re(\hat{z}) : \text{real part} \\ b = Im(\hat{z}) : \text{imaginary part} \end{cases}$

↳ rectangular or cartesian representation.



$\hat{z}$ is a vector in the complex plane

$\hat{z} = z^{\angle\theta}$: polar representation

$\boxed{\text{Rectangular}}$ $\longrightarrow$ $\boxed{\text{polar}}$

$\hat{z} = (a, b)$ $\quad$ $\hat{z} = (z, \theta)$: $\begin{cases} z = \sqrt{a^2 + b^2} & \text{Pythagoras Theorem} \\ \theta = \tan^{-1}\left(\frac{b}{a}\right) \end{cases}$

$\longleftarrow$

$\begin{cases} a = z\cos\theta \\ b = z\sin\theta \end{cases}$

So why complex numbers?

<u>Linear</u> circuits : (R, L, C) : ω is same for all elements!

→ $A, \theta \xrightarrow[\text{represented by}]{} \hat{z} = (z, \theta)$

Sinusoids & complex numbers:

$$A\sin(\omega t+\theta) \overset{\checkmark}{=} A\,\mathrm{Im}\left[e^{j(\omega t+\theta)}\right] = \mathrm{Im}\left[A e^{j(\omega t+\theta)}\right]$$

$$\boxed{e^{j\alpha} = \cos\alpha + j\sin\alpha} \quad \text{or} \quad \mathrm{Im}\left[e^{j\alpha}\right] = \sin\alpha \; ; \; \mathrm{Re}\left[e^{j\alpha}\right] = \cos\alpha$$

polar form for: $1^{\angle\alpha}$ — rectangular form $1\cos\alpha + j\,1\sin\alpha$

$$\boxed{e^{(v+s)} = e^{v}\cdot e^{s}}$$

$$\to \mathrm{Im}\left[A e^{j\theta}\cdot \underbrace{e^{j\omega t}}_{\omega \text{ is same in linear circuits}}\right]$$

$\downarrow$

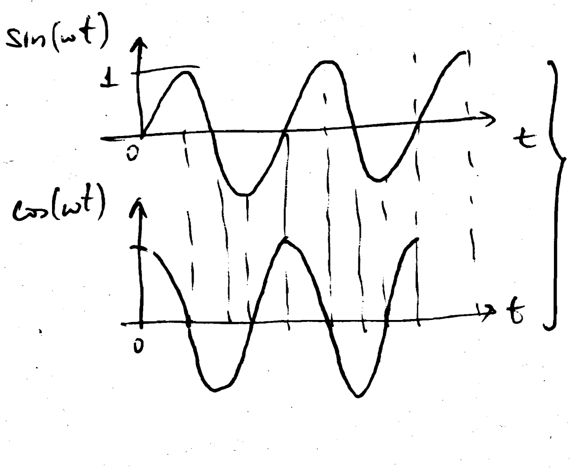
$\boxed{\text{Phasors}}$

We use phasors to represent sinusoids, phasors are clearly complex numbers: $Ae^{j\theta}$ is the polar form of a complex number of magnitude A & phase θ; is a vector in a complex plane

Let's compare 2 sinusoids

$$\begin{cases} v_1(t) = 12\sin(100t + 60^\circ) \\ v_2(t) = -6\cos(100t + 30^\circ) \end{cases}$$

$$= -6\sin(100t+120^\circ)$$
$$= 6\sin(100t - 60^\circ) =$$
$$= 6\sin(100t + 300^\circ)$$

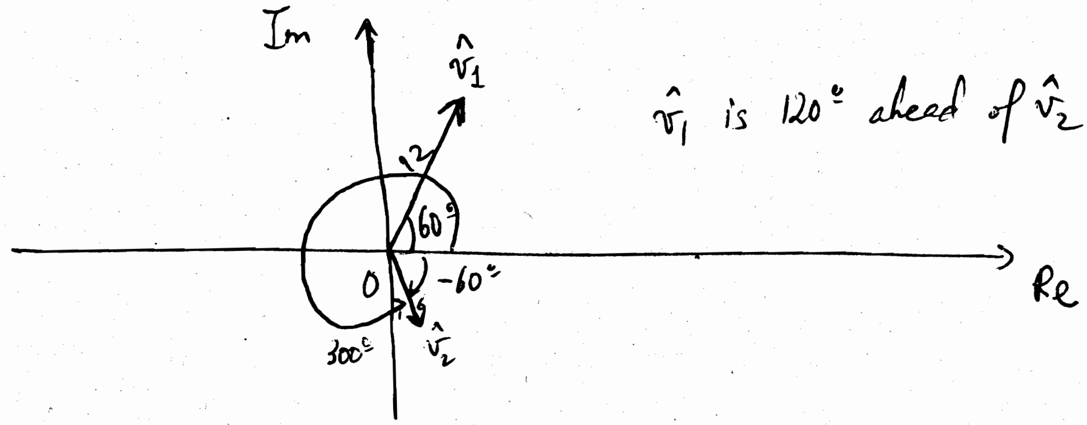


$\cos(\omega t) = \sin(\omega t + 90^\circ)$

$v_1(t) = 12 \sin(\underbrace{100}_{\omega}t + 60°)$

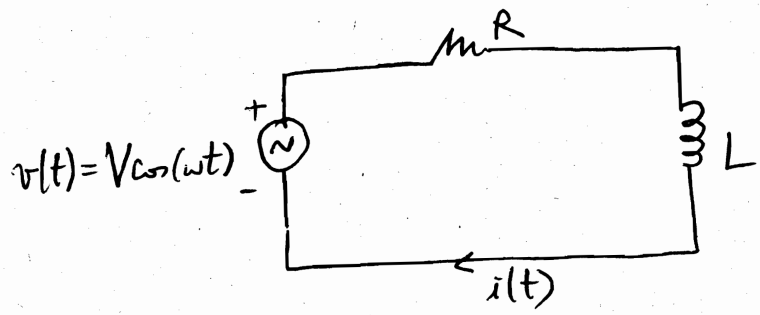
$$v_2(t) = 6 \sin(100t + 300°) = 6 \sin(\underbrace{100}_{\omega}t - 60°)$$

Represent as phasors = vectors in a complex plane:



$\hat{v}_1$ is 120° ahead of $\hat{v}_2$

Connection to circuits:



Let's solve for $i(t)$ in the time-domain or not using phasors

* Loop analysis: $V\cos\omega t = Ri + L\dfrac{di}{dt}$ (1)

→ Assume: $i(t) = \underset{?}{I} \cos(\underset{\substack{\downarrow \\ \text{same b/c linear circuit}}}{\omega} t + \underset{?}{\theta})$

$$i(t) = I\left[\cos\omega t \cos\theta - \sin\omega t \sin\theta\right]$$

$$\equiv I_1 \cos\omega t + I_2 \sin\omega t \qquad \begin{cases} I_1 \equiv I\cos\theta \\ I_2 \equiv -I\sin\theta \end{cases}$$

Now substitute into the loop equation:

$$V\cos\omega t = R\underbrace{[I_1\cos\omega t + I_2\sin\omega t]}_{i(t)} + L\omega[-I_1\sin\omega t + I_2\cos\omega t]$$

$$V\cos\omega t = \underbrace{(RI_1 + L\omega I_2}_{V})\cos\omega t + \underbrace{(RI_2 - L\omega I_1)}_{0}\sin\omega t$$

(remember $\sin(\omega t)$ & $\cos(\omega t)$ are out of phase by $90°$, which means they are perpendicular in phasors complex plane).

$$\left\{\begin{array}{l} RI_1 + L\omega I_2 = V \\ I_2 = \dfrac{L\omega}{R} I_1 \end{array}\right\} \rightarrow \boxed{\begin{array}{l} I_1 = \dfrac{R}{(L\omega)^2+R^2}V \\ I_2 = \dfrac{L\omega}{(L\omega)^2+R^2}V \end{array}}$$

$$\rightarrow i(t) = I_1\cos\omega t + I_2\sin\omega t = I\cos(\omega t + \theta) \quad \left\{\begin{array}{l} \boxed{I_1^2 + I_2^2 = I^2} \\ \dfrac{I_2}{I_1} = -\tan\theta \\ \hookrightarrow \boxed{\theta = -\tan^{-1}\dfrac{I_2}{I_1}} \end{array}\right.$$

$$= \sqrt{I_1^2+I_2^2}\cos\left(\omega t - \tan^{-1}\frac{I_2}{I_1}\right)$$

1) $$\sqrt{I_1^2+I_2^2} = \sqrt{\frac{R^2}{[(L\omega)^2+R^2]^2}V^2 + \frac{L^2\omega^2}{[(L\omega)^2+R^2]^2}V^2} = V\sqrt{\frac{R^2+L^2\omega^2}{[(L\omega)^2+R^2]^2}}$$

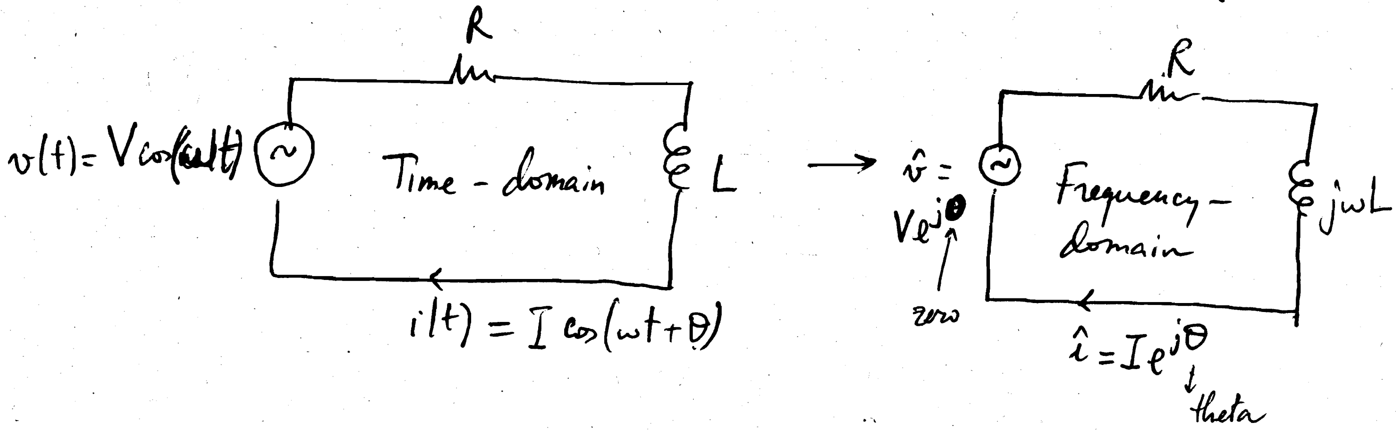
$$= \frac{V}{\sqrt{(L\omega)^2+R^2}}$$

2) $$-\tan^{-1}\frac{I_2}{I_1} = -\tan^{-1}\left(\frac{L\omega}{R}\right)$$

$$\boxed{\rightarrow i(t) = \frac{V}{\sqrt{(L\omega)^2+R^2}}\cdot\cos\left(\omega t - \tan^{-1}\left(\frac{L\omega}{R}\right)\right)}$$

Solution for the AC current in the RL circuit.

Now let's solve the same circuit using phasors $\left(A e^{j\theta}\right)$ or $\left(A \angle \theta\right)$



→ Phasor representation : same variable name, add a hat

$$v(t) = V\cos\omega t \longrightarrow \hat{v} = V e^{j0}$$

→ Impedance : $\hat{Z}$

Resistance in D.C : $R = \dfrac{V}{I}$ (Ohm's Law)

$\downarrow$

Impedance in A.C : $\hat{Z} = \dfrac{\hat{v}}{\hat{i}}$ (Ohm's Law for phasors)

At L : $\boxed{v_L = L \dfrac{di}{dt}}$

$$\underbrace{A\cos(\omega t + \theta)}_{\text{sinusoid}} = Re\left[A e^{j(\omega t + \theta)}\right] = Re\left[\overset{\text{phasor}}{A e^{j\theta}}\, e^{j\omega t}\right]$$

product of complex exponentials (Take Real part at the end).

$$v_L = L\frac{di}{dt}$$

$$\hat{v}_L e^{j\omega t} = L\frac{d}{dt}\left(\hat{i} e^{j\omega t}\right) = L\hat{i}\frac{d}{dt}\left(e^{j\omega t}\right) = Lj\omega\, \hat{i} e^{j\omega t}$$

$$\Rightarrow \hat{v}_L \cancel{e^{j\omega t}} = Lj\omega\, \hat{i}\, \cancel{e^{j\omega t}}$$

$$\frac{\hat{v}_L}{\hat{i}} = \underbrace{j\omega L}_{\downarrow\ \hat{Z}_L}$$: Ohm's law for phasors at the inductor.

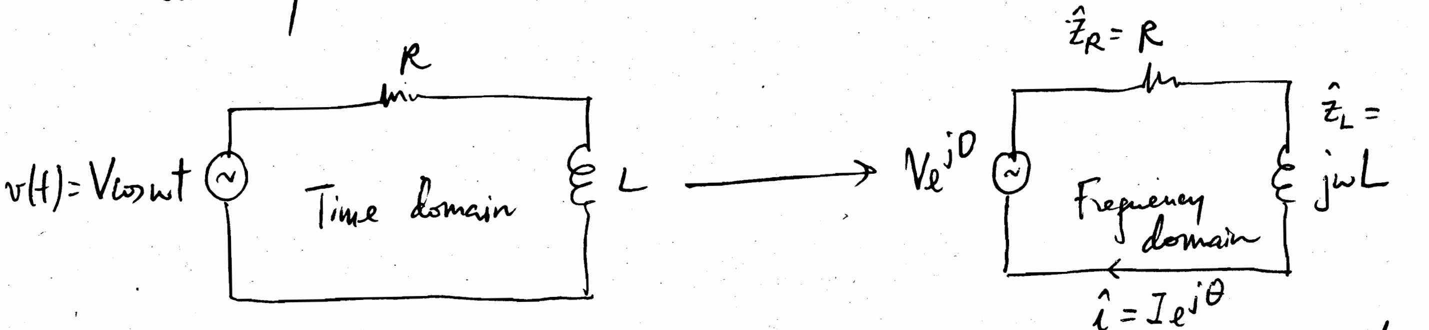
$\hat{Z}_L$: impedance at the inductor.

At R:

$$v_R = R\, i_R \longrightarrow \hat{v}_R \cancel{e^{j\omega t}} = R\, \hat{i}_R \cancel{e^{j\omega t}}$$

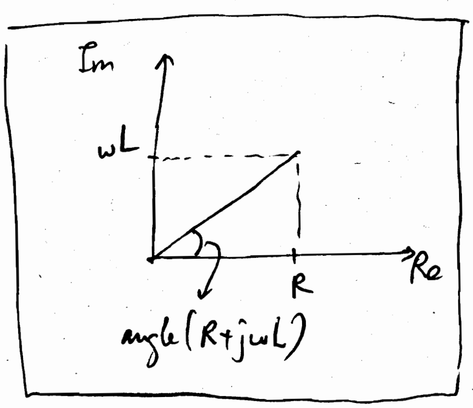
$$\hat{Z}_R = \frac{\hat{v}_R}{\hat{i}_R} = R.$$

In summary:



Ohm's law works for phasors:

$$\hat{i} = \frac{Ve^{j0}}{\hat{Z}_R + \hat{Z}_L} = \frac{V}{R + j\omega L} = \text{Magnitude of } \frac{V}{R+j\omega L}\ \angle \text{Angle of } \frac{V}{R+j\omega L}$$



$$\hat{i} = \frac{|V|}{|R+j\omega L|}\, e^{j[\text{Angle}(V) - \text{Angle}(R+j\omega L)]}$$

$$\hat{i} = \frac{V}{\sqrt{R^2 + (L\omega)^2}}\, e^{j[0 - \tan^{-1}\frac{\omega L}{R}]}$$

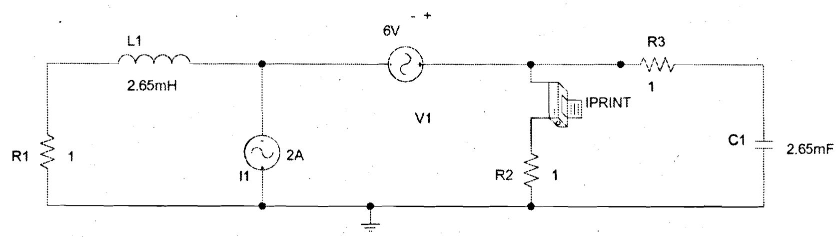
$$(\, i(t) = \text{Re}[\hat{i}\, e^{j\omega t}]\,)$$

$i(t) = ?$

$$i(t) = \text{Re}\left[\frac{V}{\sqrt{R^2+(L\omega)^2}}\, e^{-j\tan^{-1}\frac{\omega L}{R}} \cdot e^{j\omega t}\right] = \text{Re}\left[\underbrace{\frac{V}{\sqrt{R^2+(L\omega)^2}}}\, e^{j(\omega t - \tan^{-1}\frac{\omega L}{R})}\right]$$

$$\boxed{i(t) = \frac{V}{\sqrt{R^2 + (L\omega)^2}} \cos\left(\omega t - \tan^{-1} \frac{\omega L}{R}\right)}$$

We will find current through R2 for f=60Hz using PSPICE



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□
**** 02/06/07 13:45:58 *********** Evaluation PSpice (Nov 1999) **************

 * C:\Temp\232s07\Pspice01.sch

 ****     CIRCUIT DESCRIPTION

******************************************************************************



* Schematics Version 9.1 - Web Update 1
* Tue Feb 06 13:45:58 2007



** Analysis setup **
.ac LIN 1 60 60
.OP


* From [PSPICE NETLIST] section of pspiceev.ini:
.lib "nom.lib"

.INC "Pspice01.net"

**** INCLUDING Pspice01.net ****
* Schematics Netlist *



L_L1          $N_0001 $N_0002  2.65mH
R_R3          $N_0004 $N_0003  1
C_C1          0 $N_0003  2.65mF
R_R1          0 $N_0001  1
R_R2          0 $N_0005  1
I_I1          0 $N_0002 DC 0A AC 2A
V_V1          $N_0004 $N_0002 DC 0V AC 6V
V_PRINT1          $N_0004 $N_0005 0V

.PRINT        AC
+ IM(V_PRINT1)
+ IP(V_PRINT1)

**** RESUMING Pspice01.cir ****
.INC "Pspice01.als"



**** INCLUDING Pspice01.als ****
* Schematics Aliases *

.ALIASES
L_L1             L1(1=$N_0001 2=$N_0002 )
R_R3             R3(1=$N_0004 2=$N_0003 )
C_C1             C1(1=0 2=$N_0003 )
R_R1             R1(1=0 2=$N_0001 )
R_R2             R2(1=0 2=$N_0005 )
I_I1             I1(+=0 -=$N_0002 )
V_V1             V1(+=$N_0004 -=$N_0002 )
```

```
V_PRINT1          PRINT1(1=$N_0004 2=$N_0005 )

.ENDALIASES

**** RESUMING Pspice01.cir ****
.probe

.END

**** 02/06/07 13:45:58 *********** Evaluation PSpice (Nov 1999) **************

 * C:\Temp\232s07\Pspice01.sch

 ****     SMALL SIGNAL BIAS SOLUTION       TEMPERATURE =   27.000 DEG C

******************************************************************************

 NODE   VOLTAGE     NODE   VOLTAGE     NODE   VOLTAGE     NODE   VOLTAGE

($N_0001)    0.0000                   ($N_0002)    0.0000

($N_0003)    0.0000                   ($N_0004)    0.0000

($N_0005)    0.0000

    VOLTAGE SOURCE CURRENTS
    NAME         CURRENT

    V_V1         0.000E+00
    V_PRINT1     0.000E+00

    TOTAL POWER DISSIPATION   0.00E+00  WATTS

**** 02/06/07 13:45:58 *********** Evaluation PSpice (Nov 1999) **************

 * C:\Temp\232s07\Pspice01.sch

 ****     OPERATING POINT INFORMATION      TEMPERATURE =   27.000 DEG C

******************************************************************************

**** 02/06/07 13:45:58 *********** Evaluation PSpice (Nov 1999) **************

 * C:\Temp\232s07\Pspice01.sch

 ****     AC ANALYSIS                      TEMPERATURE =   27.000 DEG C
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**********************************************************************************

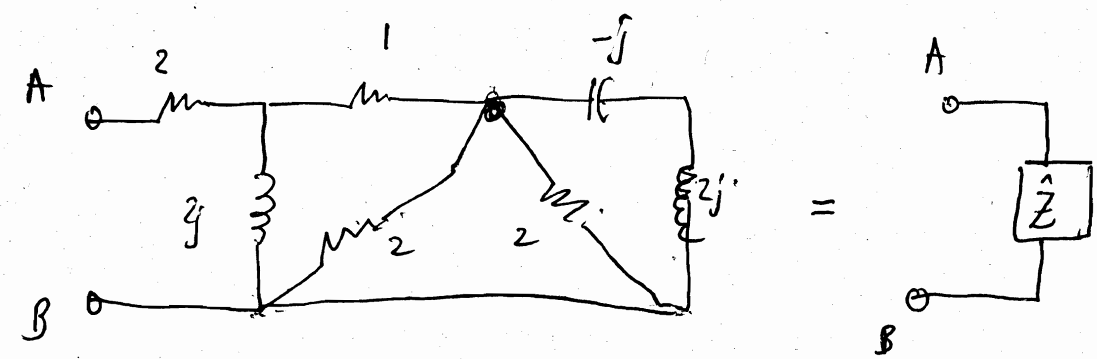
  FREQ        IM(V_PRINT1)IP(V_PRINT1)

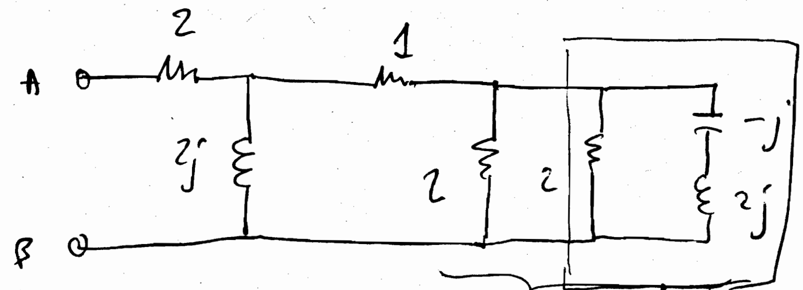
   6.000E+01   2.917E+00  -3.095E+01

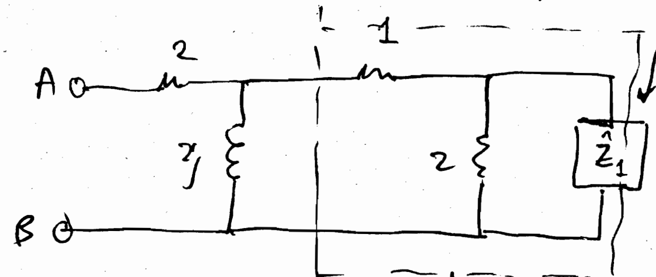
          JOB CONCLUDED

          TOTAL JOB TIME              .02
```

HW#1:

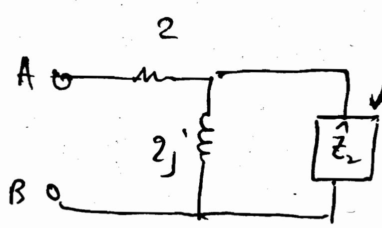






$$\hat{Z}_1 = 2 \parallel j = \frac{2j}{2+j}$$

$$\left(R_1 \parallel R_2 \Rightarrow \frac{1}{R_{\parallel}} = \frac{1}{R_1} + \frac{1}{R_2} \qquad R_{\parallel} = \frac{R_1 R_2}{R_1 + R_2} \right)$$



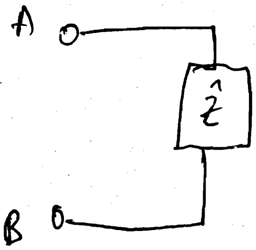
Now implement this using impedances

$$\hat{Z}_2 = 1 + \left(2 \parallel \frac{2j}{2+j} \right) = 1 + \frac{4j/(2+j)}{2 + \frac{2j}{2+j}}$$

$$= 1 + \frac{4j}{2(2+j) + 2j}$$

$$= 1 + \frac{4j}{4 + 4j} = 1 + \frac{j}{1+j}$$

$$\to \hat{z}_2 = \frac{1+j+j}{1+j} = \frac{1+2j}{1+j}$$



$$\to \hat{Z} = 2 + \left(2j \parallel \frac{1+2j}{1+j}\right) = 2 + \frac{\frac{2j-4}{1+j}}{2j + \frac{1+2j}{1+j}}$$

$\boxed{j^2 = -1}$ $\boxed{\frac{1}{j} = -j}$

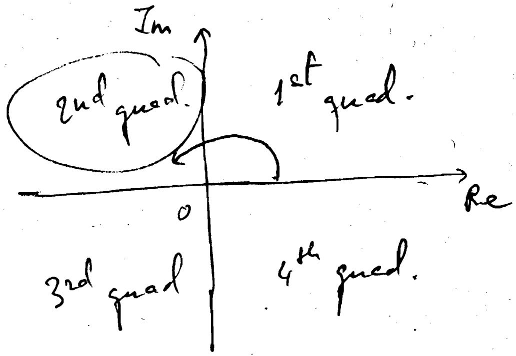
$$= 2 + \frac{2j-4}{2j-2+1+2j} = 2 + \frac{2j-4}{4j-1}$$

$$= \frac{-2+8j+2j-4}{4j-1} = \cancel{\frac{10j}{4j}} \ne \cancel{2.5\angle\tan^{-1}}$$

$$= \cancel{\sqrt{2.5^2+1^2}\angle\tan^{-1}\frac{1}{2.5}}$$

$$= \frac{-6+10j}{-1+4j} = \frac{\sqrt{36+100}\ \angle\left(\tan^{-1}\left(-\frac{10}{6}\right)+180\right)}{\sqrt{1+16}\ \angle\tan^{-1}(-4)+180}$$

2nd quadrant



angle: measured from Re axis, going counterclockwise
If going clockwise → negative angle.

$$= \frac{11.66\ \angle -59^\circ + 180^\circ}{4.12\ \angle -75.96+180^\circ} = 2.827\ \angle -59^\circ + 75.96^\circ$$

$$\boxed{\hat{Z} = 2.83\ \angle 16.96^\circ}$$

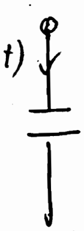
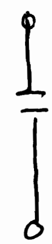
$$\boxed{\frac{\hat{z}'}{\hat{z}''} = \frac{z' e^{j\theta'}}{z'' e^{j\theta''}} = \frac{z'}{z''} e^{j(\theta' - \theta'')}}$$

Minus!

Back to circuit analysis using phasors:

R  $\longrightarrow$ $\hat{Z}_R = R$  (using Ohm's law with phasors)

L  $\longrightarrow$ $\hat{Z}_L = j\omega L$  "

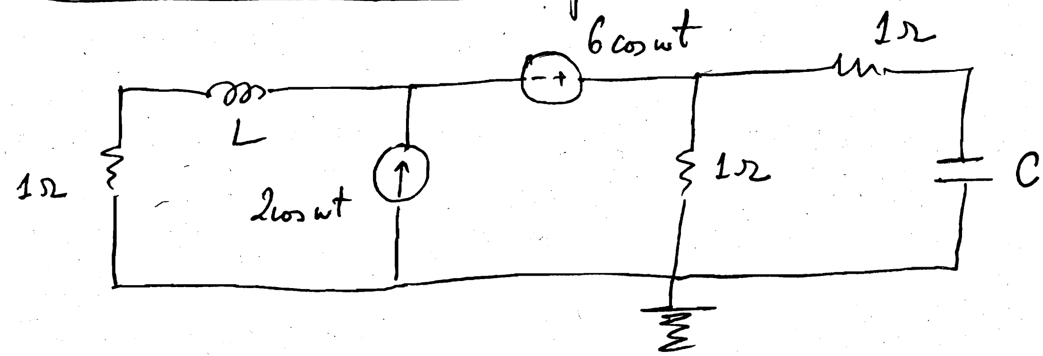
C  $i(t)$ $+$ $v(t)$ $-$ $\longrightarrow$ $\hat{Z}_C \stackrel{?}{=} \frac{1}{j\omega C}$ 

$i(t) = C\frac{dv}{dt}$ $\longrightarrow$ phasors: $\hat{I}\cancel{e^{j\omega t}} = C\underbrace{\frac{d}{dt}\left(\hat{V}e^{j\omega t}\right)}_{\hat{V}j\omega\cancel{e^{j\omega t}}}$

$$\hat{I} = C\hat{V}j\omega$$

$$\hat{Z}_C = \frac{\hat{V}}{\hat{I}} = \frac{1}{j\omega C}$$

AC Circuit Analysis Techniques:



$L = 2.65\,mH$

$C = 2.65\,mF$

$\omega = 2\pi f\;;\quad f = 60\,Hz$

Redraw same circuit in frequency-domain using phasors.