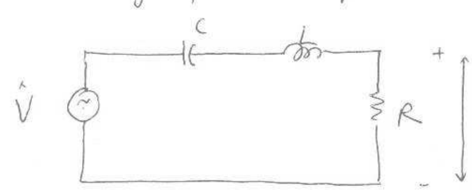
Variable Frequency Network Performance:

$$\hat{Z}_R = R$$

$$\hat{Z}_L = j\omega L$$

$$\hat{Z}_C = \frac{1}{j\omega C}$$

Find the transfer function for: $\frac{\hat{V}_o}{\hat{V}} =$ $(j\omega \to s)$



Say it has a poles of what order? (pole = zero of denominator)

By voltage division: $$\frac{\hat{V}_o}{\hat{V}} = \frac{R}{\frac{1}{j\omega C} + j\omega L + R} \xrightarrow[j\omega \to s]{} \frac{R}{\frac{1}{sC} + sL + R}$$

$$= \frac{RCs}{1 + s^2 LC + sRC} = \frac{\frac{R}{L}s}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$

Zeroes of denominator: $s^2 + s\frac{R}{L} + \frac{1}{LC} = 0 \to s = \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2}$

$$S = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\frac{R^2C - 4L}{L^2C}} = -\frac{R}{2L} \pm \frac{1}{2L}\sqrt{\frac{R^2C-4L}{C}}$$

- If $R^2C - 4L = 0$ or $\frac{R^2C}{L} = 4 \Rightarrow S_\pm = -\frac{R}{2L}$

  Then $s^2 + s\frac{R}{L} + \frac{1}{LC} = \left(s + \frac{R}{2L}\right)^2 \rightarrow$ Transfer function has a pole of 2nd order

- If $R^2C - 4L > 0 \Rightarrow s_\pm$

$$s^2 + s\frac{R}{L} + \frac{1}{LC} = \left(s + \frac{R}{2L} - \frac{1}{2L}\sqrt{\frac{R^2C-4L}{C}}\right)\left(s + \frac{R}{2L} + \frac{1}{2L}\sqrt{\frac{R^2C-4L}{C}}\right)$$

  $\rightarrow$ Transfer function has two poles of 1st order

- If $R^2C - 4L < 0 \Rightarrow S_\pm = -\frac{R}{2L} \pm j\frac{1}{2L}\sqrt{\frac{4L-R^2C}{C}}$

$$s^2 + s\frac{R}{L} + \frac{1}{LC} = \left(s + \frac{R}{2L} - j\frac{1}{2L}\sqrt{\frac{4L^2-R^2C}{C}}\right)\left(s + \frac{R}{2L} + j\frac{1}{2L}\sqrt{\frac{4L^2-R^2C}{C}}\right)$$

  $\rightarrow$ Transfer function has a pair of complex conjugate poles.

We will learn about frequency variation in connection with the type of poles of the transfer function.

Bode plots

Transfer function : $\hat{H}(j\omega)$ or $\hat{H}(s)$ (e.g. $\hat{H}(j\omega) = \frac{\hat{V}_0(j\omega)}{\hat{V}(j\omega)}$)

→ Bode Plots $\begin{cases} \text{Magnitude : } 20\log_{10}|\hat{H}(j\omega)| \text{ versus } \log_{10}\omega \\ \text{Phase : } \angle\hat{H}(j\omega) \text{ versus } \log_{10}\omega \end{cases}$

We now learn how to make Bode plots from a transfer function. (two way connection). Let's start with a standard form of $\hat{H}$: (which is in general a ratio of polynomials for linear circuits containing R, L, C)

$$\hat{H}(j\omega) = \frac{K_0 (j\omega)^{\pm N} (1 + j\omega\tau_1)\left[1 + 2\xi_3 j\omega\tau_3 + (j\omega\tau_3)^2\right] \ldots}{(1 + j\omega\tau_a)\left[1 + 2\xi_b j\omega\tau_b + (j\omega\tau_b)^2\right] \ldots}$$

(τ_1 "tau sub one")

(ξ_3 "xi sub three")

From our example: $\hat{H} = \frac{\hat{V}_0}{\hat{V}} = \frac{RCj\omega}{1 + (j\omega)^2 LC + j\omega RC}$

$\begin{cases} K_0 = RC;\ N = 1; \\ \tau_1, \tau_3, \xi_3 : \text{none} \\ \tau_a : \text{none} \\ \tau_b^2 = LC \rightarrow \tau_b = \sqrt{LC} \\ 2\xi_b\tau_b = RC \\ \rightarrow \xi_b = \frac{RC}{2\tau_b} = \frac{RC}{2\sqrt{LC}} \\ = \frac{R}{2}\sqrt{\frac{C}{L}} \end{cases}$

Review: 1) $\frac{\hat{z}_1 \cdot \hat{z}_2}{\hat{z}_3} \rightarrow \left|\frac{\hat{z}_1 \cdot \hat{z}_2}{\hat{z}_3}\right| \underset{\sqrt{\hat{z}\cdot\hat{z}^*}}{\underset{\downarrow}{=}} \sqrt{\frac{\hat{z}_1 \hat{z}_2 \hat{z}_1^* \hat{z}_2^*}{\hat{z}_3 \cdot \hat{z}_3^*}}$

$= \frac{\sqrt{(\hat{z}_1 \hat{z}_1^*)}\sqrt{\hat{z}_2 \hat{z}_2^*}}{\sqrt{\hat{z}_3 \hat{z}_3^*}} = \frac{|\hat{z}_1||\hat{z}_2|}{|\hat{z}_3|}$

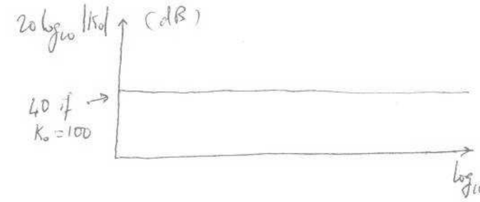
2) $\log_{10} \frac{|\hat{z}_1||\hat{z}_2|}{|\hat{z}_3|} = \log_{10}|\hat{z}_1| + \log_{10}|\hat{z}_2| - \log_{10}|\hat{z}_3|$

Magnitude Bode Plot

$$20\log_{10}|\bar{H}| = \overset{1)}{20\log_{10}|K_0|} \pm \overset{2)}{20N\log_{10}\omega} + \overset{3)}{20\log_{10}|1+j\omega\tau_1|} + \overset{4)}{20\log_{10}|1+2\xi_3 j\omega\tau_3+(j\omega\tau_3)^2|}$$
$$- \overset{5)}{20\log_{10}|1+j\omega\tau_a|} - \overset{6)}{20\log_{10}|1+2\zeta_b j\omega\tau_b+(j\omega\tau_b)^2|} - \cdots$$

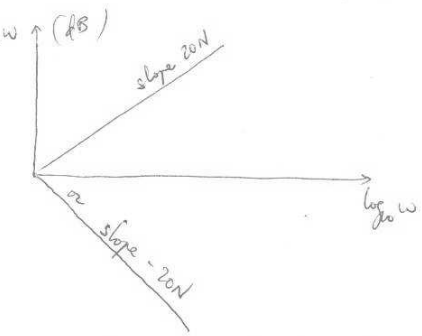
Let's see how to plot different terms:

1) $20\log_{10}|K_0|$ (dB)



(frequency in logarithmic scale)

2) $\pm 20N\log_{10}\omega$ (dB)



3) Asymptotic analysis: $20\log_{10}|1+j\omega\tau_1|$

$\omega \to 0$ (low freq): sufficiently low such that $j\omega\tau_1 \ll 1$

$$20\log_{10}|1+j\omega\tau_1| \simeq 0$$

$\omega \to \infty$ (high freq): sufficiently high such that $j\omega\tau_1 \gg 1$

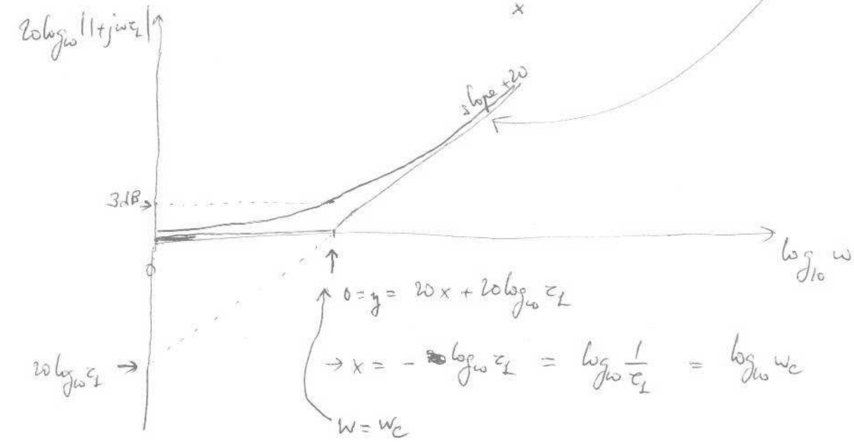
$$20\log_{10}|1+j\omega\tau_1| \simeq 20\log_{10}\omega\tau_1 = 20\log_{10}\omega + 20\log_{10}\tau_1$$

$$|j\omega\tau_1| = \omega\tau_1$$

$\omega_c\tau_1 \simeq 1$ or $\omega_c = \frac{1}{\tau_1} \rightarrow 20\log_{10}|1+j| = 20\log_{10}\sqrt{2} = 3\ \text{dB}$

critical frequency

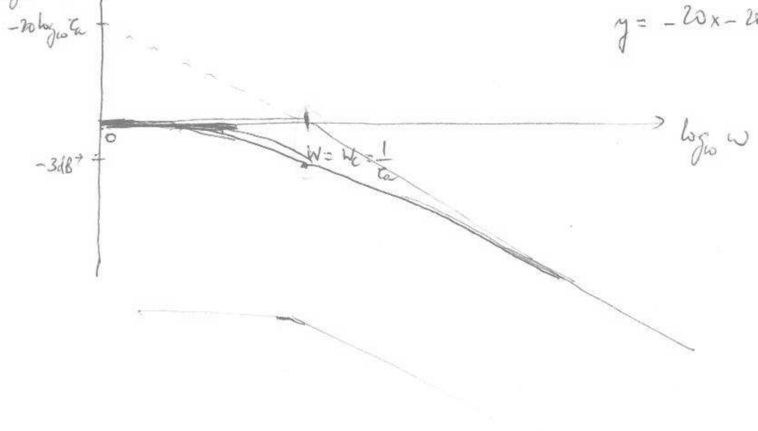
$$20\log_{10}|1+j\omega\tau_1| = \begin{cases} \omega \to 0 & 0 \\ \omega = \frac{1}{\tau_1} = \omega_c & 3\text{ dB} \\ \omega \to \infty & \underbrace{20\log_{10}\omega}_{x} + 20\log_{10}\tau_1 \quad \text{or} \quad y = 20x + \underbrace{20\log_{10}\tau_1}_{\text{negative when } \tau_1<1} \end{cases}$$



5) This term is similar to 3) except for the negative slope.

$-20\log_{10}|1+j\omega\tau_a|$

$$y = -20x - 20\log_{10}\tau_a$$



(11.13) 7th ed.

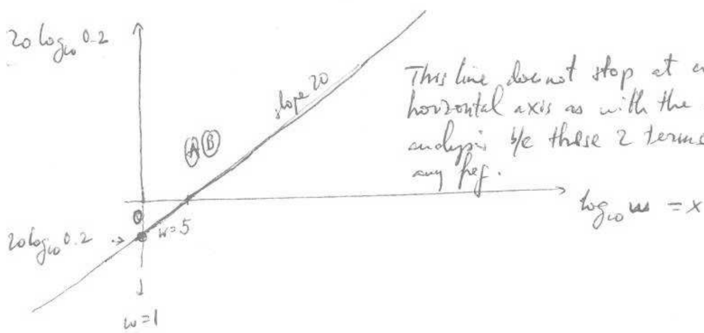
$$\hat{H}(j\omega) = \frac{100\, j\omega}{(1+j\omega)(10+j\omega)(50+j\omega)}$$ Magnitude Bode Plot?

To apply previous results → we put all factors in standard format

$$\hat{H}(j\omega) = \frac{\frac{100}{500}\, j\omega}{(1+j\omega)(1+j\frac{\omega}{10})(1+j\frac{\omega}{50})}$$

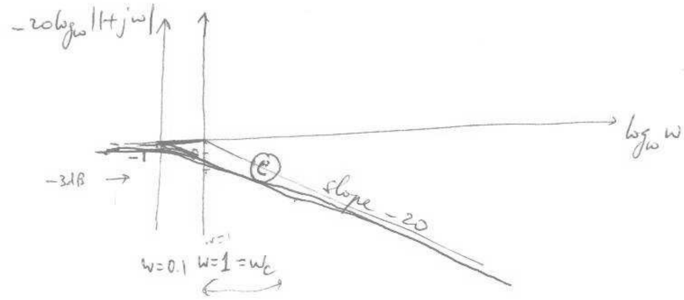
$$\rightarrow 20\log_{10}|\hat{H}| = \overset{A}{20\log_{10} 0.2} + \overset{B}{20\log_{10}\omega} - \overset{C}{20\log_{10}|1+j\omega|} - \overset{D}{20\log_{10}|1+j\tfrac{\omega}{10}|} - \overset{E}{20\log_{10}|1+j\tfrac{\omega}{50}|}$$

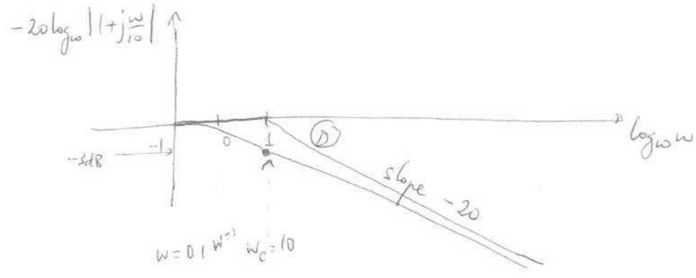
$y = \underbrace{20\log_{10}\omega}_{x} + 20\log_{10} 0.2$

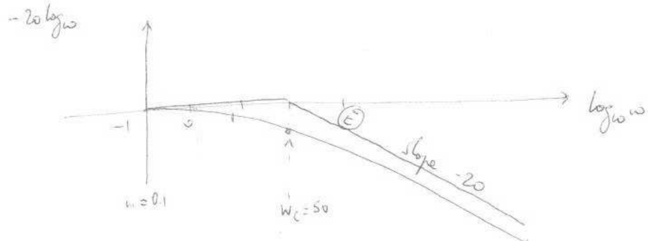


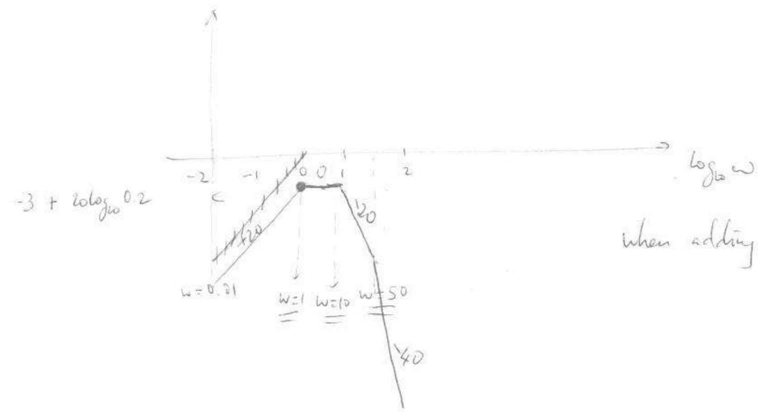
This line does not stop at crossing with horizontal axis as with the asymptotic analysis b/c these 2 terms are valid for any freq.

$-20\log_{10}|1+j\omega|$









when adding: do by pieces of frequency intervals

<u>In summary:</u> we are plotting the magnitude Bode plot for

$$\hat{H}(j\omega) = \frac{0.2\, j\omega}{(1+j\omega)(1+\frac{j\omega}{10})(1+j\frac{\omega}{50})}$$

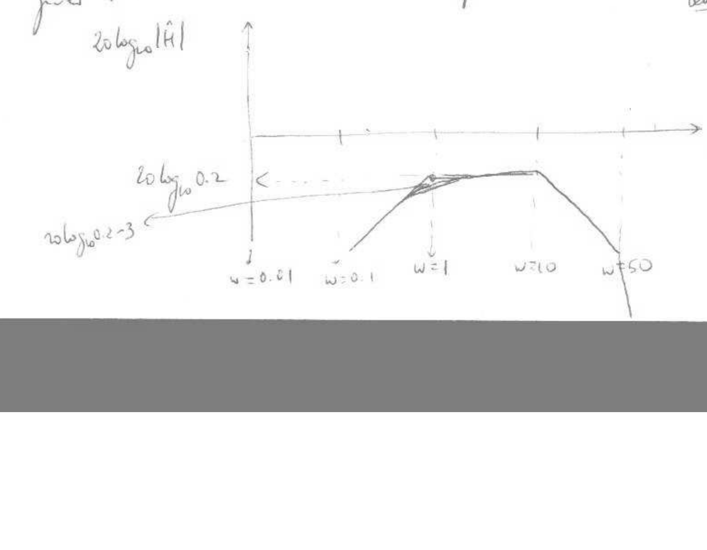
The numerator gives a line up at slope +20 at all ω (there is no asymptotic analysis on this term). The critical ω's are 1, 10, 50.

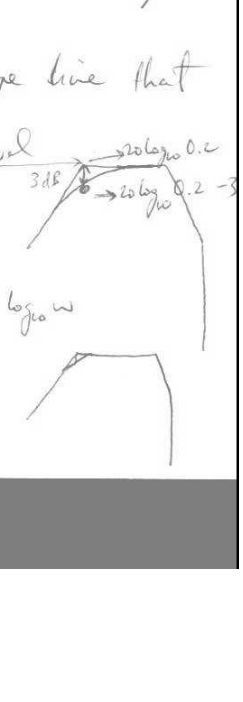
→ Before $\omega_c = 1$: only contribution from the +20 slope line that ends at $20\log_{10} 0.2 - 3$ dB.

→ At $\omega_c = 1$ term $(1+j\omega)$ provides a -20 slope line (from asymptotic analysis of large ω), this combines with the +20 slope line to give a horizontal piece. No contribution from $(1+j\frac{\omega}{10})$ or $(1+j\frac{\omega}{50})$ because they are 0 to the left of their critical points (from asymptotic analysis of small ω). This continues until $\omega_c = 10$

→ At $\omega_c = 10$ $(1+j\frac{\omega}{10})$ turns down as a -20 slope line while $(j\omega)$ & $(1+j\omega)$ continue their same behavior as before. This gives total behavior of a -20 slope line until $\omega_c = 50$. Since we are b/w 10 & 50, i.e. to the left of the $(1+j\frac{\omega}{50})$'s critical point, it has zero contribution.

→ At $\omega_c = 50$ $(1+j\frac{\omega}{50})$ kicks in with a -20 slope line that gives a total behavior: -40 slope line.





Bode Plot with quadratic poles and zeroes:

$$\overset{\text{zero}}{+}20\log_{10}\left|1+2\xi_3\, j\omega\tau_3+(j\omega\tau_3)^2\right| \quad \overset{\text{pole}}{-}\underbrace{20\log_{10}\left|1+2\xi_b(j\omega\tau_b)+(j\omega\tau_b)^2\right|}_{\text{asymptotic analysis}}$$

4)   6)

1) Small ω. $\omega\tau_b \ll 1 \rightarrow$

$$-20\log_{10}|\quad| \approx -20\log_{10}|1| = 0$$

2) Large ω. $\omega\tau_b \gg 1 \rightarrow$

$$-20\log_{10}|\quad| \approx -20\log_{10}|(j\omega\tau_b)^2|$$

$$= -20\log_{10}\omega^2 - 20\log_{10}\tau_b^2$$

$$= \underbrace{-40}\log_{10}\omega - 40\log_{10}\tau_b$$

3) $\omega\tau_b \approx 1$ or $\omega = \omega_c$

$$-20\log_{10}|\quad| \approx -20\log_{10}(2\xi_b)$$

ξ_b	$-20\log_{10}2(\xi_b)$
0.2	7.96
0.4	1.94
0.5	0 (no rounding at corner)
0.6	−1.58
0.8	−4.1

Corner is now controlled by ξ_b, not a constant of −3dB any more.