$$\left.\begin{aligned} M_{\text{on 1 by 2}} &= \frac{N_1 N_2 \mu S_1}{\ell_2} \\ M_{\text{on 2 by 1}} &= \frac{N_1 N_2 \mu S_2}{\ell_1} \end{aligned}\right\} \text{ Same } M \quad \left(\frac{S_1}{\ell_2} = \frac{S_2}{\ell_1}\right)$$

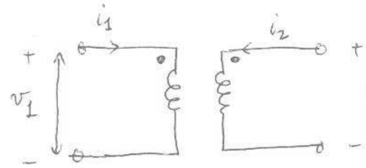
$$\text{or } S_1 \ell_1 = S_2 \ell_2$$

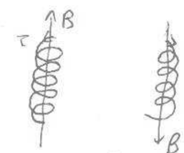
(same volume in both solenoids)

Rest of chapter: use correct sign for mutual inductance term.

$\downarrow$

assisted by the "dot convention"





this internal (to solenoid) info is coded by "dot convention".

$\downarrow$

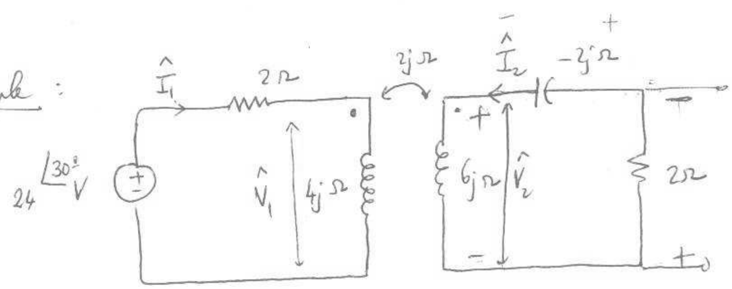
Correct equation (signs)

$$\boxed{\begin{aligned} v_1 &= L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \\ v_2 &= L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} \end{aligned}} \longrightarrow \boxed{\begin{aligned} \hat{V}_1 &= j\omega L_1 \hat{I}_1 + M j\omega \hat{I}_2 \\ \hat{V}_2 &= j\omega L_2 \hat{I}_2 + j\omega M \hat{I}_1 \end{aligned}} \checkmark$$

T.D. equations F.D. equations

$v_1 \rightarrow \hat{V}_1 e^{j\omega t}$; $i_1 \rightarrow \hat{I}_1 e^{j\omega t}$

Example :



$\hat{V}_o = -2\hat{I}_2$

Left loop: $24\angle 30^\circ = j4\,\hat{I}_1 + j2\,\hat{I}_2 + 2\,\hat{I}_1 = \hat{I}_1(2+4j) + j2\,\hat{I}_2$

Right loop: $0 = j6\,\hat{I}_2 + j2\,\hat{I}_1 + \hat{V}_o - 2j\,\hat{I}_2 + 2\,\hat{I}_2 = \hat{I}_2(2+4j) + j2\,\hat{I}_1$

$$\hat{I}_1 = -\hat{I}_2\,\frac{(2+4j)}{2j} = -\hat{I}_2(-j+2)$$

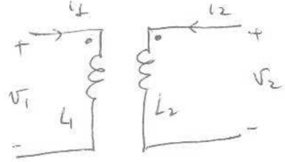
$$24\angle 30^\circ = -\hat{I}_2\,\underbrace{(2-j)(2+4j)}_{8+6j} + \hat{I}_2\,2j = \hat{I}_2(2j-8-6j) = \hat{I}_2(-8-4j)$$

$$\hat{I}_2 = \frac{6\angle 30^\circ}{-2-j} = \frac{6\angle 30^\circ}{\sqrt{5}\,\angle \tan^{-1}\frac{1}{2}} = \frac{6\angle 30^\circ}{\sqrt{5}\,\angle 206.56^\circ} = 2.68\angle -176.56^\circ \text{ A}$$

$$\hat{V}_o = -2 \times 2.68\angle -176.56^\circ = 5.36\angle 3.43^\circ \quad ✓$$

(23)

Energy of two magnetically coupled inductors:



$$\mathcal{E} = \int \underbrace{P(t)}_{\text{power (energy per unit time)}} dt$$

Total time interval: $\underbrace{0 \longrightarrow t_1}_{\text{current building up in } L_1} \underbrace{\longrightarrow t_2}_{\text{current building up in } L_2}$

1) i_1 increases from 0 to I_1
2) $i_2 = 0$

1) $i_1 = \text{constant} = I_1$
2) i_2 increase from 0 to I_2

$$\mathcal{E}_{0\to t_2} = \int_0^{t_1} P(t)\,dt + \int_{t_1}^{t_2} P(t)\,dt$$

$$P(t) = v_1(t)\,i_1(t) + v_2(t)\,i_2(t) = \left(L_1\frac{di_1}{dt} + M\frac{di_2}{dt}\right)i_1 + \left(L_2\frac{di_2}{dt} + M\frac{di_1}{dt}\right)i_2$$

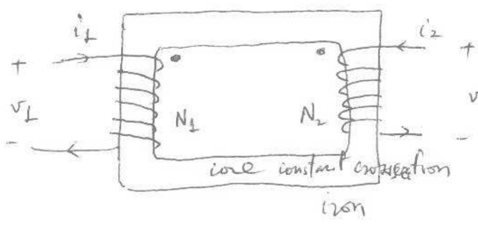
$$= \int_0^{t_1}\left(L_1\frac{di_1}{dt}\,i_1 + M\frac{di_1}{dt}\underset{0}{(i_2)}\right)dt + \int_{t_1}^{t_2}\left(M\frac{di_2}{dt}\underset{I_1}{i_1} + L_2\frac{di_2}{dt}\,i_2\right)dt$$

$$= L_1\int_0^{t_1} i_1\,di_1 \qquad + MI_1\int_{t_1}^{t_2} di_2 + L_2\int_{t_1}^{t_2} i_2\,di_2$$

$$= L_1\left(\frac{i_1^2}{2}\right)_0^{t_1} \qquad + MI_1\left[i_2\right]_{t_1}^{t_2} + L_2\left(\frac{i_2^2}{2}\right)_{t_1}^{t_2}$$

$$= \frac{L_1I_1^2}{2} \qquad + MI_1I_2 \qquad + \frac{L_2I_2^2}{2}$$

Ideal Transformer : (change a voltage)



The two solenoids can have different diameters but iron core thin then has same cross-section → same magnetic flux

$$\Phi_1 = B_1 S_1 \qquad \Phi_2 = B_2 S_2$$
$$v_1 = N_1 \frac{d\phi_1}{dt} \qquad v_2 = N_2 \frac{d\phi_2}{dt}$$

Since $\phi_1 = \phi_2$

$$\Rightarrow \boxed{\frac{v_1}{N_1} = \frac{v_2}{N_2}} \text{ or } \boxed{\frac{v_1}{v_2} = \frac{N_1}{N_2}}$$

Ampere's law : $\oint_{loop} \vec{H} \cdot d\vec{l} = i_{enclosed}$ by the loop $= N_1 i_1 + N_2 i_2$

$\vec{B} = \mu \vec{H}$

iron core : $\mu = \infty$; for finite values for $\vec{B}$ $\Rightarrow$ $\vec{H}$ needs to be 0 ⟶

$$N_1 i_1 = - N_2 i_2 \rightarrow \boxed{\frac{i_1}{i_2} = -\frac{N_2}{N_1}}$$

Power Analysis

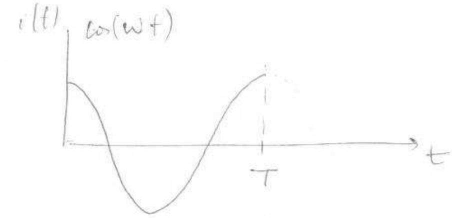
$\boxed{\text{Instantaneous Power}}$: $p(t) = i(t)\, v(t)$

AC circuits: $i(t) = I_M \cos(\underbrace{\omega t + \theta_i}_{\alpha})$; $v(t) = V_M \cos(\underbrace{\omega t + \theta_v}_{\beta})$

$p(t) = I_M V_M \frac{1}{2}\left[\cos(2\omega t + \theta_i + \theta_v) + \cos(\theta_i - \theta_v)\right]$

trig id: $\cos\alpha \cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$

$$p(t) = \underbrace{\frac{1}{2} I_M V_M \cos(\theta_v - \theta_i)}_{D.C.} + \underbrace{\frac{1}{2} I_M V_M \cos(2\omega t + \theta_v + \theta_i)}_{A.C.}$$

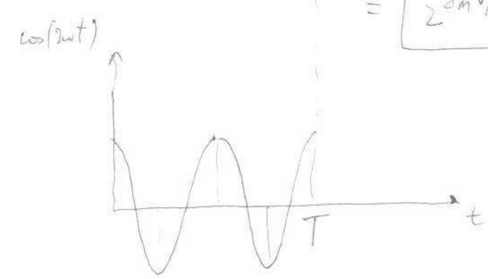


$\overline{i} = \overline{v} = 0$ but $\overline{p} \neq 0$
(need to pay utility bill)

Average power: $P = \frac{1}{T}\int_{t_0}^{t_0+T} p(t)\, dt$

(T → period)

$$= \boxed{\frac{1}{2} I_M V_M \cos(\theta_v - \theta_i)} + \frac{1}{2} I_M V_M \underbrace{\frac{1}{T}\int_T \cos(2\omega t + \theta_v + \theta_i)}_{\text{area under curve} = 0}$$



$\hat{Z} = R + jX \quad \begin{cases} R: \text{resistive} \\ X: \text{reactive} \end{cases}$ (26)

$p(t) = \frac{1}{2} I_m V_m + \frac{1}{2} I_m V_m \cos(2\omega t + 2\theta_v)$

at pure-resistive (impedance is real #) $\rightarrow$ voltage & current are in phase $\theta_v = \theta_i$

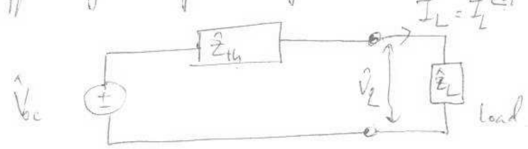
$p(t) = 0 + \frac{1}{2} I_m V_m \cos(2\omega t + \theta_i + \theta_v)$

at pure-reactive (impedance is imaginary #) voltage & current are $\pm 90°$ out of phase $= \theta_v - \theta_i = \pm 90°$

Maximum power I can get out of a circuit:

Suppose you represent your circuit with a Thevenin's equivalence

$\hat{I}_L = I_L \angle \theta_i$



$\hat{V}_L = V_L \angle \theta_v$

Need to find $\hat{Z}_{L_{max}}$ that will draw the most power from this circuit. Average power at load is

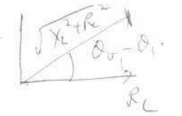
$P_L = \frac{1}{2} V_L I_L \cos(\theta_v - \theta_i) \leftarrow ①②③$

$\hat{I}_L = \frac{\hat{V}_{oc}}{\hat{Z}_{th} + \hat{Z}_L} = \frac{\hat{V}_{oc}}{R_{th} + R_L + j(X_{th} + X_L)} \rightarrow \boxed{I_L = \frac{V_{oc}}{\sqrt{(R_{th}+R_L)^2 + (X_{th}+X_L)^2}}}$ ①

$\underbrace{\hat{V}_L =}_{\text{voltage div.}} \hat{V}_{oc} \frac{\hat{Z}_L}{\hat{Z}_{th} + \hat{Z}_L} \longrightarrow \boxed{V_L = \frac{V_{oc}\sqrt{R_L^2 + X_L^2}}{\sqrt{(R_{th}+R_L)^2 + (X_{th}+X_L)^2}}}$ ②

$\cos(\theta_v - \theta_i) =$

$\hat{Z}_L = \frac{\hat{V}_L}{\hat{I}_L} = \frac{V_L}{I_L} \angle \theta_v - \theta_i = R_L + jX_L = \sqrt{R_L^2 + X_L^2} \angle \tan^{-1}\frac{X_L}{R_L}$



$\theta_v - \theta_i = \tan^{-1}\frac{X_L}{R_L}$

$\boxed{\cos(\theta_v - \theta_i) = \frac{R_L}{\sqrt{R_L^2 + X_L^2}}}$ ③

(27)

Then $P_L = \frac{1}{2} \frac{V_{oc}\cancel{\sqrt{R_L^2+X_L^2}}}{\sqrt{(R_{th}+R_L)^2+(X_{th}+X_L)^2}} \cdot \frac{V_{oc}}{\sqrt{(R_{th}+R_L)^2+(X_{th}+X_L)^2}} \frac{R_L}{\cancel{\sqrt{R_L^2+X_L^2}}}$

$= \frac{1}{2} \frac{V_{oc}^2 R_L}{(R_{th}+R_L)^2+(X_{th}+X_L)^2}$ Average Power at load.

Max P_L in 2 steps:

1) $X_L = wL$ $X_C = \frac{-1}{wC}$ $\Rightarrow$ X's (reactances) can have + or − signs) while R's can only be +

$\rightarrow P_L$ is higher when $\boxed{X_L = -X_{th}}$ (1)

2) Maximize: $\underbrace{\frac{1}{2} \frac{V_{oc}^2 R_L}{(R_{th}+R_L)^2}}_{f(R_L)}$ with respect to R_L

$\rightarrow \frac{\partial f}{\partial R_L} = 0 = \frac{V_{oc}^2}{2(R_{th}+R_L)^2} - \frac{2}{2} \frac{V_{oc}^2 R_L}{(R_{th}+R_L)^3} =$

$\rightarrow 0 = \frac{\cancel{V_{oc}^2}}{2} - \frac{\cancel{V_{oc}^2} R_L}{R_{th}+R_L}$

$\rightarrow \frac{R_L}{R_{th}+R_L} = \frac{1}{2} \rightarrow 2R_L = R_{th}+R_L$

$\boxed{R_L = R_{th}}$ (2)

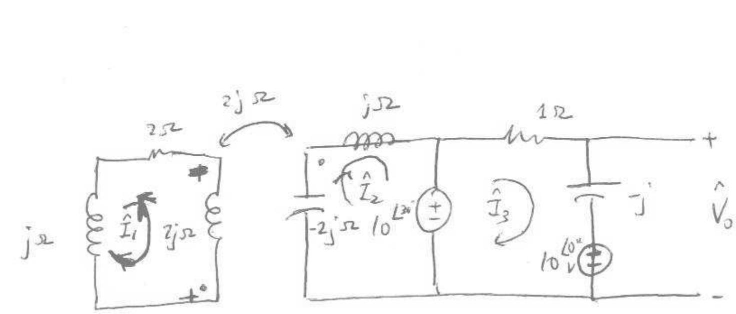
(1) & (2) $\Rightarrow \hat{Z}_L = \hat{Z}_{th}^{*}$ to draw max power out of the circuit

"*" Complex conjugate: $\hat{z} = a + jb$: $\hat{z}^* = a - jb$ (change sign of imaginary term).

$\rightarrow \boxed{P_{L_{max}} = \frac{1}{2} \frac{V_{oc}^2 R_{th}}{4 R_{th}^2} = \frac{V_{oc}^2}{8 R_{th}}}$ max power I can draw from a given circuit.

(28)

$\boxed{8.29}$



Loop analysis : 3 loops.

$\underbrace{\hat{I}_1 : CCW ; \ \hat{J}_2 : CW}_{\text{for + for Mutual Inductance Term}} ; \ \hat{I}_3 : CW$

1) $2j\hat{I}_1 + 2j\hat{I}_2 + 2\hat{I}_1 + \hat{I}_1 j = 0 = \hat{I}_1(2+3j) + 2j\hat{I}_2$ ✓

2) $10^{\angle 30°} - 2j\hat{I}_2 + j\hat{I}_2 + 2j\hat{I}_1 = 0 \Rightarrow 10^{\angle 30°} = j\hat{I}_2 - 2j\hat{I}_1$ ✓

3) $-10^{\angle 30°} + \hat{I}_3(1-j) + 10 = 0$ ✓

$$\hat{I}_3 = \frac{10^{\angle 30°} - 10}{1-j} = \frac{10(\cos 30° - 1) + j\,10\sin 30°}{\sqrt{2}^{\angle -45°}}$$

$$= \frac{-1.34 + j5}{\sqrt{2}^{\angle -45°}} = \frac{5.17^{\angle 105°}}{\sqrt{2}^{\angle -45°}}$$

$$= 3.67^{\angle 150°} \text{ A}$$

$$\hat{V}_0 = -j\hat{I}_3 + 10 = 3.67^{\angle 60°} + 10 = (3.67\underbrace{\cos 60°}_{0.5} + 10) + j\,3.67\sin 60°$$

$$= \overset{11.835}{\cancel{12.57}} + j\,3.18 = 12.25^{\angle 15.8°} \text{ V}$$